Bull. Astr. Soc. India (1988) 16, 145–158

Propagation of magnetohydrodynamic waves in the solar atmosphere. Alfven p-waves in sunspots

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Received 1988 June 1; accepted 1988 August 12

Abstract. Numerical calculations are given for the propagation of Alfvèn *p*-waves in the solar atmosphere for Geronicolas' sunspot model. It is shown that Alfvèn *p*-waves with a period > 400s do not, in fact, penetrate into upper layers of a sunspot. This is due to the reflection of Alfvén *p*-waves from the upper layers and to coupling with magneto-gravity waves trapped at a particular layer of a sunspot. Proper resonance periods for magneto-gravity waves in a sunspot are calculated. Attenuation of eigenmodes is the weakest for periods near 300s. This indicates that the low-frequency part of the spectrum of umbral oscillations is not a passive response to the excitation of the sunspot by the 5min solar oscillations. Rather, it is caused by resonance properties in the sunpot.

Key words : oscillations—sunspots—MHD waves

1. Introduction

Although the problem of propagation of linear magnetohydrodynamic waves in an inhomogeneous medium is of great interest in solar physics, it has not been investigated in sufficient detail. The propagation of MHD waves in an isothermal atmosphere with a homogeneous magnetic field has been investigated only in the case where Alfvén waves decouple from magneto-acoustic-gravity waves (Ferraro & Plumpton 1958; Brengauz 1970; Zugzda & Dzhalilov 1983; Leroy & Schwartz 1982; Schwartz & Bel 1984). The main difficulty in studying the propagation of MHD waves in an inhomogeneous medium is the following. When the WKB approximation is inapplicable (the case of special interest) the problem of propagation of MHD waves is reduced, in the general case, to a solution of an ordinary sixth order differential equation with variable coefficients. Analytical solutions of this equation can be derived only for certain simple models, e.g. the above mentioned isothermal atmosphere with a homogeneous magnetic field.

Numerical methods have to be used for other cases, and as a rule this involves significant difficulties (Hazel 1967; Scheuer & Thomas 1981). Moreover, when using numerical methods it is often difficult to understand the physical meaning of the processes taking place. However, despite these shortcomings the numerical methods are the only ones which enable us to investigate MHD waves in an inhomogeneous medium.

At present, the most popular method is that in which the stratified atmosphere is approximated by a large number of sufficiently thin homogeneous layers (Brekhovskikh 1973; Gossard & Hooke 1975). However, this method has certain disadvantages. For instance, this method fails at such an interesting and important point as resonant absorption (Ionson 1978; Rae & Roberts 1982; Zhukov 1986).

In the present paper we demonstrate the capabilities of an algorithm for the calculation of wave propagation in a one-dimensional inhomogeneous medium (Hjalmarson 1967). This is done using computations of wave motions in a simplified umbra model as an example. The model was used by Geronicolas (1977). We believe that this algorithm, when combined with the method of double passes (Betchov & Criminale 1967), enables one to solve any problem involving propagation and linear transformation of waves in a one-dimensional inhomogeneous medium.

In section 3 we give the results of numerical calculations of the coefficient of Alfvén wave transmission. These results show that Alfvén waves with periods $>$ 400s do not penetrate into the upper layers of a sunspot's atmosphere. This is due to an intensive coupling of Alfvén *p*-modes with magneto-gravity waves trapped at a given layer of a sunspot's umbra. It is also due to the reflection of the waves from this layer with a density gradient.

In section 4 the results of numerical calculations of eigenfrequencies of a magneto-gravity wave resonator in a sunspot's umbra are given. This resonator has a very low quality due to intensive emission of Alfvén *p*-waves. Attenuation due to emission of Alfvén waves is the weakest for eigenmodes with periods $\sim$ 300s. This suggests that the observed low-frequency part of the umbral oscillation spectrum (Lites & Thomas 1985) consists of eigenmodes of umbral oscillations and is not a passive response to the excitation of the sunspot by the 5min oscillation in the quiet photosphere.

