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EFFECTS OF PARTIAL FREQUENCY REDISTRIBUTION WITH DIPOLE SCATTERING ON THE FORMATION OF SPECTRAL LINES IN EXPANDING MEDIA

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ABSTRACT

Line formation in expanding spherical atmospheres using partial frequency redistribution with dipole scattering has been studied by using a non LTE two level atom model. Lines with zero natural line width are treated by using the angle dependent and angle independent redistribution function R_I (see Unno 1952, Field 1959, Hummer 1962, and Mihalas 1978). Lines formed by partial and complete redistribution with isotropic scattering also have been calculated for the sake of comparison with those formed by dipole scattering. The ratios of outer to inner radii of the atmosphere have been taken to be 1, 10 and 100 so that the effects of sphericity are clearly separated from those of plane parallel approximation. Velocities upto 2 thermal units are considered in the rest frame of the star. Two cases have been considered (1) $\epsilon = 10^{-3}$ and $\beta = 0$ and (2) $\epsilon = \beta = 10^{-3}$ where ϵ is the probability per scatter that the photon is destroyed by collisional de excitation and β is the ratio K_C/K_L of opacity due to continuous absorption per unit interval of frequency to that in the line. The total optical depth T_L at the line centre is taken to be approximately 10^3

Several important differences have been observed among the lines calculated using the five redistribution functions. However, for all parameters ϵ, β, B/A and v, the differences between the lines formed by the angle independent and angle dependent R_I with dipole scattering are substantially small so that it is not possible to resolve them graphically. When the velocity at the outermost layer is 2, the P Cygni type profiles are obtained (i.e.) with red emission and blue absorption. This effect is more pronounced in the extended spherical medium than in the plane parallel situation. However, in all situations, the lines formed by dipole scattering show less emission and absorption

Key Words radiative transfer- partial frequency redistribution function—dipole scattering

1. Introduction

Effects of photon frequency redistribution on the formation of spectral lines in stellar atmospheres are of considerable importance (Hummer 1962, 1969, Shine *et al* 1975, Vardavas 1976, Mihalas 1978, Peraiah 1978 etc) The frequency redistribution of photons after several scatterings and absorptions in the line, will change the photon escape probability through the outer surface of the stellar atmosphere. When the matter in the atmosphere is expanding radially, the line emitted by this gas shifts continuously. This means that the wing photons which would have escaped the atmosphere had the medium been stationary, will be absorbed and re-emitted with redistribution in both angle and frequency at a different radial point in the medium. The result of this is that in a moving medium (radially outwards) the source function is changed and the redistribution of photons in both angle and frequency becomes extremely complicated to understand. The situation becomes more complex when sphericity is introduced. Redistribution in frequency is influenced by the velocity gradients in the gas while the redistribution in angle coupled with the sphericity will affect both photon frequency redistribution and the motion of the gas itself through the radiation pressure in the line. So, it is important to treat the problem of transfer of line radiation by taking into account angle dependent frequency redistribution in an expanding spherical atmosphere

We wish to investigate in this paper, the effects of angle dependent partial redistribution functions on the formation of spectral lines in an expanding spherical medium. Considerable amount of work has been done by using isotropic scattering but the effects of dipole scattering on the spectral line formation are yet to be investigated We shall, therefore, consider the effects of angle averaged and angle dependent partial frequency redistribution on line formation with dipole scattering For the sake of simplicity, we consider the redistribution function R_I (See Hummer 1962) and solve the line transfer in the rest frame of the star (Wehrse and Peraiah 1979 and Peraiah and Wehrse 1978, Peraiah 1978) In this event, we have the frequency of the line photon shifted by

$$x = x' \pm v\mu$$

where $x' = (\nu-\nu_0)/\Delta\nu_D$, $\Delta\nu_D$ being the Doppler width and v is the velocity of the gas in thermal units and μ is the cosine of the angle between the ray and the radius vector The $\pm$ signs represent the oppositely directed beams of the photons of frequency x' We shall assume that the gas is expanding radially outwards with a maximum of 2 thermal units of velocity.

We shall describe briefly the redistribution function in section 2 and in sections 3-7 details of the method shall be given. The results are discussed in section 8 and the coding for computation of the lines is listed in Appendix.

2 Redistribution Function

If we consider the absorption of a photon of frequency ν with direction n within the elements $d\nu$ and $d\Omega$ then the probability of the subsequent emission of this photon with frequency ν' and direction n' within the element $d\nu'$ and $d\Omega'$ is given by

$$R(\nu\ n,\ \nu',\ n')\ d\nu'\ d\Omega'\ d\nu\ d\Omega \tag{1}$$

subjected to the normalization that

$$\iiiint R(\nu,\ n,\ \nu',\ n')\ d\nu',\ d\Omega'\ d\nu\ d\Omega = 1 \tag{2}$$

If $\phi(\nu')\ d\nu'$ is the probability that a photon with frequency in the interval $(\nu',\ \nu'+d\nu')$ is absorbed and as each absorbed photon must be emitted, we must have,

$$4\pi \iint R(\nu',\ n',\ \nu,\ n)\ d\nu\ d\Omega = \phi(\nu',\ n') \tag{3}$$

which again is subjected to the normalization condition,

$$\iint \phi(\nu',\ n')\ d\nu'\ d\Omega' = 1 \tag{4}$$

The angle dependent redistribution function for the lines with zero natural line width in the case of isotropic scattering is given by (see Hummer 1962, Mihalas 1970, Unno 1952, Field 1959).

$$R_I\ (x,\ n,\ x',\ n') = \frac{1}{16\ \pi^3 \sin\gamma} \exp\left[-x'^2 - (x-x'\cos\gamma)^2 \operatorname{cosec}^2\gamma\right] \tag{5}$$

and for the dipole scattering,

$$R_{I-D}\ (x,\ n,\ x',\ n') = \frac{3\ (1+\cos^2\gamma)}{64\ \pi^3 \sin\gamma} \exp\left[-x'^2 - (x-x\cos\gamma)^2 \operatorname{cosec}^2\gamma\right] \tag{6}$$

Correspondingly, the angle averaged redistribution functions are

$$R_{I-A}\ (x,\ x') = \frac{1}{\sqrt{\pi}} \int_{|x|}^{\infty} e^{-t^2}\, dt \tag{7}$$

for isotropic scattering and for dipole scattering

$$R_{I-AD}(x, x') = \frac{3}{8}\left\{\frac{1}{\sqrt{\pi}}\int_{|\bar{x}|}^{\infty} e^{-t^2}\,dt\,[3+2(x^2+x'^2)+4x^2x'^2] - \frac{e^{-|\bar{x}|^2}}{\sqrt{\pi}}|\bar{x}|\,(2|\underline{x}|^2+1)\right\} \tag{8}$$

Here $|\bar{x}|$ and $|\underline{x}|$ are the maximum and minimum values of $|x|$ and $|x'|$

In a static medium, the functions described in equations (7-10) follow certain symmetry relations (see Hummer 1962)

$$R(-x, n, -x', n') = R(x, n, x', n') \tag{9}$$

$$R(-x, -n, x', n') = R(-x, n, x', -n') = R(x, n, x', n') \tag{10}$$

$$\text{and } R(x, n, x', n') = R(x', n', x, n') \tag{11}$$

However, the last relation does not hold in the case of non-coherence in the atom's frame In the case of angle averaged functions, we have

$$R(-x, -x') = R(x, x') \tag{12}$$

$$\text{and } R(x, x') = R(x', x) \tag{13}$$

As the photon redistribution is symmetric in the line in a static medium, it is enough if we calculate the functions for one set of frequencies and angles However, in a moving medium, the photon redistribution is asymmetric and consequently, we have to calculate all the four asymmetric redistribution functions in the medium to represent the gas velocity at the given point as the presence of velocity gradients of the gas and the angular redistribution, particularly in a spherically symmetric media, will change the photon redistribution in the line. The redistribution functions have been calculated following the procedures described in Milkey *et al* (1975). However, to calculate angle averaged R_I functions a simple numerical integration is used as this is not time consuming

3 Interaction Principle

In the following sections, we shall describe the solution of radiative transfer in detail First, we shall introduce the Interaction Principle which explains the relationship between the input and output radiation fields from a given medium irrespective of its physical properties. We shall follow closely the two papers of Grant and Hunt (1969a, b)

Consider a medium stratified with 1-parameter family of surfaces with radial boundaries $r_1, r_2, \ldots r_n, r_{n+1}$ At any level we define two oppositely directed specific intensities or simply intensities $U^+(r_n)$, $U^-(r_n)$. Let μ be the cosine of the angle made by a ray with the radius vector in the direction in which r decreases or n increases and the optical depth increases

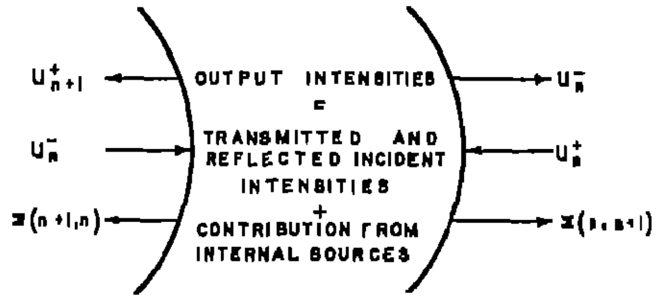


Fig. 1. Interaction Principle

we shall write,

$$U^+(r_n) = \{U(r_n, \mu) \quad 0<\mu \leq 1\}$$

$$U^-(r_n) = \{U(r_n, -\mu) \quad 0<\mu \leq 1\}$$

where U's represent the intensities of the radiation specified by the direction μ. We select a finite set of values of μ, $\{\mu_j \; 1 \leq j \leq m, \; 0 \leq \mu_1 < \mu_2 < \mu_m \leq 1\}$ and write $U^+(r_n)$ and $U^-(r_n)$ as vectors in m-dimensional Eucleadian space

$$U^+(r_n) = \begin{bmatrix} U(r_n, \mu_1) \\ U(r_n, \mu_2) \\ \\ U(r_n, \mu_{m-1}) \\ U(r_n, \mu_m) \end{bmatrix}, \quad U^-(r_n) = \begin{bmatrix} U(r_n, -\mu_1) \\ U(r_n, -\mu_2) \\ \\ U(r_n, -\mu_{m-1}) \\ U(r_n, -\mu_m) \end{bmatrix} \tag{14}$$

Consider now, a surface bounded by r_n and r_{n+1} as shown in figure 1. Let $U^+(r_n)$ and $U^-(r_{n+1})$ be the incident intensities and $U^+(r_{n+1})$ and $U^-(r_n)$ be the emergent intensities which are linearly dependent on the former and on the sources present in the layer. Therefore, we shall write, (hereafter, we shall omit r and retain its subscripts only)

$$U^+_{n+1} = t(n+1, n)\,U^+_n + r(n, n+1)\,U^-_{n+1} + \Sigma^+(n+1, n)$$

$$U^-_n = r(n+1, n)\,U^+_n + t(n, n+1)\,U^-_{n+1} + \Sigma^-(n, n+1) \tag{15}$$

or

$$\begin{bmatrix} U^+_{n+1} \\ U^-_n \end{bmatrix} = S(n, n+1) \begin{bmatrix} U^+_n \\ U^-_{n+1} \end{bmatrix} + \Sigma(n, n+1) \tag{16}$$

The pair $t(n+1, n)$ and $t(n, n+1)$ are the linear operators of diffuse transmission and $r(n, n+1)$, $r(n+1, n)$ are of diffuse reflection. These operators can be physically interpreted as follows. For example in $a \leq r_n \leq r_{n+1} \leq b$, we define $r(n, n+1)$ as an integral operator

$$r(n, n+1)\,U^-_{n+1} = \{\int_0^1 r(n, \mu, n+1, -\mu')\,U_{n+1}(-\mu')\,d\mu', \; 0 < \mu \leq 1 \tag{17}$$

or if we descretize the angle variable,

$$[r(n, n+1)\,U^-_{n+1}]_j = \sum_{k=1}^{J} r(n, \mu_j, n+1, -\mu_k)\,U_{n+1}(-\mu_k), \; 1 \leq j \leq J$$

$$r(n, n+1) = \{r(n, \mu_j, n+1, -\mu_k)\} \tag{18}$$

Equations (15) and (16) are called the Principle of Interaction, (Preisendorfer 1965). Redheffer (1962) developed a theory based on this principle but without the source terms. Grant and Hunt (1969a, b) introduced the source terms which are of considerable importance in the astrophysical context.

The Principle of Interaction derived here is most general and the r and t operators include the geometry and the physical properties of the medium. One can apply them to any partitioning of a medium by a suitable 1-parameter family of surfaces as has been demonstrated by Grant and Hunt (1969a, b) for plane parallel layers and by Peraiah and Grant (1973) for spherical shells. Now that we have obtained the response functions for a layer of specified boundaries with given inputs, we shall proceed to calculate the response functions for two or more consecutive layers, a process termed as "Star Product" (see Redheffer 1962, Grant and Hunt 1969a, Preisendorfer 1965).

4 Star Product

Let there be two layers with boundaries r_n, r_{n+1} and r_{n+2} where $a \le r_n \le r_{n+1} \le r_{n+2} \le b$. Then from equation (16)

$$\begin{bmatrix} U^+_{n+1} \\ U^-_n \end{bmatrix} = S\,(n, n+1) \begin{bmatrix} U^+_n \\ U^-_{n+1} \end{bmatrix} + \Sigma\,(n, n+1)$$

and (19)

$$\begin{bmatrix} U^+_{n+2} \\ U^-_{n+1} \end{bmatrix} = S\,(n+1, n+2) \begin{bmatrix} U^+_{n+1} \\ U^-_{n+2} \end{bmatrix} + \Sigma\,(n+1, n+2)$$

As r_n, r_{n+1}, r_{n+2} are arbitrary, we can again write using the principle of interaction,

$$\begin{bmatrix} U^+_{n+2} \\ U^-_n \end{bmatrix} = S\,(n, n+2) \begin{bmatrix} U^+_n \\ U^-_{n+2} \end{bmatrix} + \Sigma\,(n, n+2) \tag{20}$$

Equations (20) can be obtained by eliminating U^+_{n+1} and U^-_{n+1} from (19).

The relation between S (n, n+1), S (n+1, n+2) and S (n, n+2) is called 'Star Product' of the two S-matrices,

$$S\,(n, n+2) = S\,(n, n+1) * S\,(n+1, n+2) \tag{21}$$

We recall from equation (16) that,

$$S\,(n, n+1) = \begin{bmatrix} t\,(n+1, n) & r\,(n, n+1) \\ r\,(n+1, n) & t\,(n, n+1) \end{bmatrix} \tag{22}$$

so that S (n, n+2) is given by

$$S\,(n, n+2) = \begin{bmatrix} t\,(n+2, n) & r\,(n, n+2) \\ r\,(n+2, n) & t\,(n, n+2) \end{bmatrix} \tag{23}$$

where,

$$t\,(n+2, n) = t\,(n+2, n+1)\,[I - r\,(n, n+1)\, r\,(n+2, n+1)]^{-1}\, t\,(n+1, n)$$

$$t\,(n, n+2) = t\,(n, n+1)\,[I - r\,(n+2, n+1)\, r\,(n, n+1)]^{-1}\, t\,(n+1, n+2)$$

$$r\,(n+2, n) = r\,(n+1, n) + t\,(n, n+1)\, r\,(n+2, n+1)\,[I - r\,(n, n+1)\, r\,(n+2, n+1]^{-1}\, t\,(n+1, n)$$

$$r\,(n, n+2) = r\,(n+1, n+2) + t\,(n+2, n+1)\, r\,(n, n+1)\,[I - r\,(n+2, n+1)\, r\,(n, n+1]^{-1}\, t\,(n+1, n+2) \tag{24}$$

where I is the identity operator. The star product exists whenever either of the inverses in equation (24) exists. The physical meaning of these operators are clearly explained in Grant and Hunt (1969a).

It is clear that the star multiplication is non-commutative (*i e*)

$$S(i) * S(j) \neq S(j) * S(i) \tag{25}$$

for i^{th} and j^{th} layers Furthermore, since the final result cannot depend on the order in which superposition takes place, star multiplication is associative or,

$$S[i * (j * k)] = S[(i * j) * k] = S[i * j * k] \tag{26}$$

Finally, let us consider the source term Σ The result of adding two layers may be written in terms of two linear operators L (n, n+1, n+2) and L' (n, n+1, n+2) and

$$\Sigma(n, n+2) = L(n, n+1, n+2)\, \Sigma(n, n+1) + L'(n, n+1, n+2)\, \Sigma(n+1, n+2) \tag{27}$$

where,

$$L(n, n+1, n+2) = \begin{bmatrix} t(n+2, n+1)[I - r(n, n+1)\, r(n+2, n+1)]^{-1} & 0 \\ t(n, n+1)\, r(n+2, n+1)[I - r(n, n+1)\, r(n+2, n+1)]^{-1} & I \end{bmatrix}$$

and

$$L'(n, n+1, n+2) = \begin{bmatrix} I & t(n+2, n+1)\, r(n, n+1)\, [I - r(n+2, n+1)\, r(n, n+1)]^{-1} \\ 0 & t(n, n+1)[I - r(n+2, n+1)\, r(n, n+1)]^{-1} \end{bmatrix} \tag{28}$$

It is quite obvious that there is close similarity between the relations of star product and the above relations

Usually, in practical problems, one divides the medium into N layers or shells and calculates S for each shell and adds them up by the star product Clearly,

$$S(1, N) = S(1, 2) * S(2, 3) * \cdots * S(n, n+1) * S(n+1, n+2) * \cdots * S(N-1, N) \tag{29}$$

A corresponding equation can be written for the source terms. Adding layer by layer at a time one can calculate the complete external response.

6 Calculation of the Internal Diffuse Radiation Field

One should be able to calculate, the fluxes at any point inside the medium bounded by radii r_1 and r_N where N represents the number of partitions of the medium The details of its derivation is given in Grant and Hunt (1968) and we shall quote only the results

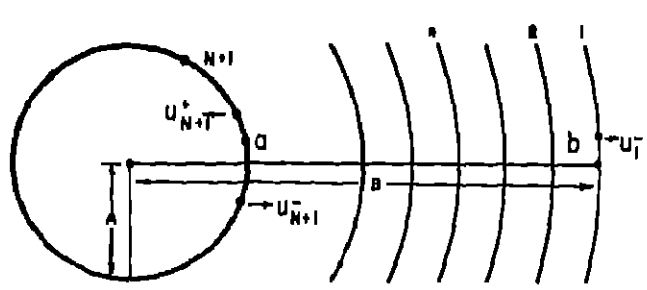


Fig. 2 Geometry of the Diffuse Radiation Field

Consider an atmosphere with radius B around a star of radius A (see Peraiah and Grant 1973). Let us divide the atmosphere into N shells or N+1 surfaces. Calculate the r and t operators (see section 3) for each shell

We wish to calculate the fluxes at the boundary of each shell inside the medium. Compute, sequentially, for $n = 1, 2, \ldots N$, the matrices $r(1, n)$ and vectors $V^{+}_{n+\frac{1}{2}}$, $V^{-}_{n+\frac{1}{2}}$ from

$$r(1, n) = r(n, n+1) + t(n+1, n)\, r(1, n)\, [I - r(n+1, n)\, r(1, n)]^{-1}\, t(n, n+1) \tag{30}$$

$$V^{+}_{n+\frac{1}{2}} = \hat{t}(n+1, n)\, V^{+}_{n-\frac{1}{2}} + \Sigma^{+}(n+1, n) + R_{n+\frac{1}{2}}\, \Sigma^{-}(n, n+1) \tag{31}$$

$$V^{-}_{n+\frac{1}{2}} = \hat{r}(n+1, n)\, V^{+}_{n-\frac{1}{2}} + T_{n+\frac{1}{2}}\, \Sigma^{-}(n, n+1)$$

with the initial conditions $r(1, 1) = 0$, $V^{+}_{\frac{1}{2}} = U^{+}(b)$ and

where

$$\hat{t}(n+1, n) = t(n+1, n)\, [I - r(1,n)\, r(n+1, n)]^{-1}$$

$$\hat{r}(n+1, n) = r(n+1, n)\, [I - r(1,n)\, r(n+1, n)]^{-1} \tag{32}$$

$$R_{n+\frac{1}{2}} = \hat{t}(n+1, n)\, r(1, n)$$

$$T_{n+\frac{1}{2}} = [I - r(n+1, n)\, r(1, n)]^{-1}$$

and

$$\hat{t}(n, n+1) = T_{n+\frac{1}{2}}\, t(n, n+1) \tag{33}$$

On this forward sweep, we need to store the quantities $r(1, n)$, $\hat{t}(n, n+1)$ which represent the diffuse reflection and transmission for each shell and $V^{+}_{n+\frac{1}{2}}$, the diffuse source vectors

Now, we shall calculate the intensities at each step by computing sequentially for $n = N, N-1, N-2, \ldots, 2, 1$

$$U^{+}_{n+1} = r(1, n+1)\, U^{-}_{n+1} + V^{+}_{n+\frac{1}{2}} \tag{34}$$

$$U^{-}_{n} = \hat{t}(n, n+1)\, U^{-}_{n+1} + V^{-}_{n+\frac{1}{2}} \tag{35}$$

with the initial condition $U^{-}_{N+1} = U^{-}(a)$

If we have a reflecting surface at $r = A$ with the operator r_{a} then,

$$U^{-}_{N+1} = r_{a}\, U^{+}_{N+1} \tag{36}$$

and for $n = N$,

$$U^{+}_{N+1} = [I - r(1, N+1)\, r_{a}]^{-1}\, V^{+}_{N+\frac{1}{2}} \tag{37}$$

from which U^{-}_{N+1} is calculated from (36) and is given by

$$U^{-}_{N+1} = r_{a}\, [I - r(1, N+1)\, r_{a}]^{-1}\, V^{+}_{N+\frac{1}{2}} \tag{38}$$

we can calculate the net flux toward the surface of the atmosphere at each boundary r_n by the relation

$$F_{net} = 2\pi \int_{-1}^{+1} U\mu\, d\mu = 2\pi \sum_{j=1}^{J} (U^{-}_{n} - U^{+}_{n})\, \mu_j\, C_j \tag{39}$$

and the mean intensity

$$J = \frac{1}{2} \int_{-1}^{+1} U\, d\mu = \frac{1}{2} \sum_{j=1}^{J} (U^{-}_{n} + U^{+}_{n})\, C_j \tag{40}$$

We have laid down the framework to calculate the diffuse radiation field for a medium of general physical and geometrical properties. As we have seen in the previous sections, the calculation of diffuse field requires the correct estimation of reflection and transmission matrices for each shell or partition of the medium. It is through these matrices that the whole physics of the medium enters, we shall now try to calculate r and t matrices for differentially expanding spherical medium in which the photon redistribution occurs in a line with zero, natural width.