2. Basic equations : A sunspot model

The subject of this paper is the calculation of Alfvén *p*-wave propagation in a sunspot without the simplifying assumptions used by Zhukov (1985). However, although the algorithm for calculating wave propagation in an inhomogeneous medium (see appendix) enables us to study the propagation of MHD waves in a sunspot's atmosphere using a minimum number of simplifying assumptions, we restrict ourselves here to the calculation of wave motions in the approximation of an incompressible medium. This is due to the fact that, on the one hand, this approximation is satisfactory for analysis of low frequency wave motions in a sunspot umbra, which is the problem of primary interest to us [High-frequency 3 min oscillations have been well studied (Scheuer & Thomas 1981; Thomas & Scheuer 1982; Zugzda, Locans & Staude 1983; Zugzda, Staude & Locans 1984; Thomas 1985; Zhukov, Efremov & Nuraliev 1987)]. On the other hand, it somewhat simplifies the problem and enables one to see the difficulties that arise in using this algorithm for numerical computations. This is particularly important because although this algorithm (see appendix) was elaborated about two decades ago, it has not so far been used for a full calculation of wave propagation.

It has been shown (Zhukov 1985) that the main equation describing wave propagation in an incompressible, inhomogeneous, ideally conducting medium with a uniform vertical magnetic field H_0 in a rectangular coordinate system whose z axis is vertical has the form

$$\Omega^2 \frac{d^4 V_z}{d\xi^4} + \Lambda^2 \frac{d}{d\xi}\left[(f(\xi) - \Omega^2) \frac{dV_z}{d\xi} \right] - \Lambda^4 \left(f(\xi) + G \frac{df(\xi)}{d\xi} \right) V_z = 0, \quad \ldots(1)$$

where

$$\xi = \frac{z}{L}, \quad \Lambda = K_x \cdot L, \quad \Omega^2 = \frac{V_A^2 \cdot K_x^2}{\omega^2}, \quad G = \frac{g}{\omega^2 L}$$

$$V_A^2 = \frac{H_0^2}{4\pi\rho}, \quad \rho_0 = \rho \cdot f(\xi), \quad \rho = \text{const.}$$

Here L is a density scale height and V_z is a vertical velocity component, harmonically depending on x and t ($V_z \sim \exp\,[i(k_x x + \omega t)]$).

In numerical calculations we use the same sunspot density profile as in Geronicolas (1977), i.e.,

$$f(\xi) = \frac{2}{3}\left(1 - \tfrac{1}{2}\, th\left(\frac{z}{L}\right)\right), \quad \ldots(2)$$

where $L = 250$ km ($\rho = 1.5 \times 10^{-7}$ g.cm^{-3}, $H_0 = 2 \times 10^3$G).

Thomas (1978) noted that such a density profile does not approximate very well the variations of density with height in a real sunspot. However, in our case an introduction into the model of a transition layer that strongly reflects Alfvén waves is not of primary importance, because we are mainly interested in the trapped magneto-gravity waves, or, strictly speaking, in the possibility of excitation of magneto-gravity waves by sunspot-generated Alfvén p-waves and in the possibility of capturing magneto-gravity waves in a sunspot's atmosphere.

Equation (1) describes the propagation of waves of two types : Alfvén p-waves and magneto-gravity waves (Zhukov 1985). It is straightforward to find asymptotic solutions of equation (1) at $z \to \pm\infty$ for the density profiles given by equation (2). They are

$$V_z \sim \exp\left(\pm i \frac{\Lambda}{\Omega}\sqrt{f}\, \xi \right), \quad \ldots(3)$$

$$V_z \sim \exp(\pm \Lambda\xi), \qquad \ldots(4)$$

correponding to Alfvén p-modes and magneto-gravity waves, respectively. It follows from equation (4) that magneto-gravity waves are evanescent with $|\xi| \gg 1$, i.e., they are trapped at a particular layer of a sunspot's umbra.

Thus two types of problems are worth considering. If we assume that Alfvén waves are generated somewhere at $z \to -\infty$, it is instructive to calculate the coefficient of transmission of Alfvén p-wave into the upper layers of a sunspot ($z \to +\infty$). It was shown by Zhukov (1985) that Alfvén p-waves should excite magneto-gravity waves, transferring the energy only in the direction perpendicular to the density gradient (because at $|\xi| \gg 1$ they do not propagate) and that is why the computation of the excitation efficiency of magneto-gravity waves is also of great interest (section 3).

Then, if magneto-gravity waves are trapped at a particular sunspot layer, naturally the task is to calculate proper oscillation frequencies for this magneto-gravity wave resonator (section 4).

3. Propagation of Alfvèn p-waves in a sunspot's umbra

In this section we will assume that Alfvén waves are generated at $Z \to -\infty$, i.e., at sufficiently deep layers in the sunspot. Let us analyse their upward propagation.