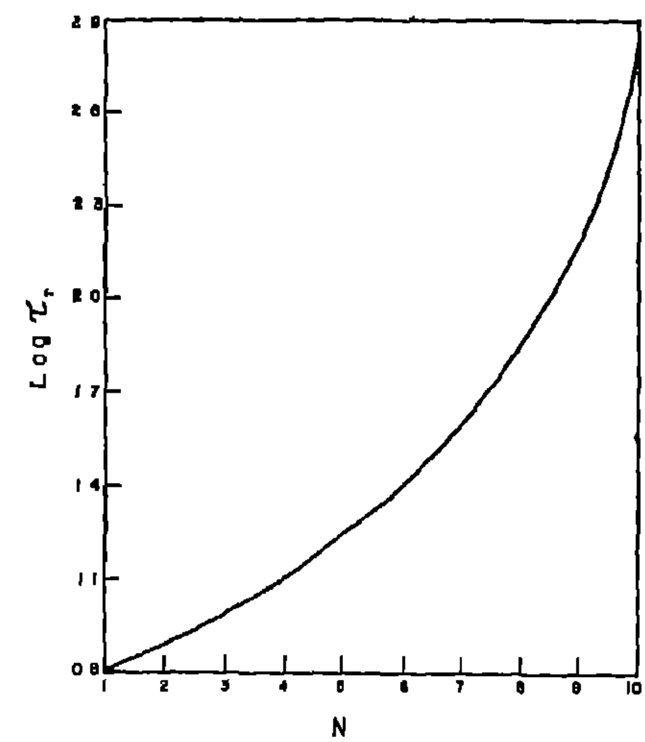


Fig 3 The optical depth is plotted against the shell number N. N = 1 and N = 10 represent the top and bottom of the atmosphere respectively

6. Calculation of Transmission and Reflection operators in a shell of given physical properties.

As the equation of line transfer describes the physical and geometrical properties of the medium in question, we shall integrate this equation with partial frequency redistribution. The equation of line transfer for a two level atom in spherical symmetry is given by

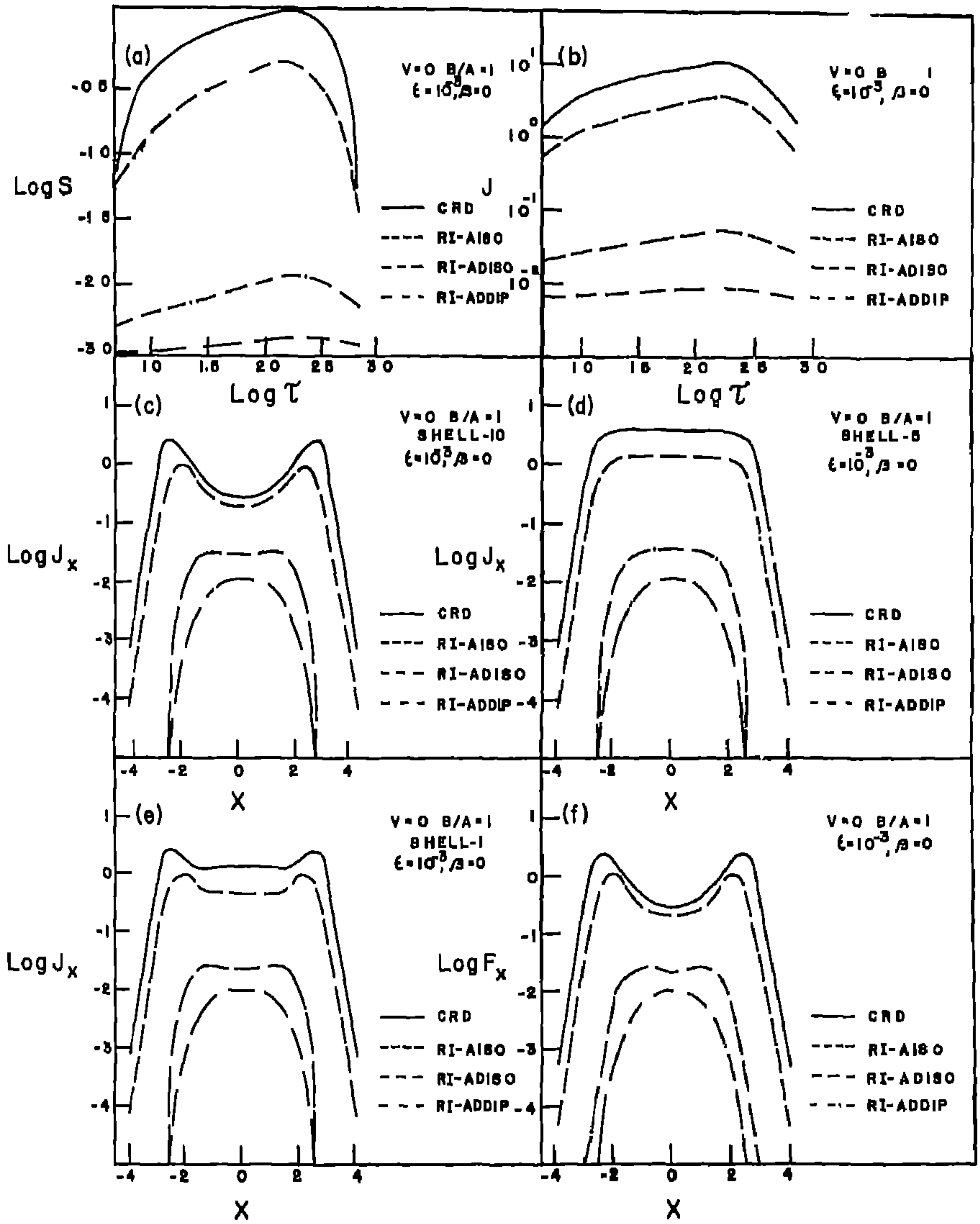
$$\mu \frac{\partial I(x, \mu, r)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I(x, \mu, r)}{\partial \mu} = k_L [\beta + \phi(x, \mu, r)] [S(x, \mu, r) - I(x, \mu, r)] \tag{41}$$

and for the oppositely directed beam,

$$-\mu \frac{\partial I(x, -\mu, r)}{\partial r} - \frac{1-\mu^2}{r} \frac{\partial I(x, -\mu, r)}{\partial \mu} = k_L [\beta + \phi(x, -\mu, r)] [S(x, -\mu, r) - I(x, -\mu, r)] \tag{42}$$

where $I(x, \mu, r)$ is the specific intensity at an angle $\cos^{-1}\mu$ ($\mu \epsilon$ [0, 1]) at the radial point r and frequency x ($= (\nu - \nu_0)/\Delta_s$, Δ_s being some standard frequency interval). The quantity β is the ratio k_c/k_L of opacity due to continuous absorption per unit interval of x to that in the line. The source function $S(x, \pm\mu, r)$ is given by

$$S(x, \mu, r) = \frac{\phi(x, \mu, r) S_L(x, \mu, r) + \beta S_c(r)}{\phi(x, \mu, r) + \beta} \tag{43}$$


(a)
(b)
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Log S
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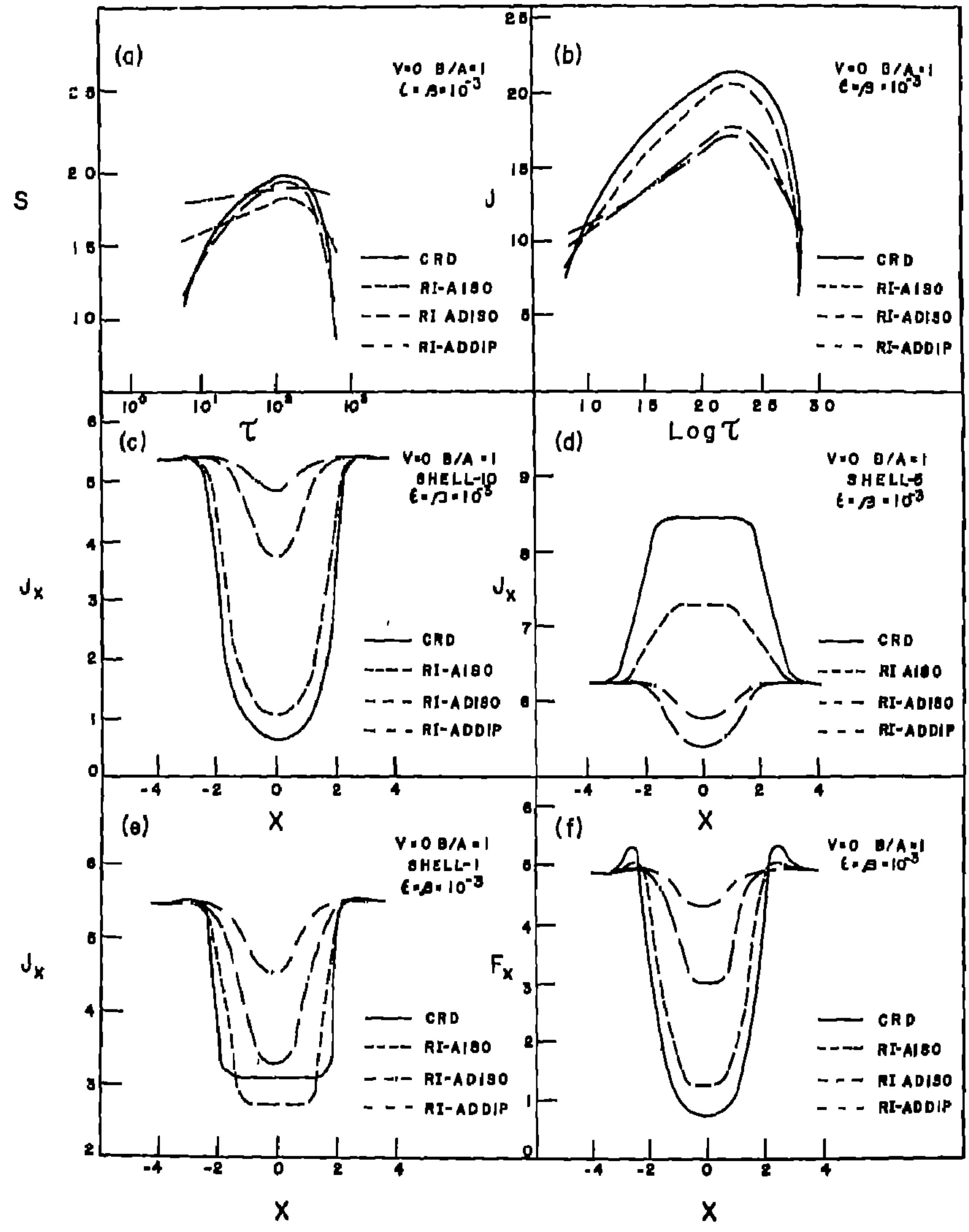
RI-AISO
RI-ADISO
RI-ADDIP
V=O B/A=1
SHELL-10
SHELL-5
SHELL-1

Fig 4.


(a)
V=0 B/A=1
ε=β=10⁻³
S
τ
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(b)
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Log τ
(c)
SHELL-10
J_x
X
(d)
SHELL-5
(e)
SHELL-1
(f)
F_x

Fig 5.

and

$$S(x, -\mu, r) = \frac{\phi(x, -\mu, r)\, S_l(x, -\mu, r) + \beta S_c(r)}{\phi(x, -\mu, r) + \beta} \tag{44}$$

S_l and S_c refer to the source functions in the line and continuum respectively and

$$S_c(r) = \rho(r)\, B(\nu_0, T_e(r)) \tag{45}$$

where B is the Planck function for frequency ν_0 at temperature T_e and both ρ and B are assumed in advance. The line source function S_l is given by,

$$S_l(x, \mu, r) = \frac{(1-\epsilon)}{\phi(x, \mu, r)} \int_{-\infty}^{+\infty} dx' \int_{-1}^{+1} R(x, \mu, x', \mu, r)\, I(x', \mu, r)\, d\mu' + \epsilon B(r) \tag{46}$$

and

$$S_l(x, -\mu, r) = \frac{(1-\epsilon)}{\phi(x, -\mu, r)} \int_{-\infty}^{+\infty} dx' \int_{-1}^{+1} R(x, -\mu, x', \mu, r)\, I(x', \mu, r)\, d\mu' + \epsilon B(r) \tag{47}$$

where $R(x, \mu, x', \mu, r)$ represents the partial frequency redistribution function and $\phi(x, \mu, r)$ is the profile function of the line (see section 2) and

$$\epsilon = \frac{C_{21}}{C_{21} + A_{21}[1 - \exp(-h\nu_0/kT_e)]^{-1}} \tag{48}$$

is the probability per scatter that a photon will be destroyed by collisional de-excitation

We shall integrate the equations (41) and (42) following Peraiah and Grant (1973) and Grant and Peraiah (1972), (hereafter referred to as PG and GP respectively) We have to discretize in frequency, angle and space coordinates For frequency discretization, we choose the discrete points x_i and weights a_i so that,

$$\int_{-\infty}^{+\infty} \phi(x)\, f(x)\, dx \simeq \sum_{i=-I}^{I} a_i f(x_i), \quad \sum_{i=-I}^{I} a_i = 1 \tag{49}$$

and for the angular discretization, we choose $\{\mu_j\}$ and weights $\{C_j\}$ such that

$$\int_0^1 f(\mu)\, d\mu \simeq \sum_{j=1}^{m} b_j f(\mu_j), \quad \sum_{j=1}^{m} b_j = 1 \tag{50}$$

and

$$B'\left(\nu_0, T_e(r)\right) = 4\pi r_0^2\, B\left(\nu_0, T_e(r)\right) \tag{51}$$

Following Carlson (1963) and Lathrop and Carlson (1967) we shall integrate the transfer equations (41 and 42) by using the so called "cell" method One integrates over an interval $[r_n, r_{n+1}] \times [\mu_{j-\frac{1}{2}}, \mu_{j+\frac{1}{2}}]$ defined on a two dimensional grid We shall discuss the choice of the set $\{r_n\}$ shortly By choosing the roots μ_j and weights C_j of Gauss Legendre quadrature formula of order J over (0, 1), we calculate the set $\mu_{j+\frac{1}{2}}$ as given by

$$\mu_{j+\frac{1}{2}} = \sum_{k=1}^{j} C_k, \quad j = 1, 2, \ldots, J. \tag{52}$$

We shall define the boundary $\mu_{\frac{1}{2}} = 0$. It is obvious that $\mu_{j-\frac{1}{2}} < \mu_j < \mu_{j+\frac{1}{2}}$.

We shall start with the angle integration of equations (41) and (42) This gives us,

$$C_j \mu_j \frac{\partial U^+_{i,j}(r)}{\partial r} + \frac{1}{r}\left\{(1-\mu^2_{j+\frac{1}{2}})\, U^+_{i,j+\frac{1}{2}}(r) - (1-\mu^2_{j-\frac{1}{2}})\, U^+_{i,j-\frac{1}{2}}(r) + C_j K_i(r)\right\}\left\{\beta + \phi^+_{i,j}(r)\right\} U^+_{i,j}(r) =$$

$$C_j K_i(r)\left\{[\rho\beta + \phi^+_{ij}(r)]\, B'(r) + \tfrac{1}{2}(1-c)\sum_{j'=1}^{J} R^{++}_{i,i',j,j'}(r)\, C_{j'} U^+_{i',j'}(r) + R^{+-}_{i,i',j,j'}(r)\, C_{j'} U^-_{i',j'}(r)]\right\} \quad (53)$$

and

$$-C_j \mu_j \frac{\partial U^-_{i,j}(r)}{\partial r} - \frac{1}{r}\left\{(1-\mu^2_{j+\frac{1}{2}})\, U^-_{i,j+\frac{1}{2}}(r) - (1-\mu^2_{j-\frac{1}{2}})\, U^-_{i,j-\frac{1}{2}}(r)\right\} + C_j K_i(r)\left\{\beta + \phi^-_{i,j}(r)\right\} U^-_{i,j}(r) =$$

$$C_j K_i(r)\left\{(\rho\beta + \phi^-_{ij}(r)\, B'(r) + \tfrac{1}{2}(1-c)\sum_{j'=1}^{J} [R^{-+}_{i,i',j,j'}(r)\, C_{j'} U^+_{i',j'}(r) + R^{--}_{i,i',j,j'}(r)\, C_{j'} U^-_{i',j'}(r)]\right\} \quad (54)$$

where

$$\begin{aligned}
U^+_{i,j}(r) &= U(x_i, \mu_j, r)\\
U^-_{i,j}(r) &= U(x_i, -\mu_j, r)\\
R^{++}_{i,i',j,j'}(r) &= R(x_i, \mu_j, x_i', \mu_j', r)\\
R^{-+}_{i,i',j,j'}(r) &= R(x_i, -\mu_j, x_i', \mu_j', r)\\
\phi^+_{i,j}(r) &= \phi(x_i, \mu_j, r)\\
\phi^-_{i,j}(r) &= \phi(x_i, -\mu_j, r)
\end{aligned} \quad (55)$$

We shall define $U^{\pm}_{j+\frac{1}{2}}$ by defining,

$$U^{\pm}_{j+\frac{1}{2}} = \frac{(\mu_{j+1} - \mu_{j+\frac{1}{2}})\, U^{\pm}_j + (\mu_{j+\frac{1}{2}} - \mu_j)\, U^{\pm}_{j+1})}{(\mu_{j+1} - \mu_j)}, j = 1, 2, \ldots, J-1 \quad (56)$$

and $U^+_{\frac{1}{2}} = U^-_{\frac{1}{2}}$ by interpolation,

$$U^+_{\frac{1}{2}} = U^-_{\frac{1}{2}} = \tfrac{1}{2}(U^+_1 + U^-_1) \quad (57)$$

By writing

$$U^+_{i,n} = \begin{bmatrix} U(x_i, \mu_1, r_n)\\ U(x_i, \mu_2, r_n)\\ \vdots \\ U(x_i, \mu_m, r_n)\end{bmatrix} \text{put, } U^-_{i,n} = \begin{bmatrix} U(x_i, -\mu_1, r_n)\\ U(x_i, -\mu_2, r_n)\\ \vdots \\ U(x_i, -\mu_m, r_n)\end{bmatrix}$$

and

$$M_m = (\mu_j\, \delta_{jk}),\ C_m = [c_j\, \delta_{jk}],$$

$$\phi^+_{i,m}(r) = \begin{bmatrix} \phi(x_i, \mu_1, r)\\ \phi(x_i, \mu_2, r)\\ \vdots \\ \phi(x_i, \mu_m, r)\end{bmatrix} \text{and } \phi^-_{i,m}(r) = \begin{bmatrix} \phi(x_i, -\mu_1, r)\\ \phi(x_i, -\mu_2, r)\\ \vdots \\ \phi(x_i, -\mu_m, r)\end{bmatrix} \quad (58)$$

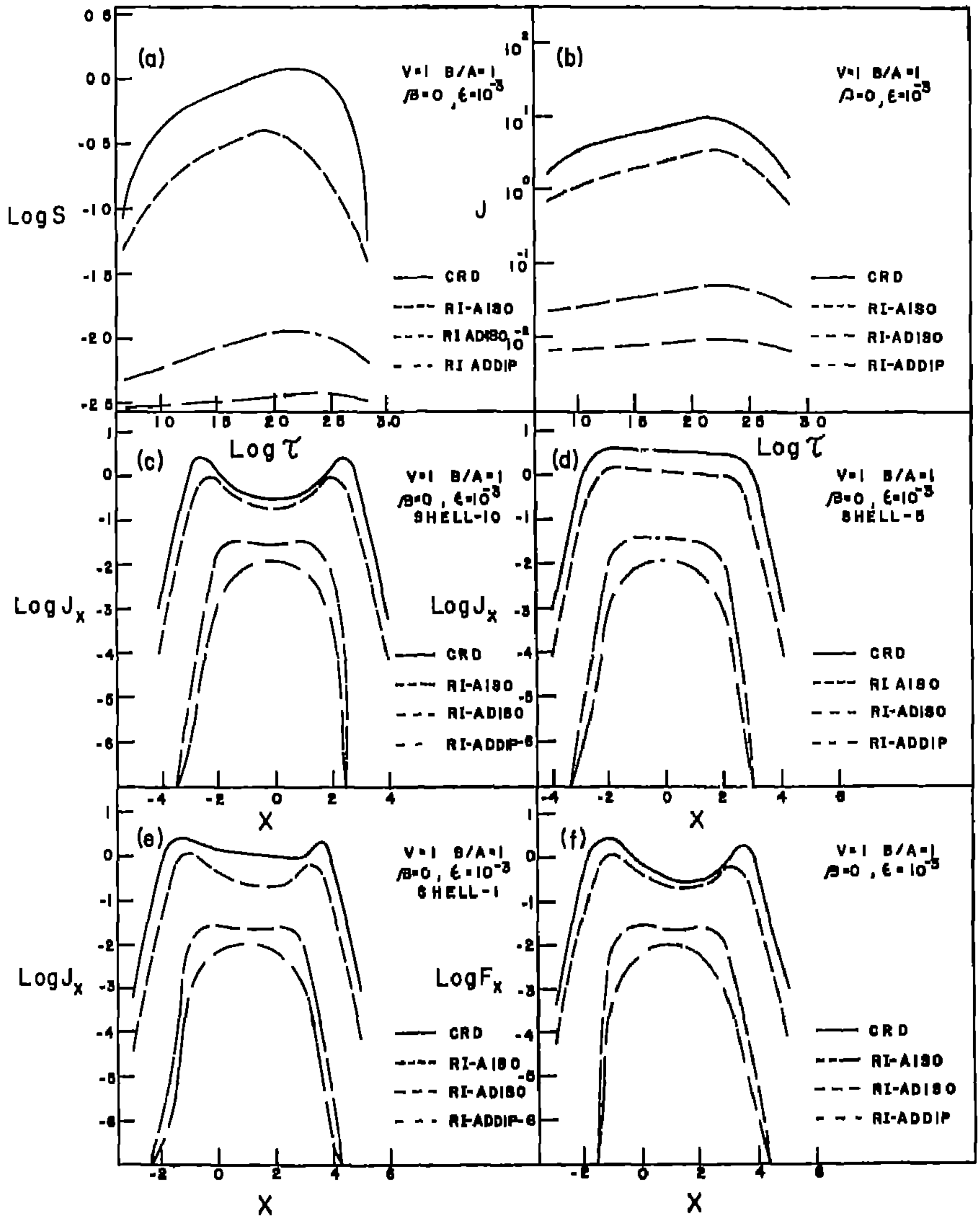

(a)
V=1 B/A=1
β=0 , ε=10⁻³
Log S
Log τ
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(b)
J
Log τ
(c)
SHELL-10
Log J_X
X
(d)
SHELL-5
(e)
SHELL-1
(f)
Log F_X

Fig 6

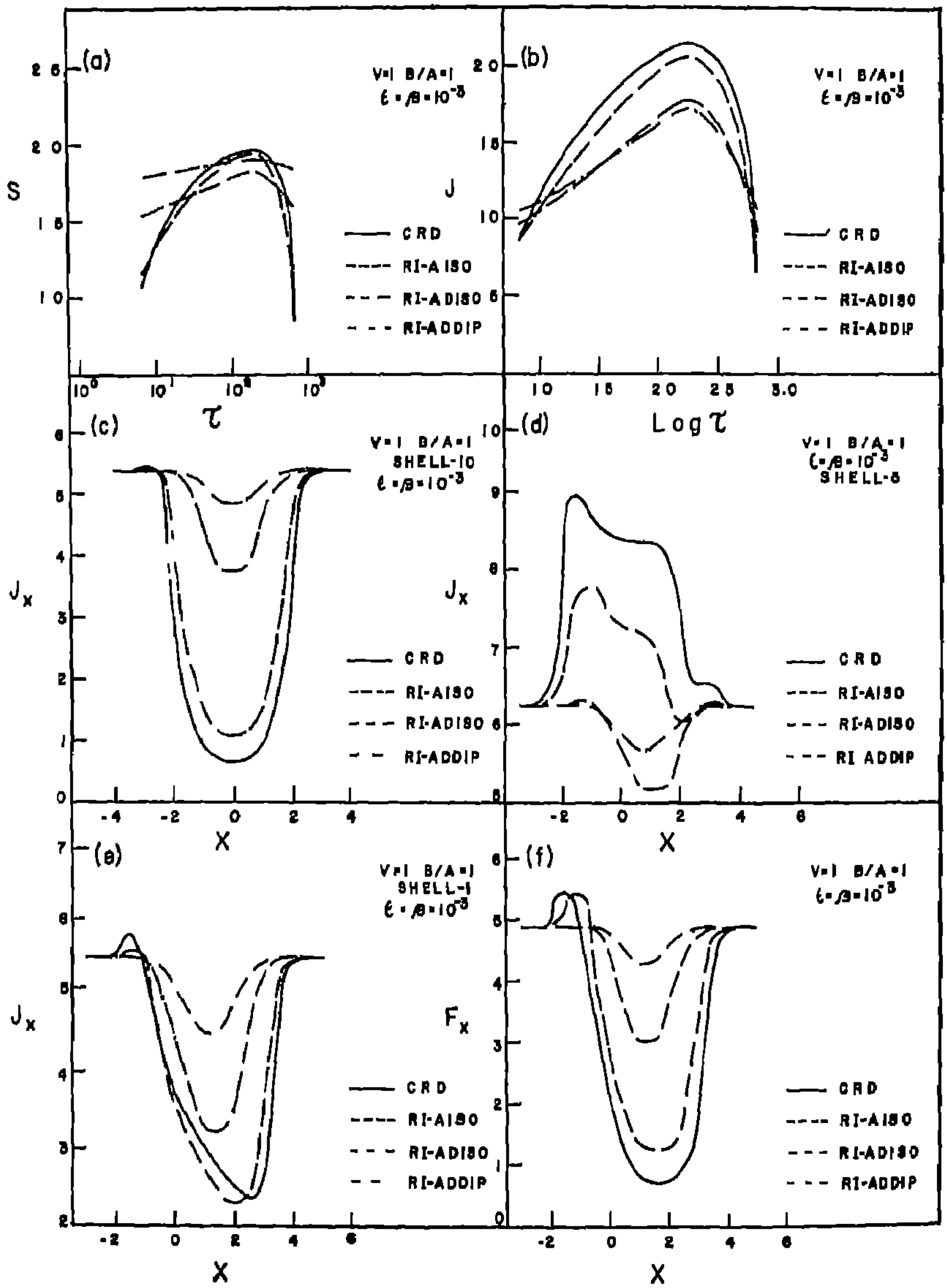

(a)
V=1 B/A=1
ε=β=10⁻³
S
τ
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(b)
J
Log τ
(c)
SHELL-10
J_x
X
(d)
SHELL-5
(e)
SHELL-1
(f)
F_x

Fig. 7.

and

$$R^{++}_{i,i'}(r) = \begin{bmatrix} R(x_i, \mu_1, x_i', \mu_1, r) \\ R(x_i, \mu_2, x_i', \mu_1, r) \\ R(x_i, \mu_m, x_i', \mu_1, r) \\ \\ R(x_i, \mu_m, x_i', \mu_m, r) \end{bmatrix} \tag{59}$$

$$R^{-+}_{i,i'}(r) = \begin{bmatrix} R(x_i, -\mu_1, x_i', \mu_1, r) \\ R(x_i, -\mu_2, x_i', \mu_1, r) \\ R(x_i, -\mu_m, x_i', \mu_1, r) \\ \\ R(x_i, -\mu_m, x_i', \mu_m, r) \end{bmatrix}$$

We can rewrite the equations (53) and (54) for the set of angles $\{\mu_j\}$ over [0, 1] as

$$M_m \frac{\partial U^+_i(r)}{\partial r} + \frac{1}{r}\left[\Lambda^+_m U^+_i(r) + \Lambda^-_m U^-_i(r)\right] + k_L(r)\left\{\beta + \phi^+_{i,m}(r)\right\} U^+_i(r) = k_L(r)\left\{(\rho\beta + \epsilon\phi^+_{i,m}(r) B'(r)\right.$$

$$\left. + \tfrac{1}{2}(1-\epsilon)\,[R^{++}_{i,i'}(r)\, a^{++}_{i'}(r)\, C_m U^+_{i'}(r) + R^{+-}_{i,i'}(r)\, a^{+-}_{i'}(r)\, C_m U^-_{i'}(r)]\right\} \tag{60}$$

Similarly for the oppositely directed beam

$$-M_m \frac{\partial U^-_i(r)}{\partial r} - \frac{1}{r}\left[\Lambda^+_m U_i(r) + \Lambda^-_m U^+_i(r)\right] + k_L(r)\left\{\beta + \phi_{i,m}(r)\right\} U_i(r) = k_L(r)\left\{(\rho\beta + \phi^-_{i,m}(r) B'(r)\right.$$

$$\left. + \tfrac{1}{2}(1-\epsilon)\,[R^{--}_{i,i'}(r)\, a^{-}_{i'}(r)\, C_m U_{i'}(r) + R^{-+}_{i,i'}(r)\, a^{-+}_{i'} C_m U^+_{i'}(r)]\right\} \tag{61}$$

where Λ^+_m and Λ^-_m are square J × J matrices defined by

$$\Lambda^+_{jk} = \frac{(1-\mu^2_{j+1})(\mu_{j+1} - \mu_j)}{C_j(\mu_{j+1} - \mu_j)},\quad k = j+1,\ j = 1, 2, \ldots, J-1.$$

$$= \frac{(1-\mu^2_{j+1})(\mu_{j+1} - \mu_{j+1})}{C_j(\mu_{j+1} - \mu_j)} - \frac{(1-\mu^2_{j-1})(\mu_j + \mu_{j-1})}{C_j(\mu_j - \mu_{j-1})}\quad k = j,\ j = 1, 2, \ldots, J-1, J.$$

$$= \frac{(1-\mu^2_{j-1})(\mu_j - \mu_{j-1})}{C_j(\mu_j - \mu_{j-1})},\quad k = j-1,\ j = 2, 3, \ldots, J \tag{62}$$

and

$$\Lambda^-_{jk} = -\frac{1}{2c_j}\,\delta_{j,1}\,\delta_{k,1} \tag{63}$$

The matrices Λ^+ and Λ^- are called curvature scattering matrices

The integration over $\{r_n, r_{n+1}\}$ of equations (60) and (61) gives us,

$$M_m(U^+_{i,n+1} - U^+_{i,n}) + \rho_c(\Lambda^+_m U^+_{i,n+1} + \Lambda^-_m U_{i,n+1}) + \tau_{n+1}(\beta + \phi^+_{m,i})\, U^+_{i,n+1} = \tau_{n+1}(\rho\beta + \phi\epsilon^+_{m,i})_{n+1} B'_{n+1}$$

$$+ \tfrac{1}{2}\tau_{n+1}(1-\epsilon)\,(R^{++}_{i,i',n+1}\, a^{++}_{i,n+1}\, C\, U^+_{i',n+1} + R^{+-}_{i,i',n+1}\, a^{+-}_{i',n+1}\, C\, U^-_{i',n+1}) \tag{64}$$

and

$$M_{m}(U^{-}_{i,n} - U^{-}_{i,n+1}) - \rho_c(\Lambda^{+}_{m}U^{-}_{i,n+\frac{1}{2}} + \Lambda^{-}_{m}U^{+}_{i,n+\frac{1}{2}}) + \tau_{n+\frac{1}{2}}(\beta + \phi^{-}_{m,i})U_{i,n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}(\rho\beta + \phi^{-}_{m,i})B'_{n+\frac{1}{2}}$$
$$+ \tfrac{1}{2}\tau_{n+\frac{1}{2}}(1-\epsilon)(R^{-+}_{i,i',n+\frac{1}{2}}a^{-+}_{i',n+\frac{1}{2}}CU^{-}_{i,n+\frac{1}{2}} + R^{--}_{i,i',n+\frac{1}{2}}a^{--}_{i',n+\frac{1}{2}}CU^{-}_{i',n+\frac{1}{2}}) \qquad (65)$$

where ρ_c is the curvature factor defined as

$$\rho_c = \frac{\Delta r}{r_{n+\frac{1}{2}}} \quad \text{and} \quad \tau_{n+\frac{1}{2}} = K_L(_{n+\frac{1}{2}})\,\Delta r \qquad (66)$$

Here the subscript n, n+1 and $n+\frac{1}{2}$ refer to the quantities at r_n, r_{n+1} and $r_{n+\frac{1}{2}}$ where $n+\frac{1}{2}$ refers to the average of the parameter over shell bounded by r_n and r_{n+1}

We shall define the weights,

$$(\phi_i W_k) = a_{i,n+\frac{1}{2}}C_j \qquad (67)$$

where the subscript k is defined as

$$(i,j) \rightarrow k = j + (i-1)J,\ 1 \leqslant k \leqslant K = IJ$$

where I and J being the number of frequency and angle points respectively and i, j their corresponding running indices We shall define as at a later stage

By letting

$$U^{+}_{n} = \begin{bmatrix} U^{+}_{1,n} \\ U^{+}_{2,n} \\ \vdots \\ U^{+}_{I,n} \end{bmatrix}, \Phi^{+}_{n+\frac{1}{2}} = [\Phi^{+}_{kk}]_{n+\frac{1}{2}} = [\beta + \phi^{+}_{k}]_{n+\frac{1}{2}}\delta_{kk'}$$

and $S^{+}_{n+\frac{1}{2}} = [\rho\beta + \epsilon\phi_k]_{n+\frac{1}{2}}B'_{n+\frac{1}{2}}\delta_{kk}$

We rewrite equations [64] and [65] to include all the frequency points as follows·

$$M[U^{+}_{n+1} - U^{-}_{n}] + \rho_c[\Lambda^{+}U^{+}_{n+\frac{1}{2}} + \Lambda^{-}U_{n+\frac{1}{2}}] + \tau_{n+\frac{1}{2}}\Phi^{+}_{n+\frac{1}{2}}U^{+}_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}S^{+}_{n+\frac{1}{2}} + \tfrac{1}{2}[1-\epsilon]\tau_{n+\frac{1}{2}} \cdot [R^{++}$$
$$W^{++}U^{+} + R^{+-}W^{+-}U^{-}]_{n+\frac{1}{2}} \qquad (68)$$

and

$$M[U^{-}_{n} - U^{-}_{n+1}] - \rho_c[\Lambda^{+}U^{-}_{n+\frac{1}{2}} + \Lambda^{-}U^{+}_{n+\frac{1}{2}}] + \tau_{n+\frac{1}{2}}\Phi_{n+\frac{1}{2}}U^{-}_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}S^{-}_{n+\frac{1}{2}} + \tfrac{1}{2}[1-\epsilon]\tau_{n+\frac{1}{2}}[R^{-+}W^{-+}$$
$$U^{+} + R^{--}W^{--}U^{-}]_{n+\frac{1}{2}} \qquad (69)$$

where

$$M = \begin{bmatrix} M_m & & \\ & M_m & \\ & & \ddots \\ & & & M_m \end{bmatrix}$$

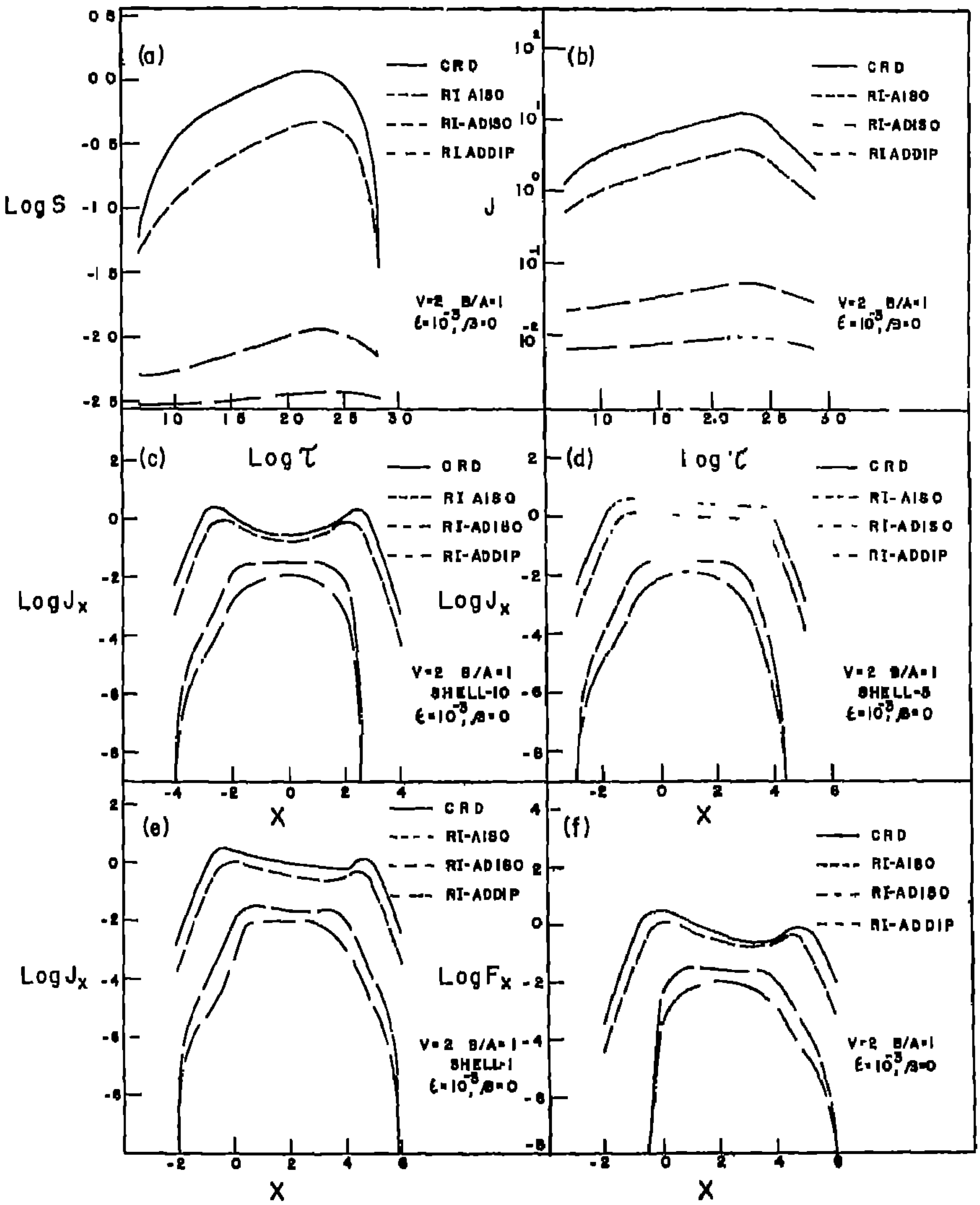

(a)
CRD
RI AISO
RI-ADISO
RI ADDIP
Log S
V=2 B/A=1
(b)
J
(c)
Log τ
LogJx
V=2 B/A=1
SHELL-10
(d)
Log τ
SHELL-5
X
(e)
SHELL-1
(f)
LogFx

Fig. 9.

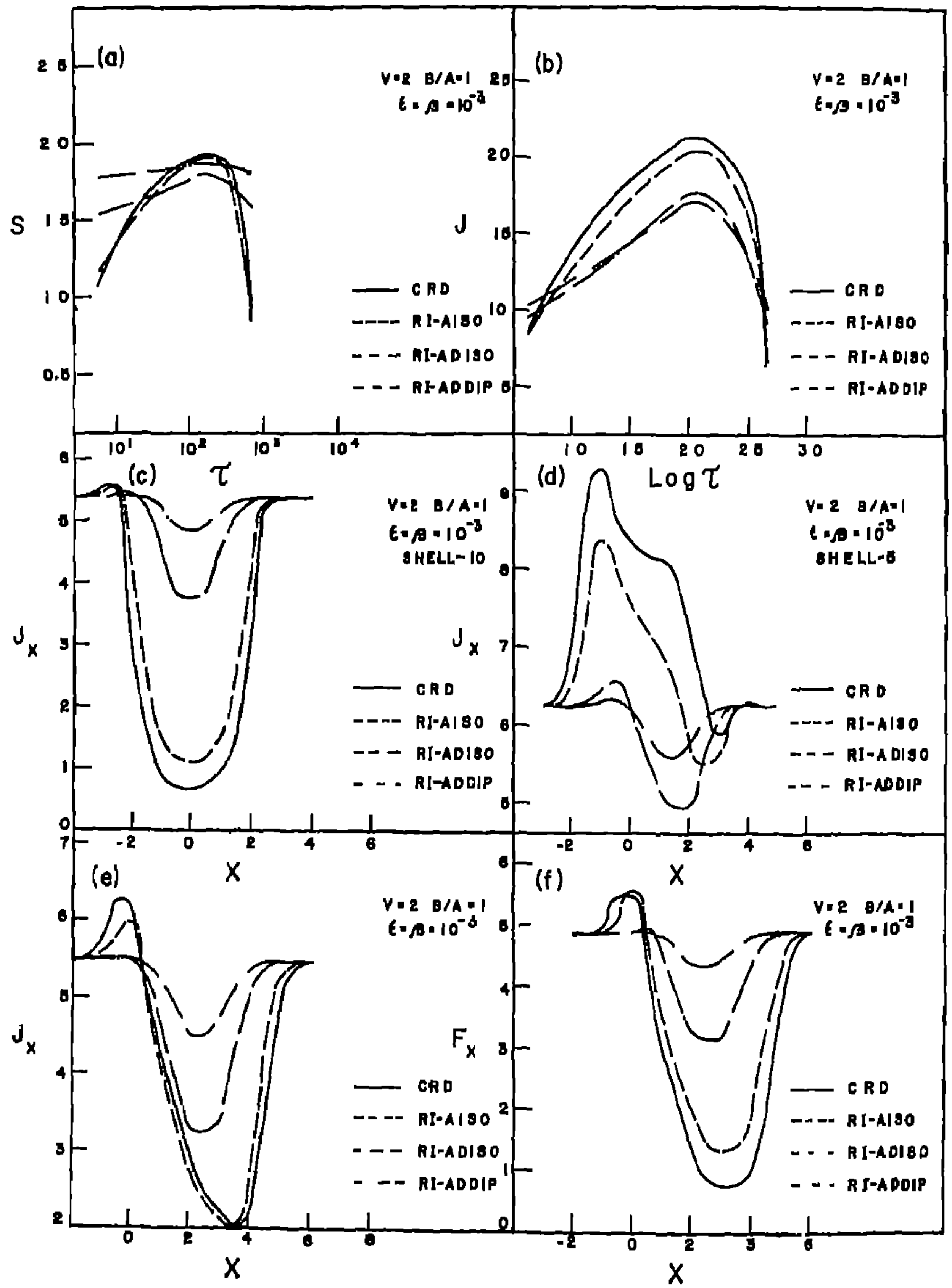

(a)
V=2 B/A=1
ε = β = 10^-3
S
CRD
RI-AISO
RI-ADISO
RI-ADDIP
τ
(b)
V=2 B/A=1
ε=β=10^-3
J
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log τ
(c)
V=2 B/A=1
ε=β=10^-3
SHELL-10
J_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X
(d)
V=2 B/A=1
ε=β=10^-3
SHELL-5
J_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X
(e)
V=2 B/A=1
ε=β=10^-3
J_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X
(f)
V=2 B/A=1
ε=β=10^-3
F_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X

Fig. 9.

and

$$\Lambda^{\pm} = \begin{bmatrix} \Lambda_m^{\pm} & & & \\ & \Lambda_m^{\pm} & & \\ & & \ddots & \\ & & & \Lambda_m^{\pm} \end{bmatrix}$$

We have to replace the average intensities $U^{+}_{n+\frac{1}{2}}$ in the above equations. For this purpose, we shall use the diamond scheme [GP equation 2.23] given by,

$$[I - x_{n+\frac{1}{2}}]\, U^{+}_{n} + x_{n}\, U^{+}_{n+1} = U^{+}_{n+\frac{1}{2}}$$

$$[I - x_{n+\frac{1}{2}}]\, U^{-}_{n+1} + x_{n+\frac{1}{2}}\, U^{-}_{n} = U^{-}_{n+\frac{1}{2}} \qquad (70)$$

with $x = \frac{1}{2} I$ for diamond scheme and I is the identity matrix. By using [70], we can write equations [68] and [69] as,

$$\begin{bmatrix} M + \frac{1}{2}\tau[\Phi^{+} - \frac{\delta}{2} R^{++} W^{++}] + \frac{1}{2}\rho_c\Lambda^{+} & -\frac{\delta\tau}{4} R^{+-} W^{+-} + \frac{1}{2}\rho_c\Lambda^{-} \\ -\frac{\delta\tau}{4} R^{-+} W^{-+} - \frac{1}{2}\rho_c\Lambda^{-} & M + \frac{\tau}{2}[\Phi^{-} - \frac{\delta}{2} R^{--} W^{--}] - \frac{1}{2}\rho_c\Lambda^{+} \end{bmatrix} \begin{bmatrix} U^{+}_{n+1} \\ U^{-}_{n} \end{bmatrix} =$$

$$\begin{bmatrix} M - \frac{\tau}{2}[\Phi^{+} - \frac{\delta}{2} R^{++} W^{++}] + \frac{1}{2}\rho_c\Lambda^{+} & \frac{\delta\tau}{4} R^{+-} W^{+-} - \frac{1}{2}\rho_c\Lambda^{-} \\ \frac{\delta\tau}{4} R^{-+} W^{-+} + \frac{1}{2}\rho_c\Lambda & M - \frac{\tau}{2}[\Phi - \frac{\delta}{2} R\; W^{--}] + \frac{1}{2}\rho_c\Lambda \end{bmatrix} \begin{bmatrix} U^{+}_{n} \\ U^{-}_{n+1} \end{bmatrix} + \begin{bmatrix} S^{+} \\ S^{-} \end{bmatrix} \qquad (71)$$

where $\delta = 1 - \epsilon$

By comparing equation [71] with the principle of interaction given in equation [15], we obtain the two pairs of transmission and reflection operators

With the following auxiliary quantities

$$\begin{aligned}
G^{+} &= [I - g^{+-} g^{-+}]^{-1} \\
G^{-} &= [I - g^{-+} g^{+-}]^{-1} \\
g^{+-} &= \tfrac{1}{2}\tau\, \Delta^{+} Y_{-} \\
g^{-+} &= \tfrac{1}{2}\tau\, \Delta^{-} Y_{+} \\
D &= M - \tfrac{1}{2}\tau Z_{-} \\
A &= M - \tfrac{1}{2}\tau Z_{+} \\
\Delta^{+} &= [M + \tfrac{1}{2}\tau Z_{+}]^{-1} \\
\Delta^{-} &= [M + \tfrac{1}{2}\tau Z_{-}]^{-1} \\
Z_{+} &= \Phi^{+} - \frac{\delta}{2} R^{++} W^{++} + \rho_c \Lambda^{+}/\tau \\
Z_{-} &= \Phi^{-} - \tfrac{1}{2}\delta R^{--} W^{--} - \rho_c \Lambda^{+}/\tau \\
Y_{+} &= \tfrac{1}{2}\delta R^{-+} W^{-+} + \rho_c \Lambda^{-}/\tau \\
Y_{-} &= \tfrac{1}{2}\delta R^{+-} W^{+-} - \rho_c \Lambda^{-}/\tau
\end{aligned} \qquad (72)$$