It is clear that at the lower layers an Alfvén wave should be reflected from the transition layer, and moreover, there should be an exponentially attenuating solution corresponding to a magneto-gravity wave, i.e., as $Z \to -\infty$

$$V_z = \exp\left(-i\frac{\Lambda}{\Omega}\sqrt{f}.\xi\right) + r.\exp\left(i\frac{\Lambda}{\Omega}\sqrt{f}.\xi\right) + B\exp(\Lambda\xi). \qquad \ldots(5)$$

Here the amplitude of an incident wave is assumed to be equal to 1, and r is the amplitude of the reflected Alfvén wave.

For the upper layers ($Z \to +\infty$) the solution should have the form

$$V_z = t.\exp\left(-i\frac{\Lambda}{\Omega}\sqrt{f}.\xi\right) + A\exp(-\Lambda\xi). \qquad \ldots(6)$$

Here t is the amplitude of the transmitted Alfvén p-wave. Equation (1) [equation (1A) in the appendix] should be integrated to determine the coefficients t, r, A, and B. This can be done as follows.

Assume some test values of r and B and integrate equation (1) up to rather large Z's, which allows the solution to be presented as equations (3) and (4). With arbitrary r and B the above solution will have not only the upward Alfvén p-wave and exponentially attenuating solution, but also a downward Alfvèn p-wave ($\sim \exp(i(\Gamma/\Omega)\sqrt{f}.\xi)$) and an exponentially growing term exp ($\sim\Lambda\xi$). Thus it would be necessary to vary r and B until one attains a solution for the upper ayers in the form of equation (6). However, this approach is far from ideal. In the present paper we use, in fact, the method of double passes used in the calculation of instability in plane parallel flows (Betchov & Criminale 1967).

The integration was carried out from top to bottom beginning with a rather large ξ, which allows the solution of equation (1) to be presented with a given accuracy as a sum of equations (3) and (4). At the upper baundary with $\xi = \xi_{up}$ the solution of equation (1) is given by

$$V_z = X_1 + X_3 = \exp\left(- i \frac{\Lambda}{\Omega} \sqrt{f}.\xi \right) + A_0 \exp(- \Lambda\xi). \qquad ...(7)$$

X_i is determined by formulae (10A)-(14A), and as is seen from equation (7) $X_{1,2}$ corresponds to Alfvén p-waves and $X_{3,4}$ to magneto-gravity waves. Equation (1) was integrated twice to determine A_0 in equation (7) : first, under the initial conditions

$$V_z = X_1 = \exp\left(- i \frac{\Lambda}{\Omega} \sqrt{f}.\xi \right),$$

$$V'_z = X'_1, \quad V''_z = X''_1, \quad V'''_z = X'''_1, \qquad ...(8)$$

with $\xi = \xi_{up}$,

and then under the initial conditions

$$V_z = X_3 = \exp(- \Lambda\xi)$$

$$V'_z = X'_3, \quad V''_z = X''_3, \quad V'''_z = X'''_3 \qquad ...(9)$$

with $\xi = \xi_{up}$.

Then A_0 was determined using the formula

$$A_0 = - \frac{\lambda_1\lambda_2\lambda_4 V_1 - (\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4) V'_1 - (\lambda_1 + \lambda_2 + \lambda_4) V''_1 - V'''_1}{\lambda_1\lambda_2\lambda_4 V_2 - (\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4) V'_2 - (\lambda_1 + \lambda_2 + \lambda_4) V''_2 - V'''_2}. \qquad ...(10)$$

Here $\lambda_q = - ik_q|_{\xi_{bot}}$, are eigenvalues of the characteristic matrix [equation (5A)] $V_{1,2}$ are solutions of equation (1) at the point $\xi = \xi_{bot}$ with the initial conditions (8) and (9) respectively, ξ_{bot} is the lower limit of the integration region chosen so that the solution of equation (1) with $\xi \ll \xi_{bot}$ could be represented with a given accuracy by asymptotic formulae (3) and (4), and the prime indicates differentiation with respect to ξ.

After the determination of A_0, equation (1) was integrated with initial conditions

$$V_z = X_1 + X_3 = \exp\left(- i \frac{\Gamma}{\Omega} \sqrt{f}.\xi \right) + A_0 \exp(- \Lambda\xi), \qquad ...(11)$$

$$V'_z = X'_1 + X'_3, \quad V''_z = X''_1 + X''_3, \quad V'''_z = X'''_1 + X'''_3$$

with $\xi = \xi_{up}$.