We can write the transmission and reflection matrices as,

$$t\,[n+1,\, n] = G^{+-}\,[\Delta^{+} A + g^{+-} g^{-+}]$$
$$t\,[n,\, n+1] = G^{-+}\,[\Delta^{-} D + g^{-+} g^{+-}]$$
$$r\,[n+1,\, n] = G^{-+} g^{-+}\,[I + \Delta^{+} A]$$
$$r\,[n,\, n+1] = G^{+-} g^{+}\,[I + \Delta^{-} D] \tag{73}$$

and the cell source vectors are given by,

$$\Sigma^{+} = G^{+-}\,[\Delta^{+} S^{+} + g^{+-}\,\Delta^{-} S^{-}]\tau$$
$$\Sigma^{-} = G^{-+}\,[\Delta^{-} S^{-} + g^{-+}\,\Delta^{+} S^{+-}]\tau \tag{74}$$

We have obtained the two pairs of transmission and reflection operators given in [73] and the source vectors given in [74] for a cell of optical depth τ and curvature factor ρ_c These operators describe the radiation field in any medium either static or moving In the case of a static medium we need not calculate all the four redistribution functions because of the [see section 2] symmetry of these functions For example in a static medium, we have

$$R\,[x,\, +\mu,\, x',\, +\mu'] = R\,[x,\, -\mu,\, x',\, -\mu'],\; R^{++} \quad R^{--}$$
$$R\,[x,\, +\mu,\, x',\, -\mu'] = R\,[x,\, -\mu,\, x',\, +\mu'],\; R^{+-} = R^{-+}$$

If the medim is in motion, then we have the frequency shifts due to Doppler effect and, therefore, the frequency changes from x to x $\pm\, \mu v$ where v is the velocity of the gas in units of thermal velocity Consequently, one has to compute all the four redistribution functions at each radial point in a moving medium

We must choose τ and ρ_c the optical depth and the curvature factor in a cell so that we obtain a stable solution For this, consider the matrices Δ^{+} and Δ^{-} given in (72] To obtain a positive matrices, we must have a positive diagonally dominant and negative off-diagonal elements of the matrices of $[\Delta^{+}]^{-1}$ and $[\Delta^{-}]^{-1}$. Therefore,

$$\tau_{n+\frac{1}{2}} \leq \tau_{crit} = \min_{k} \left| \frac{\mu_k \pm \frac{1}{2}\rho_c \Lambda^{+}_{kk} \; ---}{\frac{1}{2}\,[\Phi' - \frac{\Phi}{2}\,\delta R^{++}_{kk} W^{++}_{kk}]} \right| \tag{75}$$

for the diagonal elements and for the off-diagonal elements,

$$[\rho_c / \tau_{n+1}] < \min_{k} \left[\min_{k = k\pm 1} \left| \frac{\frac{1}{2}\Phi\,\delta R_{kk}\, W_{kk'}}{\Lambda^{+}_{kk'}} \right| \right] \tag{76}$$

The condition [75] can always be satisfied. However, the condition [76] imposes a severe restriction on the size of the curvature factor ρ_c to be used in each cell to obtain a non-negative t and r matrices. From [75] and [76], it is clear that one must divide the medium into a number of 'cells' to obtain the diffuse radiation field described in Section 5 Formally, we divide the medium into several shells [this number depends upon the capacity of the machine e g , storage space, speed etc.] and if the optical depth in each shell $\tau_{shell} > \tau_{cell}$ then we have to subdivide the shell and use the "star algorithm" given in section 4 for calculating the r and t operators for the whole shell. In such an event, we use the doubling process which is faster by choosing an extremely small value for $\rho_{subshell}$ so that the errors would be minimized in compounding the r and t operators However, one must notice that by choosing too small a curvature factor one can reduce the truncation errors but round-off errors would create problems Therefore, one has to judge oneself how to choose an optimum p. If we halve the shell p times, the star algorithm is repeated p times and in this case the curvature factor ρ_s and the optical depth τ_{s1} for the subshell or "cell" are given in terms of those for the shell (ρ_s and τ_s)

$$\rho_{ss} \approx \rho_s\, 2^{-p} / [1 - \rho_s (2^{-1} - 2^{-p})] \tag{77}$$

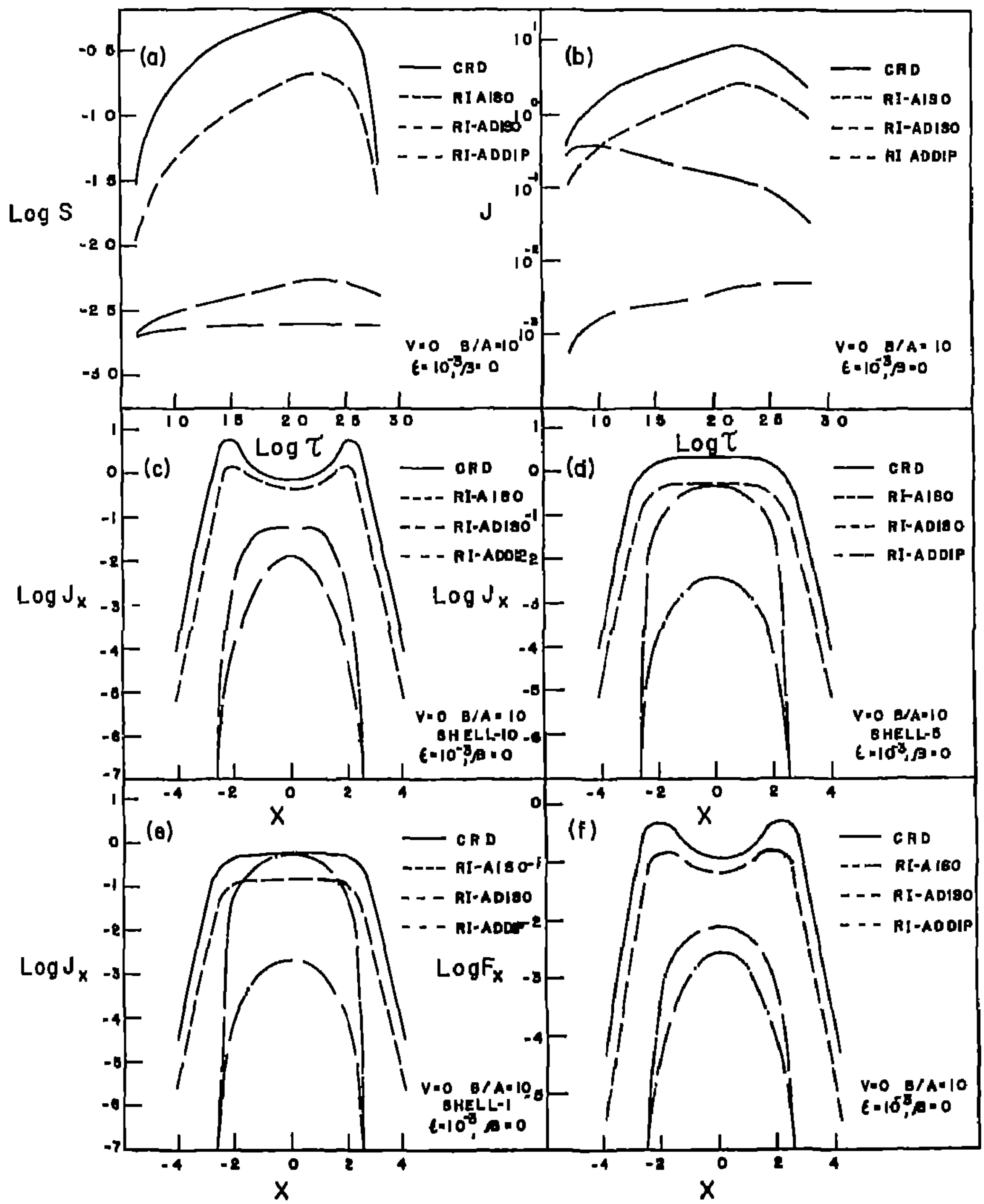
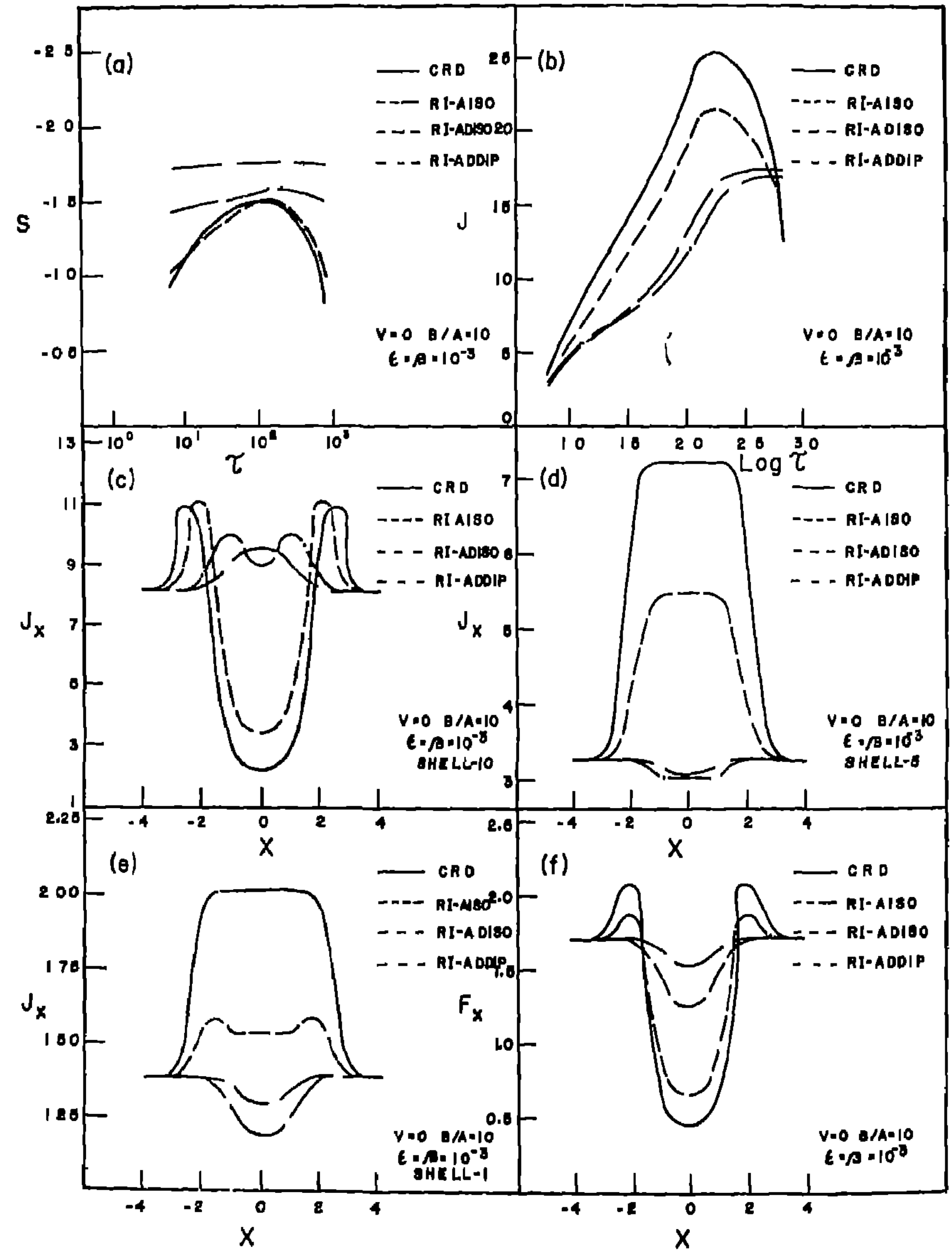

(a)
CRD
RI AISO
RI-ADISO
RI-ADDIP
Log S
V=0 B/A=10
ε=10⁻³, β=0
Log τ
(b)
CRD
RI-AISO
RI-ADISO
RI ADDIP
J
V=0 B/A=10
ε=10⁻³, β=0
Log τ
(c)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log J_X
V=0 B/A=10
SHELL-10
ε=10⁻³, β=0
X
(d)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log J_X
V=0 B/A=10
SHELL-5
ε=10⁻³, β=0
X
(e)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log J_X
V=0 B/A=10
SHELL-1
ε=10⁻³, β=0
X
(f)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log F_X
V=0 B/A=10
ε=10⁻³, β=0
X

Fig. 10.



(a)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
S
τ
V=0 B/A=10
ε=β=10^-3
(b)
J
Log τ
(c)
J_x
X
SHELL-10
(d)
SHELL-5
(e)
SHELL-1
(f)
F_x

Fig. 11.

and

$$\tau_{11} - \tau_1 2^{-p} \tag{78}$$

and the square of the mean radius of the subshell is given by

$$\bar{r^2} = R^2 \{1 - \rho_1 [K + \tfrac{1}{2}] + \tfrac{1}{3} \rho_0^2 [K^2 + K + \tfrac{1}{3}]\} \tag{79}$$

where ρ_{11} corresponds to a subshell approximately midway in the shell and ρ_1 is the curvature factor for the whole shell defined as

$$\rho_1 = \Delta r / r_{out} \tag{80}$$

τ_{11} is derived on the assumption that the optical depth in the shell is uniform R is the outer radius of the shell in terms of the inner radius of the medium and K 2 ¹ − 2^{-p} The relations set out in equations [77 79] are derived on the basis of equation [30] of Grant [1963]

One of the important checks of the method is conservation of flux In a purely scattering medium where energy is neither emitted nor absorbed, the input energy must balance the output energy In the next section we shall derive conditions for the conservation of flux

7. Flux Conservation

In this section, we shall derive certain normalization conditions for the redistribution functions For this purpose, we consider a medium which scatters and neither creates nor absorbs energy In this event, the S matrices [Grant and Hunt 1969 a, b] should give us

$$|| S (n, n+1) || = 1 + 0 (\tau) \tag{81}$$

or

$$|| t (n, n+1) + r (n, n+1) || = 1 + 0 (\tau) \tag{82}$$

In terms of [4 11] of [GHa] and equation [72] of the previous section we shall have,

$$|| t [n, n+1], r [n, n+1] || = \max_{n} \sum_{i=1}^{I} a_i \left\{ \max_{k} \sum_{j=1}^{m} 2\pi \mu_j c_j \left[\delta_{jk} - \frac{\tau}{\mu_j} (\phi^k \delta_{lm} - \tfrac{1}{2} R^{++}{}_{k\ ln} a_n c_k + \frac{\rho}{\tau} \Lambda_{n}{}^{+}{}_{lk} \quad \tfrac{1}{2} R_k{}^{-+}{}_{,ln} a_n c_k + \frac{\rho}{\tau} \Lambda_{n \quad lk} \right] \frac{1}{2\pi \mu_k c_k a_n} \right\} + 0 [\tau] \tag{83}$$

where we have put $\tau = 0$

By virtue of the identity [4.3] of [Peraiah and Grant 1973] the above equation becomes

$$|| t [n, n+1] + r [n, n+1] || = 1 + \frac{\tau}{\mu_j} \left\{ \phi^k + \tfrac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{I} [R^{++}{}_{k, ij} + R^{-+}{}_{k, ij}] a_i c_j \right\} + 0 [\tau] \tag{84}$$

However, the discrete form of equation [3] of section 2, gives us,

$$\tfrac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{m} [R^{++}{}_{k, ij} + R^{-+}{}_{k, ij}] a_i c_j = \phi_k \tag{85}$$

or

$$|| t [n, n+1] + r [n, n+1] || = 1 + 0 [\tau] \tag{86}$$

which proves the conservation of radiation The normalizing condition, therefore, for the redistribution function is given by see Peraiah [1978],

$$\tfrac{1}{2} \sum_{P=1}^{K} \sum_{Q=1}^{K} [R^{++}{}_{PQ} W^{++}{}_{P} W^{++}{}_{Q} + R^{-+}{}_{PQ} W^{-+}{}_{P} W^{-+}{}_{Q} = 1 \tag{87}$$

where

$$W_{P, Q} = a_i c_j, \quad a_i = \frac{A_i R_{PQ}}{\sum_{P, Q=1}^{K} R_{PQ} A_i C_j} \tag{88}$$

and

$$[P, Q] = j + [i - 1] J \tag{89}$$

Similarly the normalization on the curvature matrices is given by

$$\sum_{j=1}^{J} C_j (\Lambda^{-}_{jk} - \Lambda^{+}_{jk}) = 0, k = 1, 2 \quad J \tag{90}$$

It is very important that the normalization of the redistribution function and the identity [90] are satisfied to the full machine accuracy. The programme has been checked for $\epsilon = 0$ and it was found that the flux is conserved to 10^{-14} double precision of IBM 370 machine

8 Discussion of the Results

We have selected a few representative parameters to bring out the important differences between the lines formed by redistribution functions with isotropic scattering and dipole scattering The ratios of outer to inner radii (B/A, see Figure 2) are taken to be 1, for plane parallel stratification and 10 and 100 for spherically symmetric media The matter in the atmosphere is assumed to be expanding with a velocity proportional to the radius (see Peraiah and Wehrse 1978, Wehrse and Peraiah 1979) according to the relation

$$V_n = V_N + (N - n + \tfrac{1}{2}) \Delta V$$

where v's are the velocities of the gas in mean thermal units and v_n is the velocity of the gas in the n_{th} shell, v_N is the velocity at the inner surface of the atmosphere (we have set $v_N = 0$ in all cases and $n = 1$ corresponds to outermost shell and $n = N$ to that of the innermost shell) and

$$\Delta v = (v_1 - v_N)/N$$

where N is the total number of shells The quantity $\tfrac{1}{2}$ is introduced because we consider the velocity at the centre of the shell. The atmosphere is divided into 10 shells ($N = 10$) each of equal radial thickness but of unequal optical thickness and we have set $V_{10} = 0$ in all cases and $V_1 = 0$, 1 and 2 thermal units. To be consistent with equation of conservation of mass, we have set the density varying as r^{-3} The variation of the optical depth with respect to the shell number is given in Figure 3 The total optical depth T_L is taken to be 10^3

The boundary conditions are $U^{+}_{1} (X_i, \tau = 0, \mu_j) = 0$ and $U^{-}_{n+1} (X_i, \tau = T, \mu_j) = 0$, (see Figure 2) that is, no radiation is incident on either side of the medium. The Planck function B is set equal to 1 in all cases The frequency dependent mean intensities $J_n (X_i)$, total mean intensities J, total source functions S and the monochromatic emergent flux $F (X_i)$ are calculated by the following relations

$$J_n (X_i) = \tfrac{1}{2} \sum_{j=1}^{J} C_j [U^{+}_n (X_i, \mu_j) + U^{-}_n (X_i, \mu_j)]$$

$$J_n = \sum_{i=1}^{I} J (X_i, n) A_i$$

$$S_n = \sum_{i=1}^{I} A_i \sum_{j=1}^{J} S (X_i, \mu_j, \tau_n) C_j$$

and

$$F (X_i) = \left(\frac{A}{B}\right)^2 \sum_{j=1}^{J} U^{-}_1 (X_i, \mu_j, \tau = 0) C_j \mu_j$$

We have employed 20 frequency points and 4 angles ($I = 20$, $J = 4$) The coding has been checked for flux conservation (see Peraiah 1978) by putting $\epsilon = \beta = 0$ and giving incident radiation at $n = N$. This is

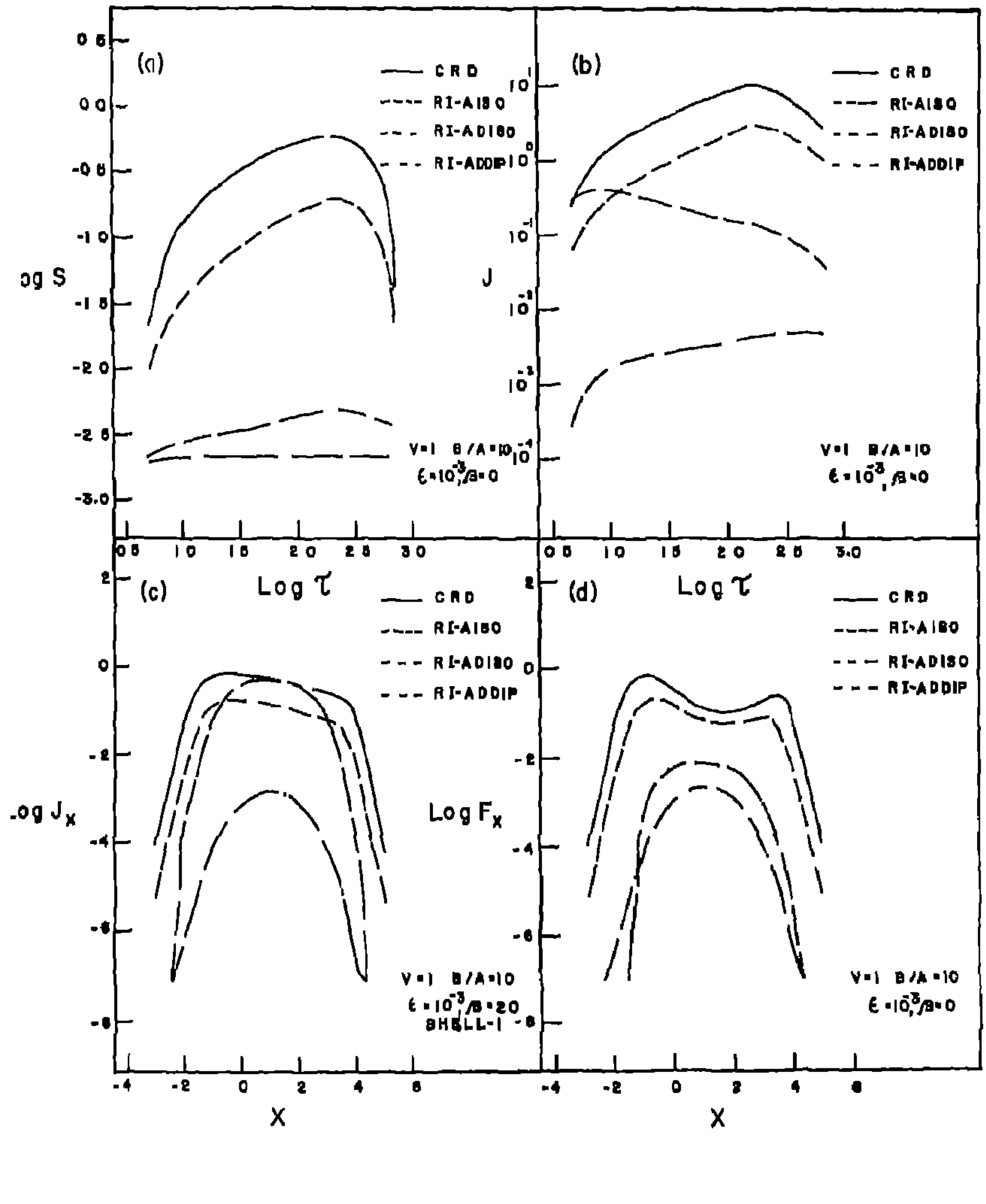

(a)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log S
V=1 B/A=10
Log τ
(b)
J
(c)
Log J_x
X
(d)
Log F_x

Fig 12

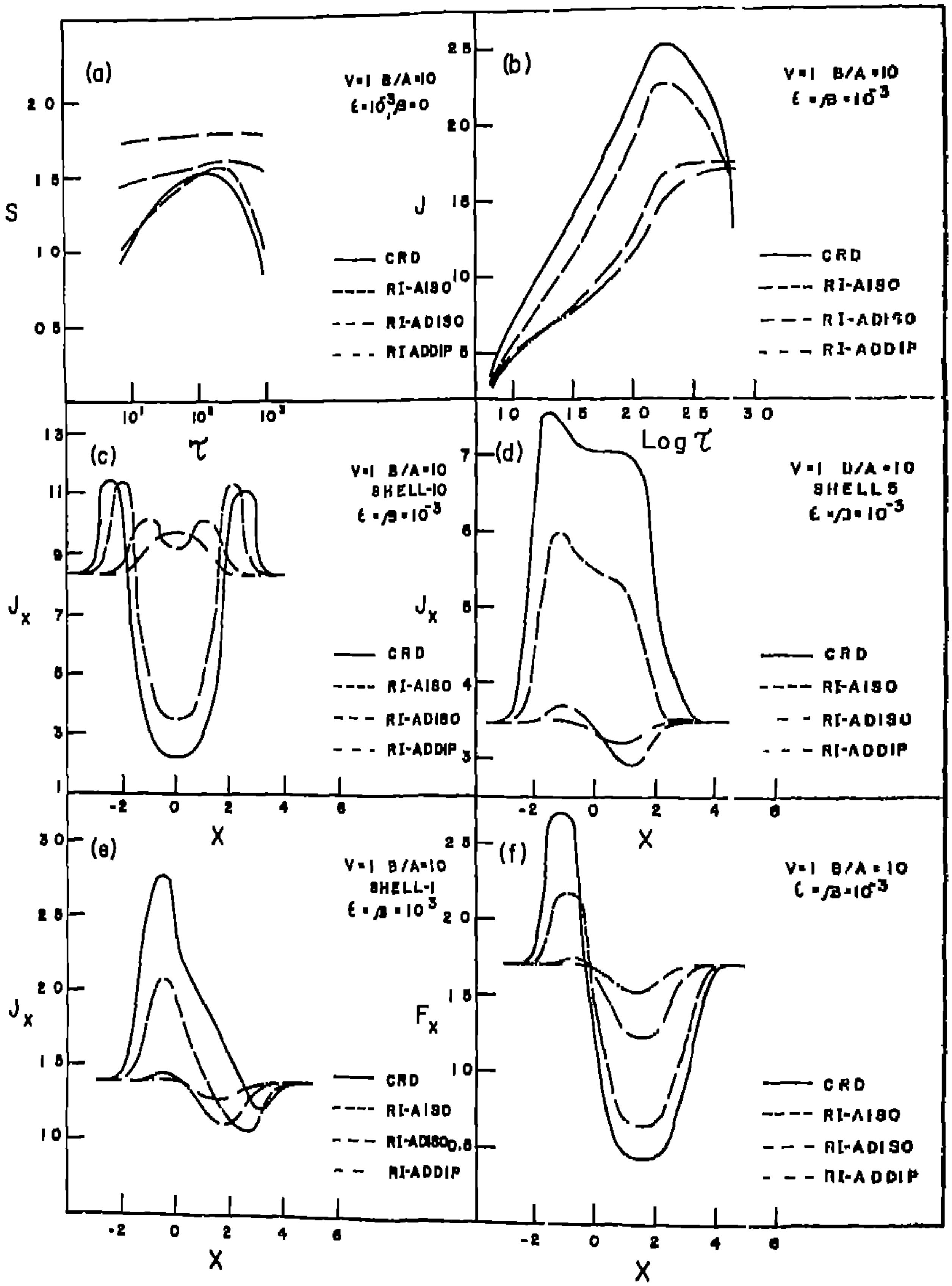

(a)
V=1 B/A=10
ε=10^-3, β=0
S
CRD
RI-AISO
RI-ADISO
RI ADDIP
τ
(b)
V=1 B/A=10
ε=β=10^-3
J
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log τ
(c)
V=1 B/A=10
SHELL-10
ε=β=10^-3
J_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X
(d)
V=1 B/A=10
SHELL 5
ε=β=10^-3
J_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X
(e)
V=1 B/A=10
SHELL-1
ε=β=10^-3
J_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X
(f)
V=1 B/A=10
ε=β=10^-3
F_X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
X

Fig 13

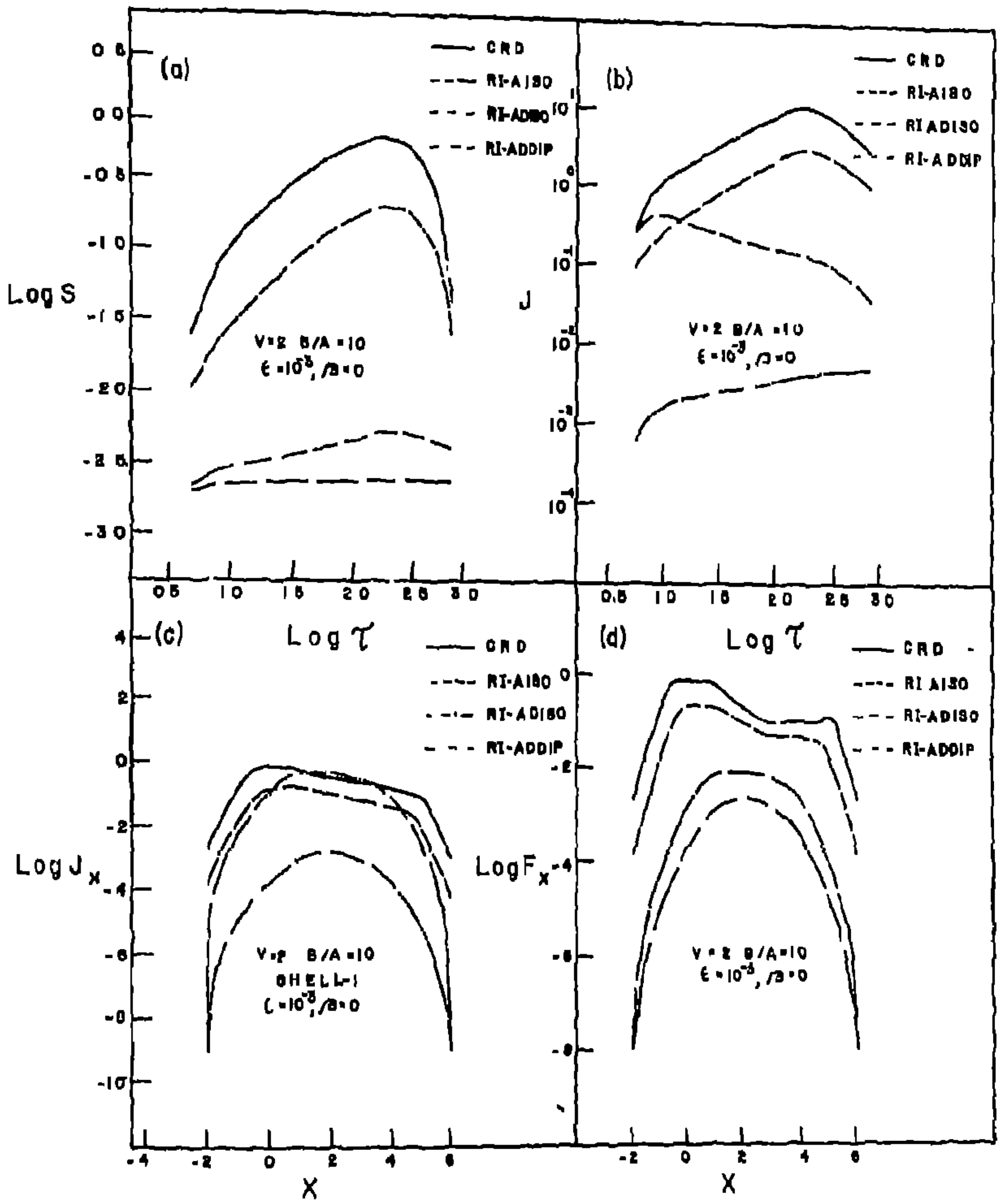

(a)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log S
V=2 B/A =10
ε =10⁻³, β=0
Log τ
(b)
CRD
RI-AISO
RI ADISO
RI-ADDIP
J
V=2 B/A =10
ε =10⁻³, β=0
Log τ
(c)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log J_x
V=2 B/A =10
SHELL-1
ε =10⁻³, β=0
X
(d)
CRD
RI AISO
RI-ADISO
RI-ADDIP
Log F_x
V=2 B/A =10
ε =10⁻³, β=0
X

Fig 14.

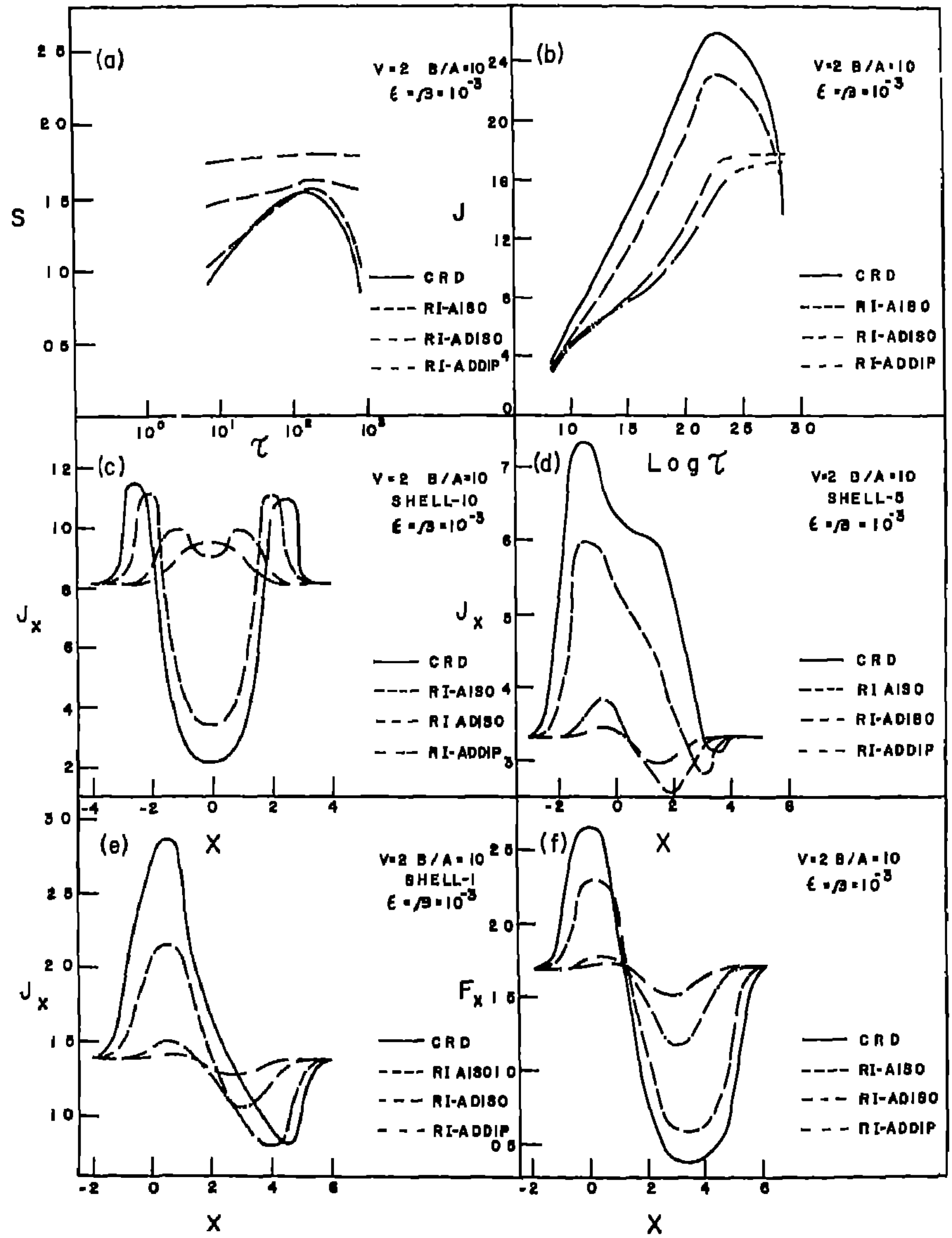
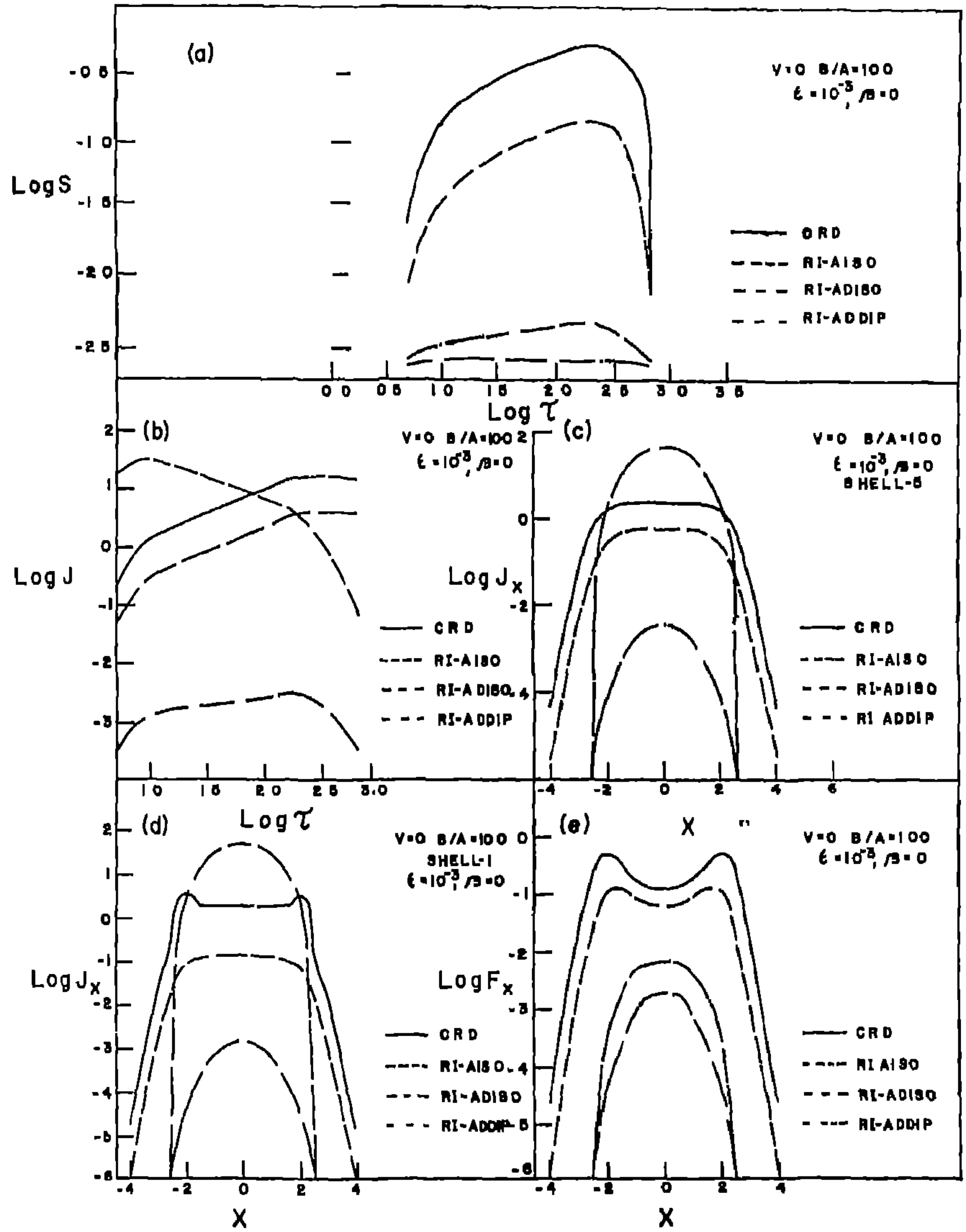

(a)
V=2 B/A=10
ε=β=10⁻³
S
τ
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(b)
V=2 B/A=10
ε=β=10⁻³
J
Log τ
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(c)
V=2 B/A=10
SHELL-10
ε=β=10⁻³
Jx
X
CRD
RI-AISO
RI ADISO
RI-ADDIP
(d)
V=2 B/A=10
SHELL-5
ε=β=10⁻³
Jx
X
CRD
RI AISO
RI-ADISO
RI-ADDIP
(e)
V=2 B/A=10
SHELL-1
ε=β=10⁻³
Jx
X
CRD
RI AISO
RI-ADISO
RI-ADDIP
(f)
V=2 B/A=10
ε=β=10⁻³
Fx
X
CRD
RI-AISO
RI-ADISO
RI-ADDIP

Fig 15



(a)
V=0 B/A=100
ε=10⁻³, β=0
Log S
Log τ
ORD
RI-AISO
RI-ADISO
RI-ADDIP
(b)
Log J
(c)
SHELL-5
Log J_x
(d)
SHELL-1
(e)
Log F_x
X

Fig. 16.

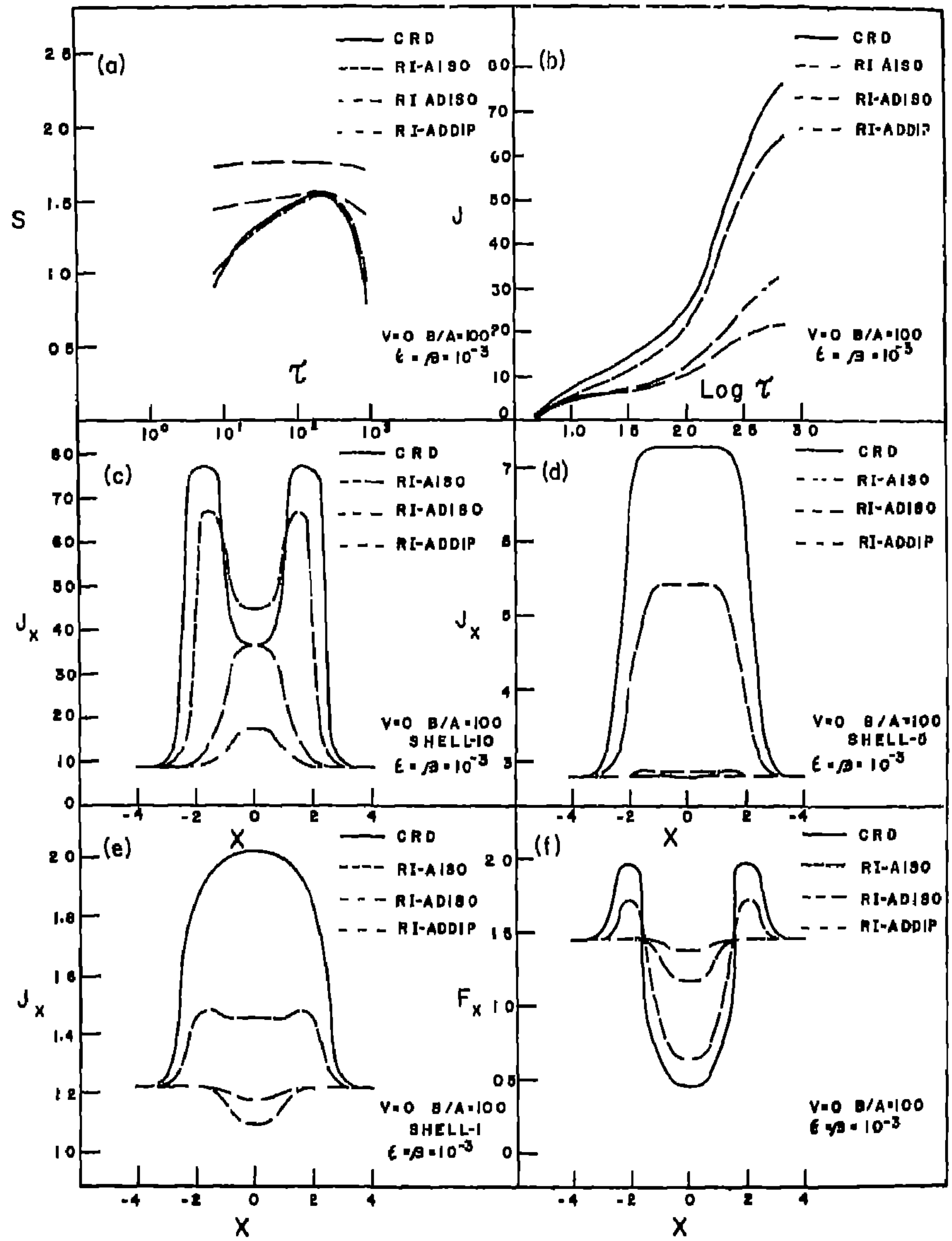

(a)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
S
V=0 B/A=100
ε = β = 10⁻³
τ
(b)
CRD
RI AISO
RI-ADISO
RI-ADDIP
J
V=0 B/A=100
ε = β = 10⁻³
Log τ
(c)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Jx
V=0 B/A=100
SHELL-10
ε = β = 10⁻³
X
(d)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Jx
V=0 B/A=100
SHELL-5
ε = β = 10⁻³
X
(e)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Jx
V=0 B/A=100
SHELL-1
ε = β = 10⁻³
X
(f)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Fx
V=0 B/A=100
ε = β = 10⁻³
X

Fig. 17.

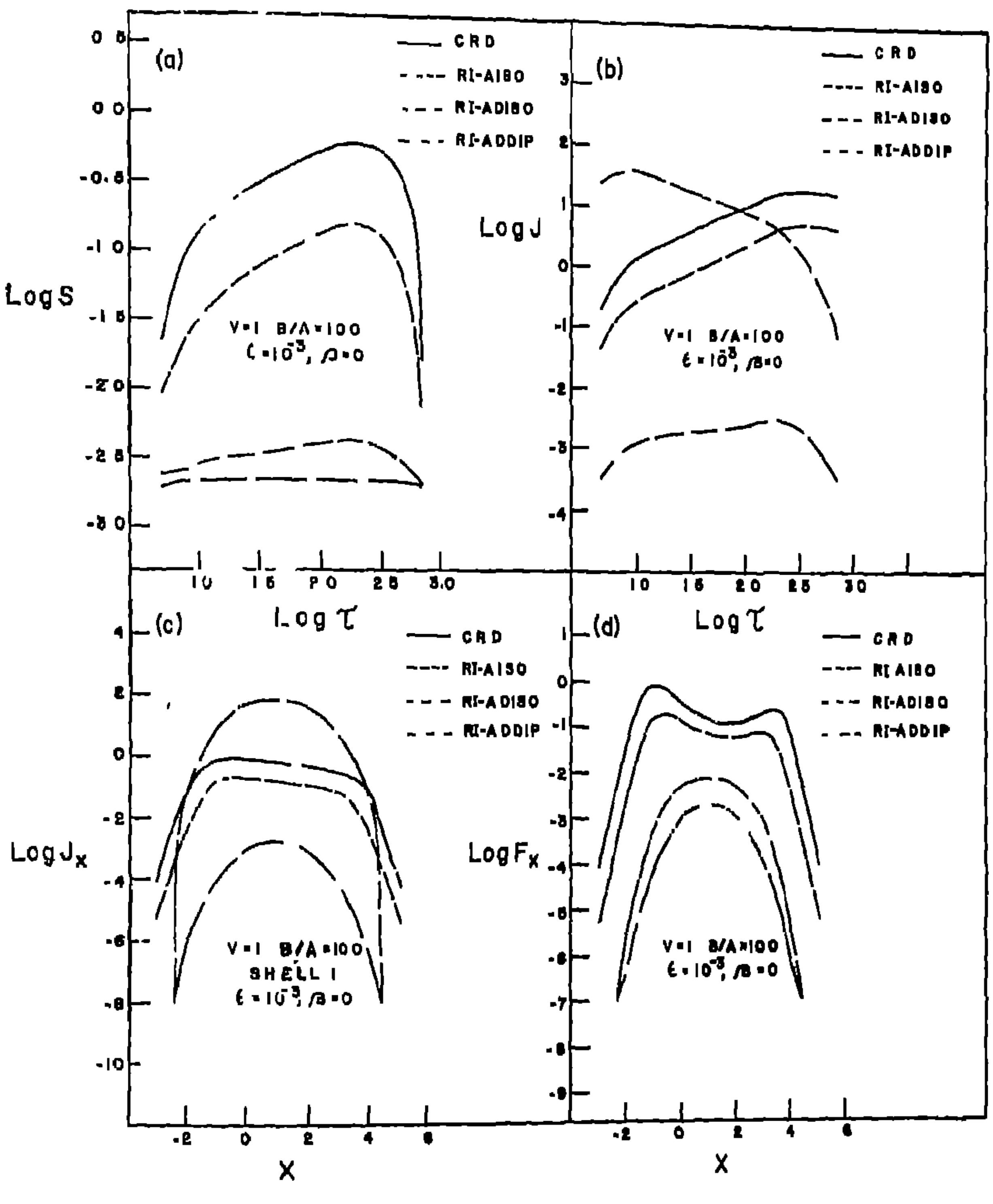

(a)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log S
V=1 B/A=100
Log τ
(b)
Log J
(c)
Log J_x
SHELL 1
X
(d)
RI AISO
Log F_x

Fig 16.

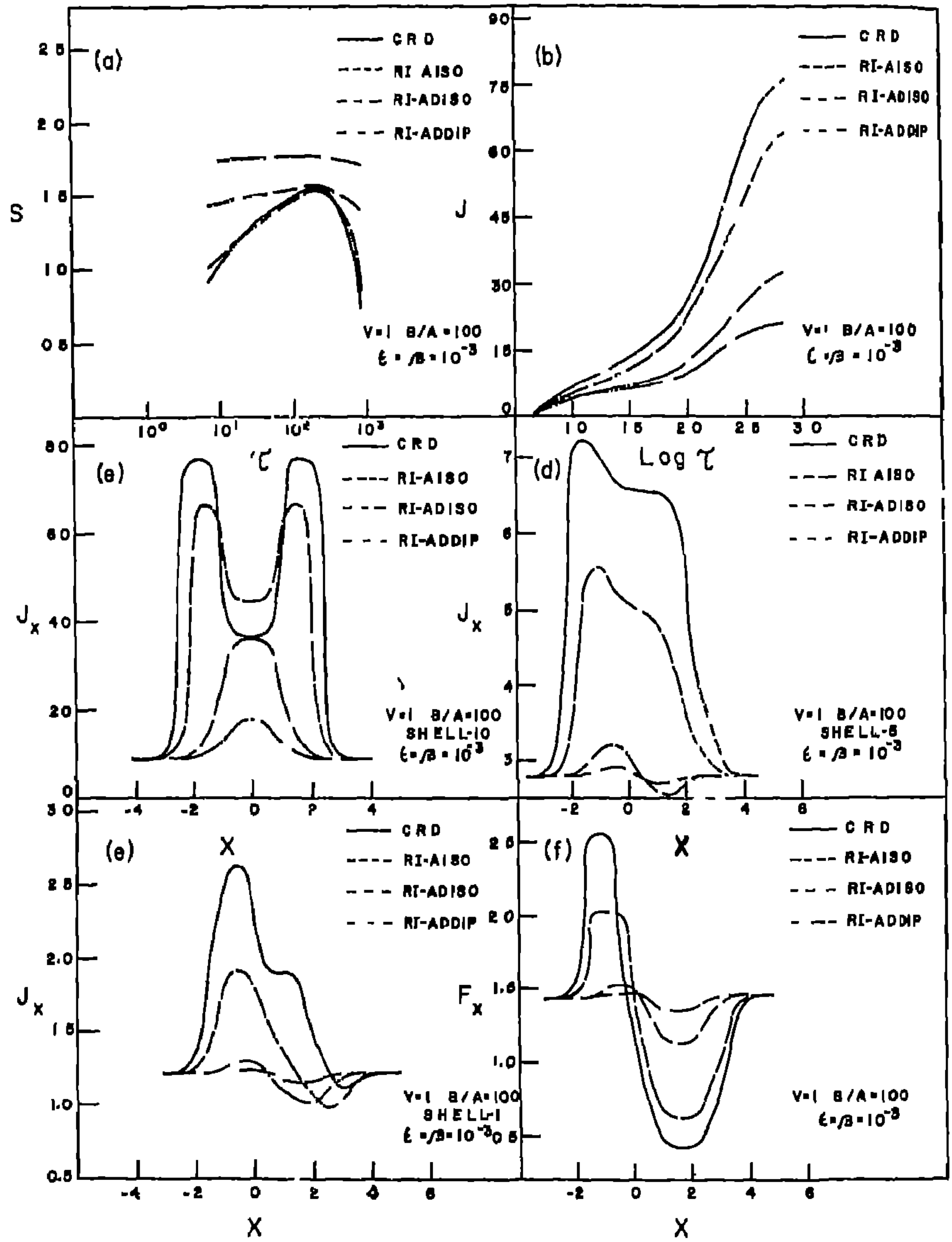

(a)
CRD
RI AISO
RI-ADISO
RI-ADDIP
S
V=1 B/A=100
ε=/β=10^-3
τ
(b)
J
Log τ
(c)
Jx
SHELL-10
X
(d)
SHELL-5
(e)
SHELL-1
(f)
Fx

Fig. 10.

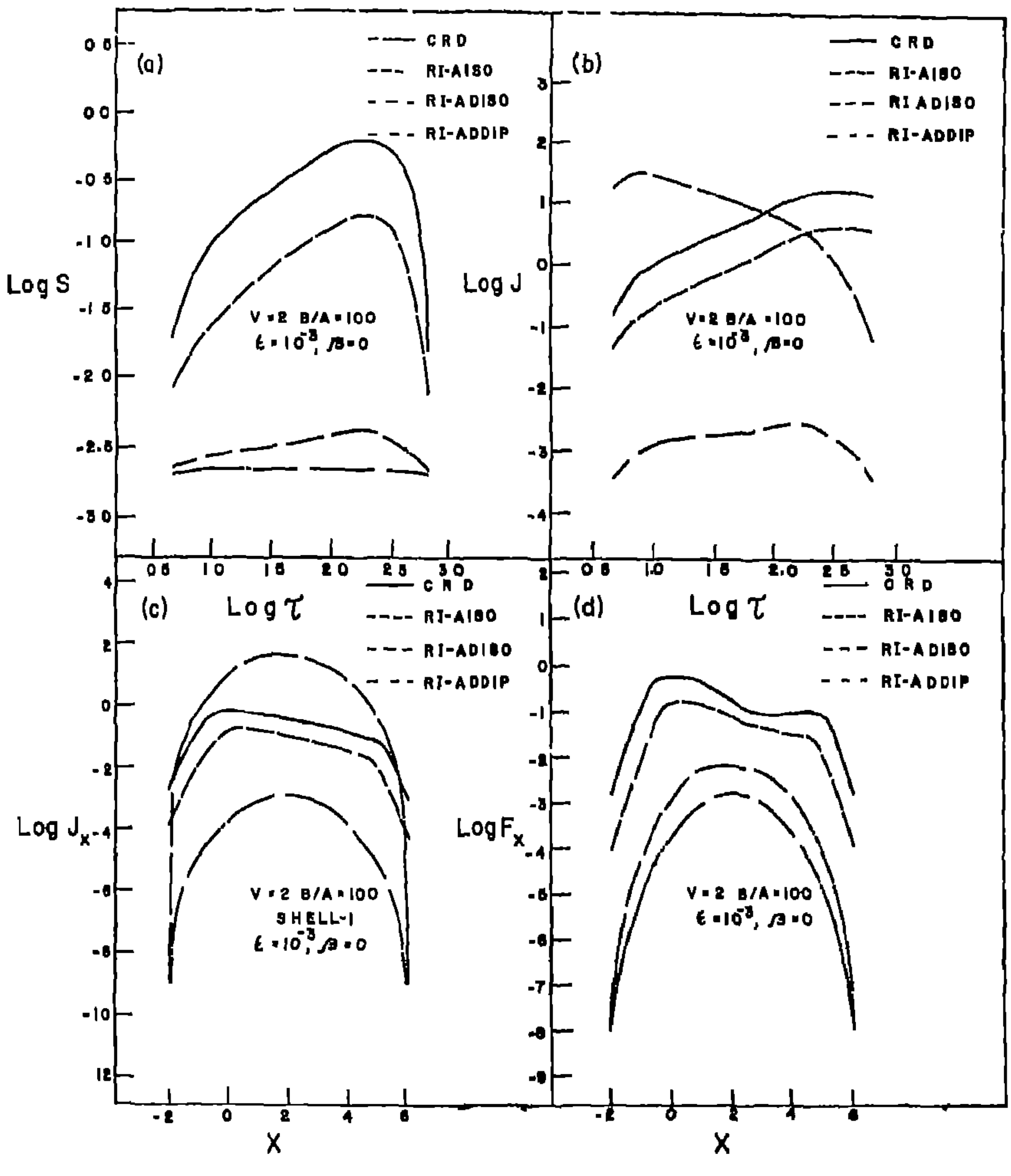

(a)
(b)
(c)
(d)
CRD
RI-AISO
RI-ADISO
RI-ADDIP
Log S
Log J
Log τ
Log J_x
Log F_x
X
V=2 B/A=100
ε=10⁻³, β=0
SHELL-I

Fig 20

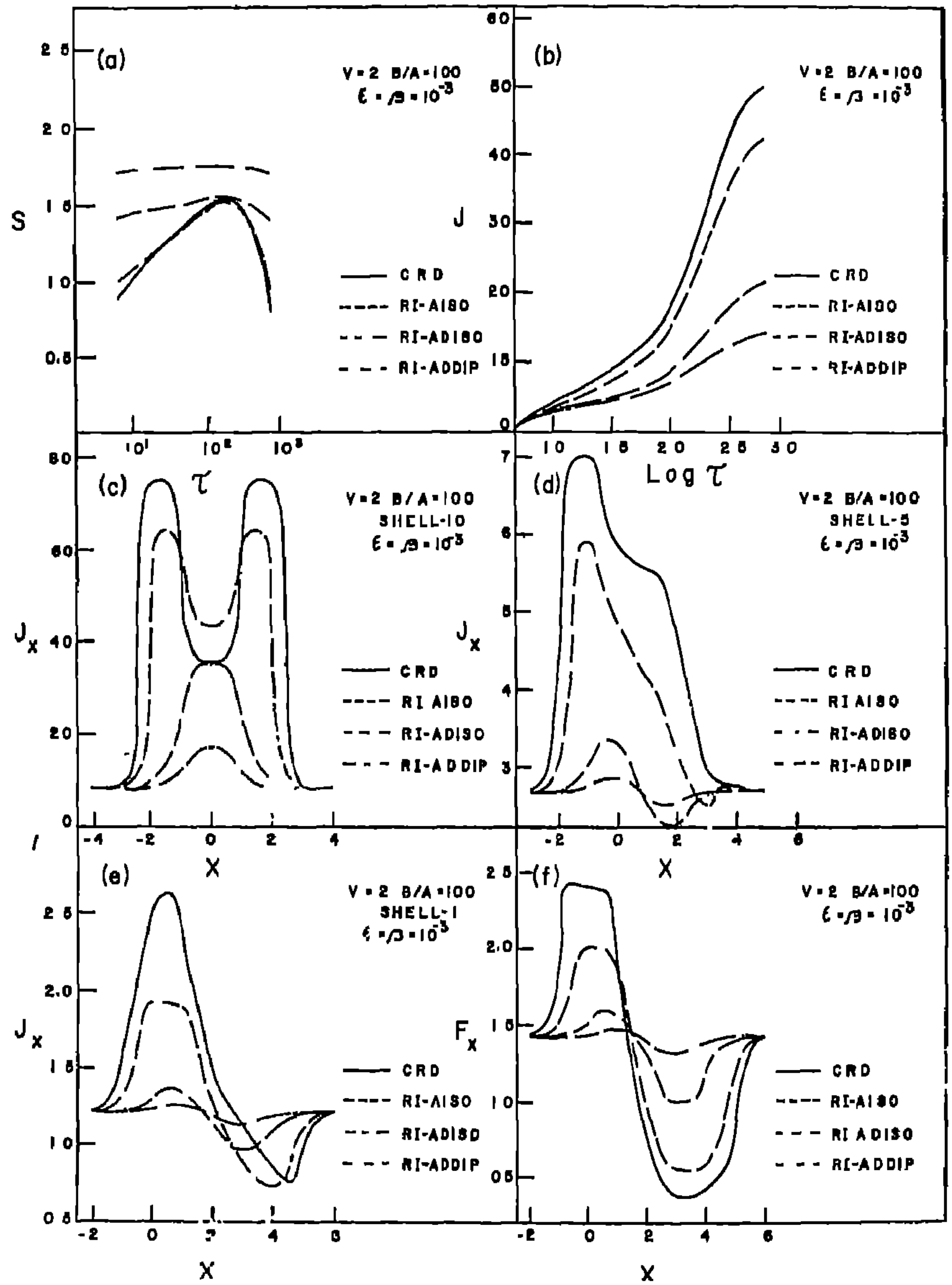

(a)
V = 2 B/A = 100
ε = β = 10⁻³
S
τ
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(b)
V = 2 B/A = 100
ε = β = 10⁻³
J
Log τ
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(c)
V = 2 B/A = 100
SHELL-10
ε = β = 10⁻³
Jx
X
CRD
RI AISO
RI-ADISO
RI-ADDIP
(d)
V = 2 B/A = 100
SHELL-5
ε = β = 10⁻³
Jx
X
CRD
RI AISO
RI-ADISO
RI-ADDIP
(e)
V = 2 B/A = 100
SHELL-1
ε = β = 10⁻³
Jx
X
CRD
RI-AISO
RI-ADISO
RI-ADDIP
(f)
V = 2 B/A = 100
ε = β = 10⁻³
Fx
X
CRD
RI-AISO
RI ADISO
RI-ADDIP

Fig. 21.

important because it will automatically check the programme for non-physical errors also We have performed calculations for two physical situations

$$\text{Case (1)} \quad \epsilon = 10^{-3} \text{ and } \beta = 0$$

$$\text{Case (2)} \quad \epsilon = \beta = 10^{-3}$$

For each set of parameters B/A, V_1, r and β, the quantities S_n J_n J (X_i, n = 10), J (X_i, n = 5), J (X_i, n = 1) and F (X_i) are presented in Figures (4-21) for complete redistribution (CRD), angle averaged redistribution function with isotropic scattering ($R_{I\ AISO}$), of angle dependent function with isotropic scattering ($R_{I\ ADISO}$) and angle dependent function with dipole scattering ($R_{I\ ADDIP}$) As the results for dipole scattering with angle averaged and angle dependent functions are graphically unresolvable only those results for angle dependent functions for dipole scattering are shown We shall hereafter refer to various curves by their corresponding simplified names such as CRD, $R_{I\ AISO}$ etc

Each of Figures [4-11], [13], [15], [17], [19] and [20] contains 6 parts a, b, c, d, e and f In parts [a] and [b], the total source function and the total mean intensities are plotted against the optical depth, and in parts [c], [d] and [e], the run of frequency dependent mean intensities are given for shells 10, 5 and 1 respectively These figures describe the mean intensities corresponding to total optical depths 10^3, 55 and 0 in the medium respectively We have plotted monochromatic emergent fluxes F [X_i] versus X_i in part [f] In Figures 12, 14, 16, 18, 20, the intermediate mean intensities are not plotted

The total source functions, mean intensities and emergent monochromatic fluxes are presented for a stationary, plane parallel medium in Figures [4] and [5] for case [1] and case [2] respectively The CRD values are larger than those of the PRD values The source functions of CRD, $R_{I\ AISO}$ become maximum at about log ϵ = 2 5 where as this maximum reduces in the case of S corresponding $R_{I\ ADISO}$, the source function of $R_{I\text{-}ADDIP}$ is almost flat in case [1] The total mean intensities reflect the source functions in both the cases The monochromatic mean intensities J_n [X_i] at n = 10, 5, 1 for the two cases are markedly different In the first case, we see emission lines with self absorption [there is only emission for $R_{I\ ADISO}$ and $R_{I\text{-}ADDIP}$ and for all types of redistributions for the shell 5] where as in the second case absorption is more prominent except in the case of J [X_i] for n = 5. A very interesting feature in case 2 is that the mean intensity J [X_i] for CRD and $R_{I\ AISO}$ is in absorption at n = 10, and appear in emission at the intermediate point n = 5 and emerges in absorption at n = 1 This can be understood from the fact that the source function for CRD and $R_{I\ AISO}$ becomes maximum at n = 5 whereas the source function for $R_{I\ ADISO}$ and $A_{I\ ADDIP}$ is almost flat.

In Figures [6-9], the results for differentially expanding media are given One can immediately see the asymmetry in the emergent flux profiles and in the frequency dependent mean intensities at intermediate points in the atmosphere However, the mean intensities at n = 10 show almost no asymmetry whereas those at n = 5 and n = 1 show a gradual increase in the asymmetry The emergent flux profiles show maximum asymmetry The red emission and blue absorption increase as the velocity is increased from v = 1 to v = 2 thermal units and a similarity to that of a P Cygni type profile can be noticed particularly in case [2] *[i e]* $\epsilon = \beta = 10^3$

We shall now consider the results in spherically symmetric media In figures [10] and [11], we present results for parameters B/A 10 and V 0 for case [1] and case [2] respectively The important difference between the plane parallel and spherically symmetric situations in case [2] is that there is strong emission in the wings which is clearly the effect of sphericity Mean intensities J [X_i] in Case 2, for CRD and $R_{I\text{-}AISO}$ show strong central absorption at n = 10, total emission at n = 5 and n = 1. The mean intensities for $R_{I\text{-}ADISO}$ and $R_{I\ ADDIP}$ show exactly the opposite behaviour The emergent flux profiles [Figure 11] show emission in the wings and absorption at the centre of the line in the case of CRD and $R_{I\text{-}AISO}$ and a totally absorption line in the case of $R_{I\ ADISO}$ and $R_{I\ ADDIP}$ is obtained In Figures [12-15], the results are given for a spherical medium expanding with gas velocity v = 1 and where the ratio of outer to inner radii is 10. The mean intensities J [X_i] for case

[2] at n - 10 show considerable amount of self absorption with a pronounced emission in the wings for CRD and R_{I-AISO}, where as for $R_{I-ADISO}$ and $R_{I\ ADDIP}$, the mean intensities show more emission than absorption At shell n 6 where the velocity is nearly ½ unit of a mean thermal velocity we notice that the lines not only are shifted but also become asymmetric about their centres The emergent monochromatic fluxes F [X_i] in Figure [13f] clearly develop into the P Cygni type profiles particularly in the case of CRD and R_{I-AISO} where as profiles calculated by the functions $R_{I\ AIDSO}$ and $R_{I\ ADDIP}$ show very little change except that their centres are shifted A similar trend can be noticed when the velocity of the gas is increased to 2 mean thermal units [see Figure 16]

In Figures [16-21], the results are presented for B/A = 100, V - 0, 1 and 2 for the two cases These results show similar characteristics as shown by the results given in Figures [10-15] However the emergent profiles become much broader and the heights of emission peaks are considerably larger than those formed in other situations In Table 1, we give the ratio of height of emission to the depth of absorption for case [2] *[i e]* $\epsilon = \beta = 10^{-4}$, for v - 0, 1 and 2 and B/A - 1, 10, and 100 for the profiles calculated with CRD

Table I Ratios of emission heights to absorption depth

V	B/A 1	10	100
0	104	287	500
1	180	743	1 032
2	167	650	,936

From Table 1, we can see that for a given velocity, as the parameter B/A increases, the emission also increases However when velocity increases for a given value of B/A the emission does not increase propotional to the velocity This is perhaps due to the fact that the line becomes broader when the gas moves with larger velocities

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APPENDIX

CODING FOR CALCULATION OF LINES IN MOVING MEDIA WITH PARTIAL FREQUENCY REDISTRIBUTION

```
C     THE PROGRAM IS STRAIGHTFORWARD AND SIMPLE.  EXPLANATION IS
C     GIVEN WHEREVER IT IS NECESSARY IN THE FORM OF COMMENTARY.  THE
C     DIMENSIONS OF VARIOUS MATRICES ARE GIVEN FOR A GENERAL SITUATION.
C     SO, IF THE DYNAMIC ARRAY FACILITY IS NOT AVAILABLE THESE DIMEN-
C     SIONS (KK,II, ETC) MUST BE GIVEN EXACT VALUES.  IN EXPLAINING
C     THE QUANTITIES, THE REFERENCE IS GIVEN TO EQUATIONS IN THE TEXT.
C     INTERMEDIATE STEPS, HOWEVER, ARE NOT EXPLAINED AS THE READER
C     WILL FIND IT EASY TO GO THROUGH.  THIS PROGRAM ENABLES US TO
C     CALCULATE THE LINE PROFILES IN SPHERICAL SYMMETRY IN THE STAR'S
C     REST FRAME WITH ASSUMED VELOCITY, DENSITY DISTRIBUTION.  THE
C     QUANTITIES, ∈ , β , ρ , B/A, THE TOTAL OR SHELL OPTICAL DEPTHS
C     ARE ALL FREE PARAMETERS.  THESE FREE PARAMETERS CAN ALSO BE
C     CALCULATED AS IF ONE HAS A MODEL, OR THIS PROGRAM CAN ALSO BE
C     JOINED WITH THE REAL HYDRODYNAMICAL PROBLEM.  THE SOURCE
C     FUNCTIONS, MEAN INTENSITIES, FLUXES AT INTERNAL POINTS CAN BE
C     OBTAINED FROM THIS PROBLEM.  VARIOUS REDISTRIBUTION FUNCTIONS
C     CAN BE USED BY CHANGING THE SUBROUTINE DIST.  NO ATTEMPT HAS
C     BEEN MADE TO OPTIMIZE THE PROGRAM.  IT IS ALWAYS POSSIBLE FOR AN
C     INTELLIGENT PROGRAMMER TO SAVE MEMORY SPACE PARTICULARLY IN THE
C     FIELD AND CELL ROUTINES BY OVER-WRITING SOME OF THE MATRICES.
C     PROGRAM FOR LINE TRANSFER IN MOVING MEDIA WITH PARTIAL FREQUENCY
C     REDISTRIBUTION.  WE HAVE TREATED A TWO-LEVEL NON-LTE ATOM WITH
C     GIVEN VELOCITY AND DENSITY DISTRIBUTIONS.

C     MAIN PROGRAM

      CØMMØN NLYR,NC,II,KK,X(II),A(II),ALF,CØ(NC),C(NC),
     1BETA,EPS,DELTA,RHØ,BP,TTAU,TCF,RCW,BA,VB,VA,
     2VD,VR,RW,CP(KK,KK),CRM,FRD(KK,KK),FRD1(KK,KK),
     3FRD2(KK,KK),FRD3(KK,KK),FRD4(KK,KK),BØUND,CR(NC,NC),TØD(NLYR),
     4PI

C     NLYR = NO. OF SHELLS, NC = NO. OF COSINES OR NO. OF ANGLES,
C     II = NO. OF FREQUENCY POINTS, KK = II* NO; X,A ARE THE FREQUENCY
C     POINTS AND THEIR CORRESPONDING WEIGHTS.  ONE CAN SELECT THE
C     ZEROS AND WEIGHTS OVER (-1, +1) EITHER OF A GAUSS-LEGENDRE
C     QUADRATURE FORMULA OR OF TRAPEZOIDAL RULE DEPENDING UPON THE
C     PROBLEM.  CØ AND C'S ARE THE ANGLE POINTS AND WEIGHTS OVER (0,1).
C     THE ZEROS AND WEIGHTS OF GAUSS-LEGENDRE QUADRATURE FORMULA ARE
C     USED. (SEE ABROMOWITZ AND STEGUN: HANDBOOK OF MATHEMATICAL
C     FUNCTIONS, PAGE 921). BETA = β , EPS = ∈ , DELTA = δ , RHØ = ρ ,
C     BP = B, BA = B/A (THE RATIO OF OUTER TO INNER RADII), VB AND VA
C     ARE THE VELOCITIES AT B AND A RESPECTIVELY (SEE SECTION 6 OF
C     THE TEXT).  CP IS THE CURVATURE MATRIX Λ+, CRM = Λ- (1,1);
C     BØUND IS THE BOUNDARY CONDITION AT A.  FRD'S ARE THE FREQUENCY
C     REDISTRIBUTION FUNCTIONS.  ALF = BAND WIDTH.

      DEFINE NLYR, II,NC,X(II),A(II),CØ(NC),C(NC),EPS,BETA,RHØ,BA,
     BØUND, BP
```

```
      VA = 0.
      VB = 2.
C     ONE CAN CHANGE VA AND VB
      DØ 1  I = 1, II
      X(I) = ALF* X(I)
    1 CØNTINUE
      PI = 3.141592654
      BP = 1.
C     BP IS THE PLANCK FUNCTION
      CALL CURV
      EPS = 0.
      BETA = 0.
      BØUND = 1.
C     WE HAVE GIVEN THE INNER BOUNDARY CONDITION BØUND = 1
C     BECAUSE ∈ = β = 0.  IF WE SET ∈ OR β  OR BOTH > 0 THEN WE MAY
C     SET BØUND = 0.
      DELTA = 1.
      RHØ = 1.
      BA = 10.
      RCW = (BA-1.)/(BA*NLYR)
      DØ 2 J = 1,KK
      DØ 2 K = 1,KK
      CP(J,K) = 0.
    2 CØNTINUE
      DØ 3 I = 1,II
      DØ 3 J = 1,NC
      K = J+(I-1)*NC
      CP(K,K) = CR(J,J)
    3 CØNTINUE
      DØ 4 I = 1,II
      DØ 4 J = 2,NC
      K = J+(I-1)*NC
      CP(K,K-1) = CR(J,J-1)
      CP(K-1,K) = CR(J-1,J)
    4 CØNTINUE
      CALL FIELD
      RETURN
      END

      SUBRØUTINE CURV
C     THIS CALCULATE THE CURVATURE SCATTERING MATRICES Λ + AND Λ -
C     (CRM) MATRICES (SEE EQUATIONS (62) AND (63), SECTION 6).
C     USE THE COMMON STATEMENTS AS IN THE MAIN ROUTINE DIMENSIØN
C     CT(NC+1), CR(NC,NC), CAZ(NC)
      EK = 1.
      EL = .5
      CT(1) = 0.
```

```
      DØ 2 JJ = 1,NC
      CFC = 0.
      DØ 1 I = 1,JJ
      CFC = CFC+C(I)
    1 CØNTINUE
      CF(JJ+1) = CFC
    2 CØNTINUE
      DØ 3 J = 1,NC
      DØ 3 K = 1,NC
      CR(J,K) = 0.
    3 CØNTINUE
      JM1 = NC-1
      DØ 4 J = 1,JM1
      CR (J,J+1)=(ZL-CF(J+1)**2)*(CF(J+1)-CØ(J)/(C(J)*(CØ(J+1)-CØ(J))
    4 CØNTINUE
      CR(1,1)=(((ZK-C(1)**2)*(CØ(2)-C(1))/(CØ(2)-CØ(1)))-ZL)/C(1)
      CR(NC,NC)=-((ZK-CF(NC)**2)*(CF(NC)-CØ(NC-1)))/(C(NC)*(CØ(NC)
     -CØ(NC-1)))
      NC1 = NC-1
      DØ 5 J = 2,NC1
      CR(J,J)=((ZK-CF(J+1)**2)*(CØ(J+1)-CF(J+1)))/
     1(C(J)*(CØ(J+1)-CØ(J)))-((ZK-CF(J)**2)*(CF(J)-CØ(J-1)))/
     2(C(J)*(CØ(J)-CØ(J-1)))
    5 CØNTINUE
      DØ 6 J = 1,NC
      CR(J,J-1)=-((ZK-CF(J)**2)*(CØ(J)-CF(J)))/(C(J)*(CØ(J)-CØ(J-1)))
    6 CØNTINUE
      CRM = -ZL/C(1)
      DØ 7 J = 1,NC
    7 PRINT 8,(CR(J,K),K=1,NC)
    8 FØRMAT(3X,4(3X,E16.8))
      PRINT 9,CRM
    9 FØRMAT(3X,'CRM=',3X,E16.8)
C     NOW WE SHALL VERIFY THE IDENTITY (90).  All CAZ'S SHOULD BE
C     EXACTLY ZERO OR SHOULD BE ACCURATE TO THE MACHINE'S PRECISION.
      DØ 10 J = 1,NC
      CSMG = CSMG+CR(J,1)*C(J)
   10 CØNTINUE
      CAZ(1)=CSMG
      DØ 11 J = 1,NC
      CSMG=CSMG+CR(J,K)*C(J)
   11 CØNTINUE
      CAZ(K)=CSMG
   12 CØNTINUE
      PRINT 13,(CAZ(K),K=1,NC)
   13 FØRMAT(3X,4(3X,E16.8))
      RETURN
      END
      SUBRØUTINE FIELD
C     THIS ROUTINE CALCULATES THE DIFFUSE RADIATION FIELD USING THE
C     SCHEME GIVEN IN SECTION 5.  THIS ROUTINE ALSO CALCULATES THE
C     MEAN INTENSITIES, SOURCE FUNCTIONS AND NET FLUXES.
C     PUT THE SAME COMMON STATEMENTS AS IN THE MAIN ROUTINE.  IN
C     ADDITION TO THE ABOVE, THE FOLLOWING SHOULD BE ADDED.
      CØMMØN/ST/PHI(KK),PEP(KK),PMEN(KK),PMM(KK),PET(KK),PET(KK),
      PMET(KK),PMT(KK),V1(KK),V2(KK),V3(KK),V4(KK),V5(KK),V6(KK),
      V7(KK),V8(KK)
      CØMMØN/PRØ/SØFM(NLYR),SUPP(NLYR)
      DIMENSIØN TA(KK,KK),RA(KK,KK),TF(KK,KK),RF(KK,KK),SIP(KK)
     1SIM(KA),AZ(2),PKZ(KK),EZ(KK,KK),VP(KK),S1(KK,KK),S2(KK,KK)
     2S3(KK,KK),THF1(KA,KK),RH(KK,KK),V1(KK),V2(KK),V3(KK),V4(KK),
     3VM(KK),UM(KK),UP(KA),FRQY(II),RNU(II),L(KA),M(KK),THF(KA,KK),
     4SARM(KK,KA),SARP(KK,KK),AMI1(II),AMI2(II),AMI3(II)
C     NOW THE ROUTINE FIELD STARS (NOTE: ITX=1 CORRESPONDS TO OUTER
C     MOST SURFACE OF THE SPHERICAL MEDIUM (i.e.) R = B AND ITX =
C     NLYR TO R = A.)
      IQ=6
      DØ 9 ITX =1,NLYR
      IF(ITX-1)1,1,4
    1 DØ 2 J =1,KK
      VP(J)=0.
      DØ 2 K =1,KK
      RZ(J,K)=0.
    2 CØNTINUE
C     RZ AND VP ARE THE BOUNDARY CONDITIONS R(1,1)=0. and V+(1/2) =
C     U+(b)=0.  THE LATTER CONDITION MEANS THAT THERE IS NO RADIATION
C     INCIDENT AT THE OUTER SURFACE OF THE MEDIUM.  SEE EQUATIONS
C     (30,31) OF SECTION 5.
    4 RW=RCW/(1.-(ITX-1)*RCW)
C     RW=CURVATURE FACTOR FOR SHELL ITX CALCULATED IN TERMS OF THE
C     CURVATURE FACTOR OF THE OUTERMOST SHELL RCW.
      TØD(ITX)=TTAU/((NLYR-ITX+1)**2)
C     TØD IS THE OPTICAL DEPTH IN EACH SHELL AND TTAU IS THE OPTICAL
C     DEPTH IN THE SHELL NEAREST TO THE STAR (i.e.) OF THE SHELL WITH
C     LARGEST CURVATURE FACTOR.  HERE OPTICAL DEPTH IS ASSUMED TO VARY
C     AS 1./(R**2) OR THE DENSITY    AS VARYING AS 1./(R**3).  THE
C     OPTICAL DEPTH CAN BE CALCULATED WITH REAL PARAMETERS.
      VD=(VB-VA)/NLYR
      VR=VA+(NLYR-ITX+.5)*VD
```

```
C       VD IS THE DV/DN AND VR = V(R(N)).  WHERE  N  IS NUMBER OF THE
C       SHELL.  HERE THE ATMOSPHERE IS ASSUMED TO EXPAND RADIALLY.  ONE
C       CAN CHANGE THE VARIATION IN VELOCITY AS ONE DESIRES.
C       NOW WE SHALL CALCULATE THE REDISTRIBUTION FUNCTIONS FOR THE
C       RADIATION IN THE OPPOSITE DIRECTIONS.  SEE SECTION 6.  WE HAVE
C       TO STORE THESE MATRICES AS WE MAY NEED THEM TO CALCULATE THE
C       SOURCE FUNCTIONS, THEREFORE, WE PRESERVE THESE ON A DISC OR
C       TAPE.  HERE WE SHALL WRITE ON THE DISC.

        IC1=1
        IC2=1
        CALL DIST (IC1,IC2)
        DØ 700 J=1,KK
        DØ 700 K=1,KK
        FRD1(J,K)=FRD(J,K)
  700   CØNTINUE
        WRITE (8) FRD1
        IC1=1
        IC2=-1
        CALL DIST(IC1,IC2)
        DØ 7001 J=1,KK
        DØ 7001 K=1,KK
        FRD2(J,K)=FRD(J,K)
 7001   CØNTINUE
        WRITE (8) FRD2
        IC1=-1
        IC2=1
        CALL DIST (IC1,IC2)
        DØ 7002 J=1,KK
        DØ 7002 K=1,KK
        FRD3(J,K)=FRD(J,K)
 7002   CØNTINUE
        WRITE (8) FRD3
        IC1=-1
        IC2=-1
        CALL DIST(IC1,IC2)
        DØ 7003 J=1,KK
        DØ 7003 K=1,KK
        FRD4(J,K)=FRD(J,K)
 7003   CØNTINUE
        WRITE (8) FRD4
        PRINT 300,ITX
  300   FØRMAT(9X,'SHELL NUMBER IS',3X,I6)
        CALL CELL (TA,RA,TF,RF,SIP,SIM,ITX,IQ,ZA,ZB)
        PRINT 301,ZA,ZB
  301   FØRMAT(3X,'ZA=',E16.8,3X,'ZB=',E16.8)
        CALL MATMUL(RA,RZ,S1,KK,KK,KK)
        CALL MATMUL(RZ,RA,S2,KK,KK,KK)
        DØ 5 J=1,KK
        DØ 5 K=1,KK
        S1(J,K)=-S1(J,K)
        S2(J,K)=-S2(J,K)
    5   CØNTINUE
        DØ 6 J=1,KK
        S1(J,J)=1.+S1(J,J)
        S2(J,J)=1.+S2(J,J)
    6   CØNTINUE
        DA=1.
        CALL MIN(S1,KK,DA,L,M)
        CALL MIN(S2,KK,DA,L,M)
        CALL MATMUL(RZ,S1,S3,KK,KK,KK)
        CALL MATMUL(S1,TF,THF,KK,KK,KK)
        KF=1
        CALL MATMUL(S1,SIM,V1,KK,KK,KF)
        CALL MATMUL(TA,S2,S1,KK,KK,KK)
        CALL MATMUL(S1,RZ,RH,KK,KK,KK)
        CALL MATMUL(S1,VP,V2,KK,KK,KK)
        CALL MATMUL(RH,SIM,V3,KK,KK,KF)
        CALL MATMUL(RA,S2,S1,KK,KK,KK)
        CALL MATMUL(S1,VP,V4,KK,KK,KF)
        CALL MATMUL(TA,S3,S2,KK,KK,KK)
        CALL MATMUL(S2,TF,S1,KK,KK,KK)
        DØ 7 J=1,KK
        VP(J)=V2(J)+SIP(J)+V3(J)
        VM(J)=V4(J)+V1(J)
        DØ 7 K=1,KK
        RZ(J,K)=HF(J,K)+S1(J,K)
    7   CØNTINUE
C       RZ = R(1,N+1),VP=V+(1/2),VM=V-(N+1/2),THF=T(N,N+1).
C       SEE EQUATIONS (30-33), SECTION 5.
C       THESE SHOULD BE PRESERVED ON DISC OR TAPE FOR THE BACKWARD
C       SWEEP.
        WRITE(9)RZ
        WRITE(9)THF
        WRITE(9)VP,VM,PHI,PEP,PMEN,PMM,PHT,PET,PMET,PMT
       1W1,W2,W3,W4,W5,W6,W7,W8
C       FOR THE EXPLANATION OF THE VECTORS PHI ETC. SEE CELL.
    9   CØNTINUE
        DØ 10 J=1,KK
        UM(J)=BØUND
```

```
   10 CONTINUE
C       HERE UM = U-(N+1), THE BOUNDARY CONDITION AT R = A.
C       NOW THE BACKWARD SWEEP STARS.  DURING THIS PROCESS, WE
C       CALCULATE THE INTENSITIES MEAN INTENSITIES, SOURCE FUNCTIONS
C       NET FLUXES OF THE DIFFUSE RADIATION FIELD AT ALL THE BOUNDARIES
C       OF THE SHELLS.
        IQ=2
        ITX=NLYR
        CALL CELL (TA,RA,TF,RF,SIP,SIM,ITX,IQ,ZA,ZB)
        BACK SPACE 9
        BACK SPACE 9
        BACK SPACE 9
        READ(9)RZ
        READ(9)THF
        READ(9)VP,VM,PHI,PEP,PMHN,PMM,PHT,PET,PMHT,PMT,
       1W1,W2,W3,W4,W5,W6,W7,W8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        READ(8)FRD1
        READ(8)FRD2
        READ(8)FRD3
        READ(8)FRD4
        DO 8900 I=1,II
        AMK=0.
        DO 8901 J=1,NC
        K=J+(I-1)*NC
        AMK=AMK+C(J)*UM(K)
 8901   CONTINUE
        AMI1(I)=AMK/(ZB*ZB)
 8900   CONTINUE
        DO 60 K=1,KK
        DO 60 I=1,KA
        S1(J,K)=FRD4(J,K)*W8(K)*(1.-EPS)
        S2(J?K)=FRD2(J,K)*W6(K)*(1.-EPS)
   60   CONTINUE
        CALL MATMUL(S1,VM,V3,KK,KK,KF)
        CALL MATMUL(S2,VM,V4,KA,KK,KF)
        GOTO 699
   11   BACK SPACE 9
        BACK SPACE 9
        BACK SPACE 9
        BACK SPACE 9
        BACK SPACE 9
        BACK SPACE 9
        READ(9)RZ
        READ(9)THF
        READ(9)VP,VM,PHI,PEP,PMHN,PMM,PHT,PET,PMHT,PMT,
       1W1,W2,W3,W4,W5,W6,W7,W8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        BACK SPACE 8
        READ(8)FRD1
        READ(8)FRD2
        READ(8)FRD3
        READ(8)FRD4
  699   CALL MATMUL(RZ,UH,V1,KA,KK,KF)
        CALL MATMUL(THF,UM,V2,KK,KK,KF)
        DO 12 J=1,KK
        UP(J)=V1(J)+VP(J)
        VM(J)=V2(J)+VM(J)
   12   CONTINUE
        CALL CELL (TA,RA,TF,RF,SIP,SIM,ITX,IQ,ZA,ZB)
        DO 8902 I = 1,II
        AMJ=0.
        DO 8903 J=1,NC
        K=J+(I-1)*NC
        AMJ=AMJ+C(J)*UP(K)
 8903   CONTINUE
        AMI2(I)=AMJ/(ZB*ZB)
 8902   CONTINUE
        DO 8903 I=1,II
        AMI3(I)=(AMI2(I)+AMI1(I))*.5
 8904   CONTINUE
        PRINT 8905
 8905   FORMAT(3X,'FREQUENCY DEPENDANT MEAN INTENSITIES AMF')
        PRINT 8906,(AMI3(I),I=1,II)
 8906   FORMAT(6(3X,E13.6))
        AMG=0.
        DO 8907 I=1,II
        AMG=AMG+AMI3(I)*A(I)
 8907   CONTINUE
        PRINT 8908, AMG
 8908   FORMAT(3X,'TOTAL MEAN INTENSITY IB',3X,E16.8)
        PRINT 99 ITX
```

```
   99 FØRMAT(9X,'SHELL NUMBER IS',3X,I6)
      PRINT 100
  100 FØRMAT(3X,'THE INTENSITY VECTØR U+IS')
      PRINT 102,(UP(J),J=1,KK)
  102 FØRMAT(6(3X,E13.6))
      PRINT 103
  103 FØRMAT(3X,'THE INTENSITY VECTØR U-IS')
      PRINT 102,(UM(J),J=1,KK)
C     CALCULATE THE FLUXES AT BOUNDARIES
      FLX=0.
      FXL=0.
      DØ 104 I=1,II
      DØ 104 J=1,NC
      K=J+(I-1)*NC
      FLX=FLX+A(I)*C(J)*CØ(J)*UP(K)
      FXL=FXL+A(I)*C(J)*CØ(J)*UM(K)
  104 CØNTINUE
      PRINT 105, FLX, FXL
  105 FØRMAT(9X,'FLUX ØF UP=',E16.8,3X,'FLUX ØF UM=',E16.8)
      DØ 61 K=1,KK
      DØ 61 J=1,KK
      S3(J,K)=FRD1(J,K)*W5(K)*(1.-EPS)
      THP1(J,K)=FRD3(J,K)*W7(K)*(1.-EPS)
   61 CØNTINUE
      CALL MATMUL(S3,UP,V1,KK,KK,KF)
      CALL MATMUL(THP1,UP,V2,KK,KK,KF)
      DØ 62 J=1,KK
      SARM(J)=((V3(J)+V2(J))/(4.*PI*ZA*ZA)+EPS*BP*
     1PMHH(J)+BETA*RHØ*BP)/(PMHH(J)+BETA)
      SARP(J)=((V4(J)+V1(J))/(4.*PI*ZA*ZA*)+EPS*HP*
     1PHI(J)+BETA*RHØ*BP)/(PHI(J)+BETA)
   62 CØNTINUE
      PRINT 117
  117 FØRMAT(3X,'ANGLE DEPENDANT SØURCE FUNCTIØNS ARE')
      PRINT 119,(SARM(J),J=1,KK)
      PRINT 119,(SARP(J),J=1,KK)
  119 FØRMAT(6(3X,E13.6))
      SØF=0.
      SUF=0.
      DØ 66 I=1,II
      DØ 66 J=1,NC
      K=J+(I-1)*NC
      SØF=SØF+A(I)*C(J)*SARM(K)
      SUF=SUF+A(I)*C(J)*SARP(K)
   66 CØNTINUE
      SØFM(ITX)=SØF
      SUFP(ITX)=SUF
      PRINT 67,SØF
   67 FØRMAT(3X,'SØFM=',E16.8)
      DØ 63 K=1,KK
      DØ 63 J=1,KK
      S1(J,K)=FRD4(J,K)*W8(K)*(1.-EPS)
      S2(J,K)=FRD2(J,K)*W6(K)*(1.EPS)
   63 CØNTINUE
      CALL MATMUL(S1,UM,V3,KK,KK,KF)
      CALL MATMUL(S2,UM,V4,KK,KK,KF)
      DØ 8909 I=1,II
      AMK=0.
      DØ 8910 J=1,NC
      K=J+(I-1)*NC
      AMK=AMK+C(J)*UM(K)
 8910 CØNTINUE
      AMI1(I)=AMK
 8909 CØNTINUE
      IF(ITX-1)36,110,36
  110 PRINT 111
  111 FØRMAT(9X,'THE FØLLØWING ARE THE EMERGENT FREQUENCY FLUX PØINTS')
      DØ 114 I=1,II
      FRQ=0.
      DØ 112 J=1,NC
      K=J+(I-1)*NC
      FRQ=FRQ+C(J)*CØ(J)*UM(K)
  112 CØNTINUE
      FRQY(I)=FRQ/(ZA*ZA)
  114 CØNTINUE
      PRINT 113,(FRQY(I),I=1,II)
  113 FØRMAT(6(3X,E13.6))
      PRINT 69
   69 FØRMAT(3X,'THE FØLLØWING ARE THE FREQUENCY INDEPENDANT
     1SØURCE FUNCTIØNS SØFM,SUFP')
      PRINT 70,(SØFM(J),J=1,NLYR)
      PRINT 70,(SUFP(J),J=1,NLYR)
   70 FØRMAT (6(3X,E13.6))
   36 ITX=ITX-1
      IF(ITX)13,13,11
   13 RETURN
      END

      SUBRØUTINE CELL (TA,RA,TF,RF,SIP,SIM,ITX,IQ,ZA,ZB)
C     THIS ROUTINE CALCULATES THE TURPAIRS OF CELL MATRICES OF REFLEC-
C     TION AND TRANSMISSION TOGETHER WITH THE INTERNAL SOURCE TERMS.
C     SEE EQUATIONS 72,73,74 OF SECTION 6.  WHEN THE OPTICAL DEPTH IN
C     EACH SHELL BECOMES GREATER THAN   CRIT,IT CALLS THE SUBROUTINE
C     STAR TA=T(N+1,N),TF=T(N,N+1),RA=R(N+1,N),RF=R(N,N+1),SIP=   +,
      SIM=   -.
```

```
C       INTRODUCE THE SAME COMMON STATEMENTS AS IN THE MAIN ROUTINE
        CØMMØN/ST/PHI(KK),PEP(KK),PMHN(KK),PMM(KK),PHT(KK),PET(KK)
       1PMHT(KK),PMT(KK),W1(KK),W2(KK),W2(KK),W3(KK),W4(KK),W5(KK)
       2W6(KK),W7(KK),W8(KK)
        DIMENSIØN TA(KK,KK),RA(KK,KK),TF(KK,KK),RF(KK,KK)
       1SIP(KK,SIM(KK),CØ1(KK),GA1(KK),GA2(KK),G(KK),
       2PHW(KK,KK),ZD(KK,KK),ZP(KK,KK),ZM(KK,KK),AM(KK,KK)
       3DM(KK,KK),GM(KK,KK),RP(KK,KK),RM(KK,KK),PM(KK,KK),
       4XP(KK,KK),A1(KK,KK),A2(KK,KK),A3(KK,KK),L(KK),M(KK),
       5AC(KK),SPØN(KK,KK),ZZ(KK,KK),GAA(KK),GAB(KK),GAC(KK),GAD(KK)
        DIMENSIØN TXY(KK,KK),TYX(KK,KK),RXY(KK,KK),RYX(KK,KK),
       1TYZ(KK,KK),TZY(KK,KK),RYZ(KK,KK),RZY(KK,KK1,
       2TXZ(KK,KK),TZX(KK,KK),RXZ(KK,KK),RZX(KK,KK),
       3SMXY(KK),SPYX(KK),SMYZ(KK),SPZY(KK),SMXZ(KK),SPZX(KK)
        IF(I4-1) 25,26,25
     25 ZA=BA-((ITX-1)*(BA-1.))/NLYR
        ZB=BA-(ITX*(BA-1.))/NLYR
C       ZA AND ZB ARE THE OUTER AND INNER RADII OF THE SHELL ITX
C       IN TERMS OF A, THE INNER RADIUS OF THE ATMOSPHERE
        TCRIT = 2.*CØ(1)
        RCW1 = RCW/(1.-(ITX-1)*RCW)
        TCF = TØD(ITX)
        TAU = TCF
        NSDB = 0
   7799 IF(TAU-TCRIT)7801,7801,7800
   7800 NSDB=NSDB+1
        TAU=TAU/2
        GØ TØ 7799
   7801 RW=RCW1/(1.+(2**NSDB)*(1.-RCW1))
C       IF THE SHELL IS TOO THICK IT IS SUBDIVIDED INTO SUBSHELLS AND
C       RW AND TAU ARE THE CURVATURE FACTOR AND THE OPTICAL DEPTH OF
C       THIS SUBSHELL.  NSDB IS THE NUMBER OF THESE SUBSHELLS.  SEE
C       RELATIONS (75),(76),(77),(78).  WE HAVE TO REPEAT THE STAR
C       ALGORITHM NSDB TIMES.
        SPI=SQRT(PI)
        PID=1./(SPI*DELTA)
C       NOW WE ARE GOING TO CALCULATE THE PROFILE FUNCTIONS USING THE
C       REDISTRIBUTION FUNCTIONS - SEE EQUATIONS (3) AND (4) OF SECTION
C       2.  FOLLOWING THIS CALCULATION THE NORMALIZATION IS DONE.
        DØ 700 I=1,II
        DØ 700 J=1,NC
        K=J+(I-1)*NC
        AC(K)=A(I)*C(J)
    700 CØNTINUE
        DØ 701 J =1,KK
        PHI(J)=0.
        PEP(J)=0.
        PMHN(J)=0.
        PMM(J)=0.
        DØ 702 K=1,KK
        PHI(J)=PHI(J)+FRD1(J,K)*AC(K)
        PEP(J)=PEP(J)+FRD2(J,K)*AC(K)
        PMHN(J)=PMHN(J)+FRD3(J,K)*AC(K
        PMM(J)=PMM(J+FRD4(J,K)*AC(K)
    702 CØNTINUE
    701 CØNTINUE
        DØ 709 K=1,KK
        PHT(K)=0.
        PET(K)=0.
        PMHT(K)=0.
        PMT(K)=0.
        DØ 713 J=1,KK
        PHT(K)=PHT(K)+FRD1(J,K)*AC(J)
        PET(K)=PET(K)+FRD2(J,K)*AC(J)
        PMHT(K)=PMHT(K)+FRD3(J,K)*AC(J`
        PMT(K)=PMT(K)+FRD4(J,K)*AC(J)
    713 CØNTINUE
    709 CØNTINUE
        SUM1=0.
        SUM2=0.
        SUM3=0.
        SUM4=0.
        SUM5=0.
        SUM6=0.
        SUM7=0.
        SUM8=0.
        DØ 99 I=1,II
        DØ 99 J=1,NC
        K=J+(I-1)*NC
        SUM1=SUM1+A(I)*PHI(K)*C(J)
        SUM2=SUM2+A(I)*PEP(K)*C(J)
        SUM3=SUM3+A(I)*PMHN(K)*C(J)
        SUM4=SUM4+A(I)*PMM(K)*C(J)
        SUM5=SUM5+A(I)*PHT(K)*C(J)
        SUM6=SUM6+A(I)*PET(K)*C(J)
        SUM7=SUM7+A(I)*PMHT(K)*C(J)
        SUM8=SUM8+A(I)*PMT(K)*C(J)
     99 CØNTINUE
        DØ 1936 K=1,KK
        W1(K)=AC(K)/SUM1
        W2(K)=AC(K)/SUM2
        W3(K)=AC(K)/SUM3
        W4(K)=AC(K)/SUM4
```

```
      W5(L)=AC(K)/SUM5
      W6(K)=AC(K)/SUM6
      W7(K)=AC(L)/SUM7
      W8(L)=AC(K)/SUM8
 1936 CØNTINUE
      IF(IQ-3)26,27,27
   26 RETURN
   27 AP1=.5-1./(2.**NSDB)
      AP2=AP1**2+.5*AP1+.25
      AP3=AP1+.5
      AP4=1.-RW*AP3+(RW**2*AP2)/3.
C     SEE EQUATION (79) OF SECTION 6.
      DØ 19 I=1,II
      DØ 11 J=1,NC
      CØ1(K)=CØ(J)
      GA1(K)=(RHØ*BETA+PHI(K)*EPS)*BP*4.*PI*ZA*ZA
      GA2(K)=(RHØ*BETA+PMHN(K)*EPS)*BP*4.*PI*ZA*ZA
   11 CØNTINUE
  601 DØ 116 K=1,KL
      DØ 116 J=1,KK
      PHW(J,L)=FRD4(J,L)*W8(K)*(1.-EPS)
      SPØN(J,K)=FRD1(J,K)*W5(K)*(1.-EPS)
  116 CØNTINUE
  609 DØ 117 J=1,KL
      DØ 117 K=1,KK
      ZD(J,K)=-SPØN(J,K)
      ZZ(J,K)=-PHW(J,K)
  117 CØNTINUE
      DØ 118 J=1,KK
      ZD(J,J)=BETA+PHI(J)+ZD(J,J)
      ZZ(J,J)=BETA+PMHN(J)+ZZ(J,J)
  118 CØNTINUE
  613 DØ 119 J=1,KK
      DØ 119 K=1,KK
      ZP(J,K)=.5*TAU*ZD(J,K)-(RW/2.)*CP(J,K)
      ZM(J,K)=.5*TAU*ZZ(J,L)+(RW/2.)*CP(J,K)
      AM(J,L)=-ZP(J,K)
      DM(J,K)=-ZM(J,K)
  119 CØNTINUE
      DØ 121 J=1,KK
      ZP(J,J)=CØ1(J)+ZP(J,J)
      ZM(J,J)=CØ1(J)+ZM(J,J)
      AM(J,J)=CØ1(J)+AM(J,J)
      DM(J,J)=CØ1(J)+DM(J,J)
  121 CØNTINUE
  617 DA=1
      CALL MIN(ZP,LK,DA,L,M)
      CALL MIN(ZM,KK,DA,L,M)
  621 PBZ=(RW/2.)*CRM
      DØ 6933 L=1,KL
      DØ 6933 J=1,LK
      PHW(J,K)=(1.-EPS)*FRD2(J,L)*W6(K)
      GM(J,L)=(1.EPS)*FRD3(J,L)*W7(K)
 6933 CØNTINUE
      DØ 123 J=1,LL
      DØ 123 K=1,KL
      PHW(J,K)=.5*TAU*PHW(J,K)
      GM(J,K)=.5*TAU*GM(J,L)
  123 CØNTINUE
      DØ 369 I=1,II
      K1=1+(I-1)*NC
      GM(K1,K1)=GM(K1,K1)-PBZ
      PHW(K1,K1)=PHW(K1,K1)+PBZ
  369 CØNTINUE
  624 CALL MATMUL(ZP,PHW,RP,LK,LL,KL)
      CALL MATMUL(ZM,GM,RM,KL,KK,LK)
      CALL MATMUL(RP,RM,PM,KK,LK,KK)
      CALL MATMUL(RM,RP,XP,KK,KK,KK)
  973 DØ 126 J=1,KK
      DØ 126 K=1,KK
      PM(J,K)=-PM(J,K)
      XP(J,K)=-XP(J,K)
  126 CØNTINUE
      DØ 127 J=1,KK
      PM(J,J)=1.+PM(J,J)
      XP(J,J)=1.+XP(J,J)
  127 CØNTINUE
      CALL MIN(PM,KK,DA,L,M)
      CALL MIN(XP,KK,DA,L,M)
      CALL MATMUL(ZP,AM,A1,KK,KK,KK)
      CALL MATMUL(RP,A2,A3,KK,KK,KK)
      DØ 128 J=1,KK
      DØ 128 K=1,KK
      A3(J,K)=A3(J,K)+A1(J,K)
  128 CØNTINUE
      CALL MATMUL(PM,A3,TA,KK,KK,KK)
      CALL MATMUL(RM,A1,A3,KK,KK,KA)
      DØ 129 J=1,KK
      DØ 129 K=1,KK
      A3(J,K)=A3(J,K)+A2(J,K)
  127 CØNTINUE
      CALL MATMUL(XP,A3,RA,KK,KK,KL)
      CALL MATMUL(ZM,DM,A1,KK,KK,LK)
      CALL MATMUL(ZP,PHW,A2,KK,KK,KK)
      CALL MATMUL(RM,A2,A3,KK,KK,KK)
```

```
      DØ 130 J=1,KK
      DØ 130 K=1,KK
      A3(J,K)=A3(J,K)+A1(J,K)
  130 CØNTINUE
      CALL MATMUL(XP,A3,TF,KK,KK,KK)
      CALL MATMUL(RP,A1,A3,KK,KK,KK)
      DØ 131 J=1,KK
      DØ 131 K=1,KK
      A3(J,K)=A3(J,K)+A2(J,K)
  131 CØNTINUE
      KF=1
      CALL MATMUL(PM,A3,RF,KK,KF,KK)
      CALL MATMUL(RP,ZM,A1,KK,KK,KK)
      CALL MATMUL(RM,ZP,A2,KK,KK,KK)
      CALL MATMUL(ZP,GA1,GAA,KK,KK,KF)
      CALL MATMUL(ZM,GA2,GAB,KK,KK,KF)
      CALL MATMUL(A1,GA2,GAC,KK,KK,KF)
      CALL MATMUL(A2,GA1,GAD,KK,KK,KF)
      DØ 132 J=1,KK
      GAA(J)=GAA(J)+GAC(J)
      GAB(J)=GAB(J)+GAD(J)
  132 CØNTINUE
      CALL MATMUL(PM,GAA,SIP,KK,KK,KF)
      CALL MATMUL(XP,GAB,SIM,KK,KK,KF)
      DØ 133 J=1,KK
      SIP(J)=TAU*SIP(J)
      SIM(J)=TAU*SIM(J)
  133 CØNTINUE
C     NOW WE SHALL EMPLOY STAR ALGORITHM REPEATEDLY TO GET THE R & T
C     OPERATORS FOR THE COMPOUND CELL OR SHELL.
      IF(MSDB)630,630,633
  633 DØ 160 J=1,KK
      SMXY(J)=SIM(J)
      SPYX(J)=SIP(J)
      SMYZ(J)=SIM(J)
      SPZY(J)=SIP(J)
      DØ 160 K=1,KK
      TXY(J,K)=TF(J,K)
      TYX(J,K)=TA(J,K)
      RXY(J,K)=RF(J,K)
      RYX(J,K)=RA(J,K)
      TYZ(J,K)=TF(J,K)
      TZY(J,K)=TA(J,K)
      RYZ(J,K)=RF(J,K)
      RZY(J,K)=RA(J,K)
  160 CØNTINUE
C     HERE WE ARE ASSUMING THAT THE TWO SUBSHELLS WHICH ARE GOING TO H
C     STAR-ADDED HAVE THE SAME PROPERTIES.
      DØ 162 IXP=1,MSDB
      CALL STAR(TXY,TYX,RXY,RYX,TYZ,TZY,RYZ,RZY,TXZ,
     1TZX,RXZ,RZX,SMXY,SPYX,SMYZ,SPZY,SMXZ,SPZX)
      DØ 161 J=1,KK
      SMXY(J)=SMXZ(J)
      SPYX(J)=SPZX(J)
      SMYZ(J)=SMXZ(J)
      SPZY(J)=SPZX(J)
      DØ 161 K=1,KK
      TXY(J,K)=TXZ(J,K)
      TYX(J,K)=TZX(J,K)
      RXY(J,K)=RXZ(J,K)
      RYX(J,K)=RZX(J,K)
      TYZ(J,K)=TXZ(J,K)
      TZY(J,K)=TZX(J,K)
      RYZ(J,K)=RXZ(J,K)
      RZY(J,K)=RZX(J,K)
  161 CØNTINUE
  162 CØNTINUE
      DØ 163 J=1,KK
      SIP(J)=SPZX(J,K)
      SIM(J)=SMXZ(J,K)
      DØ 163 K=1,KK
      TF(J,K)=TXZ(J,K)
      TA(J,K)=TZX(J,K)
      RF(J,K)=RXZ(J,K)
      RA(J,K)=RZX(J,K)
  163 CØNTINUE
  630 RETURN
      END

      SUBRØUTINE MATMUL(AX,BX,CX,L1,M1,N1)
C     THIS ROUTINE CALCULATES THE PRODUCT OF TWO SQUARE MATRICES OF
C     THE SAME DIMENSION OR THE PRODUCT OF A MATRIX AND A VECTOR WHOSE
C     COLUMN VECTOR HAS THE SAME DIMENSION AS THE ROW DIMENSION OF THE
C     MATRIX
      DIMENSIØN AX(KK,KK),BX(KK,KK),CX(KK,KK)
      DØ 1 J=1,N1
      DØ 1 I=1,L1
      CX(I,J)=0.
      DØ 1 K=1,M1
    1 CX(I,J)=CX(I,J)+AX(I,K)*BX(K,J)
      RETURN
      END
```

```
      SUBRØUTINE STAR(TXY,TYX,RXY,RYX,TYZ,TZY,
     1RYZ,RZY,TXZ,TZX,TXZ,RXZ,RZX,SMXY,SPYX,SMYZ,SPZY,SMXZ,SPZX)
C     THIS CALCULATES THE STAR PRODUCT.  SEE SECTION 4.
      DIMENSIØN TXY(KK,KK),TYX(KK,KK),RXY(KK,KK),RYX(KK,KK),
     1TYZ(KK,KK),TZY(KK,KK),RYZ(KK,KK),RZY(KK,KK),TXZ(KK,KK),
     2TZX(KK,KK),RXZ(KK,KK),RZX(KK,KK),SMXY(KK),SPYX(KK),
     3SMYZ(KK),SPZY(KK),SMXZ(KK),SPZX(KK)
      DIMENSIØN XYZ(KK,KK),ZYX(KK,KK),QA(KK,KK),QB(KK,KK),QC(KK,KK),
     1QD(KK,KK),QE(KK,KK),QF(KK,KK),QK(KK,KK),QL(KK,KK),QG(KK),QH(KK).
     2QI(KK),QJ(KK),QM(KK),QN(KK),L(KK),M(KK)
      N=KK
      D=1.
      CALL MATMUL(RXY,RZY,XYZ,N,N,N)
      CALL MATMUL(RZY,RXY,ZYX,N,N,N)
      DØ J=1,N
      DØ K=1,N
      XYZ(J,K)=-XYZ(J,K)
      ZYX(J,K)=-ZYX(J,K)
    1 CØNTINUE
      DØ 2 J=.,N
      XYZ(J,J)=1.+XYZ(J,J)
      ZYX(J,J)=1.+ZYX(J,J)
    2 CØNTINUE
      CALL MIN(XYZ,N,D,L,M)
      CALL MIN(ZYX,N,D,L,M)
      CALL MATMUL(XYZ,TYX,QA,N,N,N)
      CALL MATMUL(ZYX,TYZ,QB,N,N,N)
      CALL MATMUL(TZY,QA,TZX,N,N,N)
      CALL MATMUL(TXY,QB,TXZ,N,N,N)
      CALL MATMUL(RZY,QA,QC,N,N,N)
      CALL MATMUL(TXY,QC,QD,N,N,N)
      CALL MATMUL(RXY,QB,QE,N,N,N)
      CALL MATMUL(TZY,QE,QF,N,N,N)
      DØ 3 J=1,N
      DØ 3 K=1,N
      RZX(J,K)=RYX(J,K)+QD(J,K)
      RXZ(J,K)=RYZ(J,K)+QF(J,K)
    3 CØNTINUE
      KF=1
      CALL MATMUL(RXY,SMYZ,QG,N,N,KF)
      CALL MATMUL(RZY,SPYX,QH,N,N,KF)
      DØ 4 J=1,N
      QI(J)=SPYX(J)+QG(J)
      QJ(J)=SMYZ(J)+QH(J)
    4 CØNTINUE
      CALL MATMUL(TZY,XYZ,QK,N,N,N)
      CALL MATMUL(TXY,ZYX,QL,N,N,N)
      CALL MATMUL(QK,QI,QM,N,N,KF)
      CALL MATMUL(QL,QJ,QN,N,N,KF)
      DØ 6 J=1,N
      SPZX(J)=SPZY(J)+QM(J)
      SMXZ(J)=SMXY(J)+QN(J)
    6 CØNTINUE
      RETURN
      END

      SUBRØUTINE DIST (IC1,IC2)
C     WE GIVE THE ROUTINE TO CALCULATE RI-A FOR ISOTROPIC SCATTERING.
C     SAME COMMON STATEMENTS AS GIVEN IN MAIN ROUTINE SHOULD BE PUT
C     HERE.
      DIMENSIØN GF6(200),XXP(200),SM(200),CØ1(NC),CØ2(NC)
      M=II*NC
      DØ 14 I=1,II
      DØ 14 IK=1,II
      DØ 13 J=1,NC
      DØ 13 K=1,NC
      CØ1(J)=IC1*CØ(J)
      CO2(J)=IC2*CØ(K)
      XXQ=X(I)+VR*CØ1(J)
      XXP=X(IK)+VR*CØ2(J)
      XBAR=MAX1(ABS(XXP),ABS(XXQ))
      IEE=1
    1 ADXB=XBAR+(IEE-1)*ADEX
      IF(ADXB-10.)3,3,2
    2 ERF=1.E-45
      GØ TØ 4
    3 ERF=EXP(-(ADXB**2))
    4 GF6(IEE)=ERF
      IEE=IEE+1
      IF(ADXB-11.)1,1,10
   10 IGF=IEE-1
      IGA=IGF-1
      IGB=IGF-2
      SØM1=0.
      DØ 11 IE=2,IGA,2
      SØM1=SØM1+GF6(IE)
   11 CØNTINUE
      SØM2=0.
      DØ 12 IE=3,IGB,2
      SØM2=SØM2+GF6(IE)
```

```
   12 CØNTINUE
      GF7=(ADEX/3.)*(GF6(1)+GF6(IGF)+4.*SØM1+2.*SØM2)
      GF7=GF7/SQRT(PI)
      JI=J+(I-1)*NC
      JK=K+(IK-1)*NC
      FRD(JI,JK)=GF7
   13 CØNTINUE
   14 CØNTINUE
      RETURN
      END

      SUBRØUTINE MIN(A,N,DA,L,M)
C     THIS ROUTINE CALCULATES THE INVERSE OF MATRIX A WITH N BY N
C     DIMENSION, AND IT IS REPLACED BY ITS INVERSE.  THIS ROUTINE HAS
C     BEEN TAKEN FROM IBM SSP MANUAL PAGE 131.  FOR THE SAKE OF
C     COMPLETENESS WE SHALL GIVE THE ROUTINE HERE.

      DIMENSIØN A(1),L(1),M(1)

      DA=1.
      NK=-N
      DØ 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DØ 20 J=K,N
      IZ=N*(J-1)
      DØ 20 I=K,N
      IJ=IZ+I
   10 IF(ABS(BIGA)-ABS(A(IJ)))15,20,20
   15 BIGA=A(IJ)
      L(K)=I
      M(K)=J
   20 CØNTINUE
      J=L(K)
      IF(J-K)35,35,25
   25 KI=K-N
      DØ 30 I=1,N
      KI=KI+N
      HØLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
   30 A(JI)=HØLD
   35 I=M(K)
      IF(I-K)45,45,38
   38 JP=N*(I-1)
      DØ 40 J=1,N
      JK=NK+J
      JI=JP+J
      HØLD=-A(JK)
      A(JK)=A(JI)
   40 A(JI)=HØLD
   45 IF(BIGA)48,46,48
   46 DA=0.
      RETURN
   48 DØ 55 I=1,N
      IF(I-K)50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CØNTINUE
      DØ 65 I=1,N
      IK=NK+I
      HØLD=A(IK)
      IJ=I-N
      DØ 65 J=1,N
      IJ=IJ+N
      IF(I-K)60,65,60
   60 IF(J-K)62,65,62
   62 KJ=IJ-I+K
      A(IJ)=HØLD*A(KJ)+A(IJ)
   65 CØNTINUE
      KJ=K-N
      DØ 75 J=1,N
      KJ=KJ+N
      IF(J-K)70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CØNTINUE
      DA=DA*BIGA
      A(KK)=1./BIGA
   80 CØNTINUE
      K=N
  100 K=K-1
      IF(K)150,150,105
  105 I=L(K)
      IF(I-K)120,120,108
  108 JQ=N*(K-1)
      JR=N*(I-1)
      DØ 110 J=1,N
      JK=JQ+J
      HØLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
  110 A(JI)=HØLD
  120 J=M(K)
      IF(J-K)100,100,125
  125 KI=K-N
      DØ 130 I=1,N
      KI=KI+N
      HØLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
```