At lower layers where $\xi \ll \xi_{bot}$ the solution has the form

$$V_z = a. \exp\left(- i \frac{\Lambda}{\Omega} \sqrt{f}.\xi \right) + b. \exp\left(i \frac{\Lambda}{\Omega} \sqrt{f}.\xi \right) + c. \exp(\Lambda\xi) \qquad .. (12)$$

Since this is a linear problem, we compute t and r from the above equation normalizing all coefficients to a.

Thus it turns out that the direct method has limited capabilities and can be used only when the exponential growth of the solution [terms exp $(\pm \Lambda\xi)$] within the integration region is not very large.

Figure 1 shows the results of calculating the coefficient of Alfven p-wave transmission with respect to frequency. The transmission coefficient is given by

$$T = \frac{f(\xi_{\rm up}) + \Omega^2 \, (\rho_0 V_A \mid V_Z \mid ^2) \, \xi_{\rm up}}{f(\xi_{\rm bot}) + \Omega^2 \, (\rho_0 V_A \mid V_Z \mid ^2) \, \xi_{\rm bot}}. \qquad ...(13)$$

For comparison, figure 2 shows as a function of frequency the transmission coefficient calculated for a model consisting of two homogeneous layers with densities ρ_1 and ρ_2 (Savage 1969).

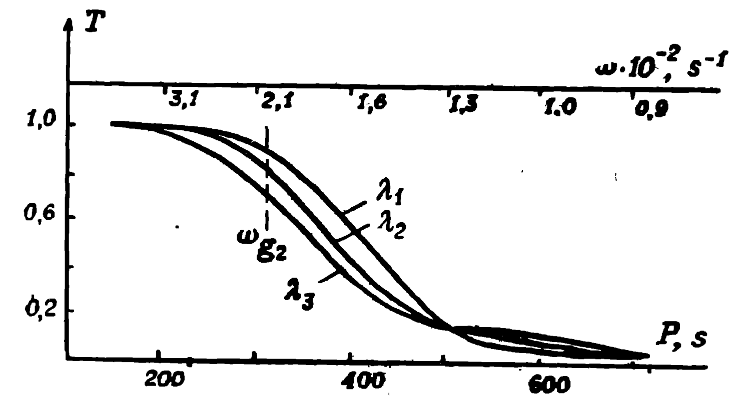

Figure 1. Alfvén p-wave transmission coefficient for three horizontal wavelengths ($\lambda_1 = 1.5 \times 10^8$ cm, $\lambda_2 = 2 \times 10^8$ cm, $\lambda_3 = 2.5 \times 10^8$ cm). The dashed line shows the frequency for λ_2 calculated by formula (14) (for ρ_1 and ρ_2 asymptotic density values were taken with $\xi \rightarrow \pm\infty$).

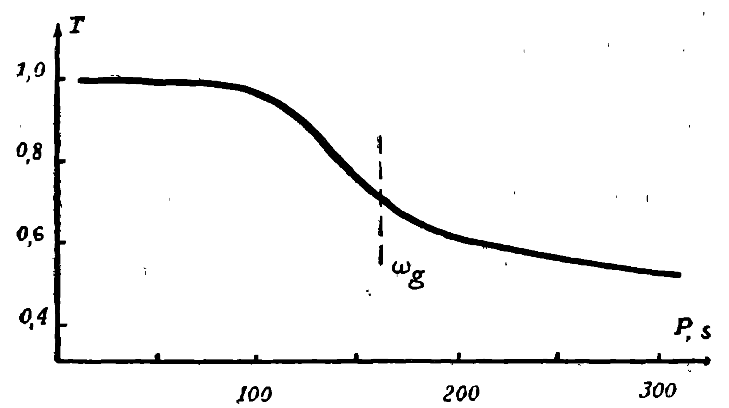

Figure 2. Dependence of the Alfvèn p-wave transmission coefficient on the period for horizontal wavelength $\lambda = 6.28 \times 10^7$ cm, $V_A = 10^6$ cm s^{-2}, $\frac{\rho_2}{\rho_1} = 3$.

Here, as in the model with a density discontinuity, the transmission coefficient is small if the Alfvén wave frequency is smaller than the gravity wave frequency

$$\omega^2 = g \cdot K_x (\rho_2 - \rho_1)/(\rho_2 + \rho_1) \qquad ...(14)$$

i.e., in the case where the coupling between magneto-gravity and Alfvén p-waves becomes significant. In order to show that coupling indeed takes place between magneto-gravity waves and Alfvén p-waves, we calculated the horizontal component F_x of the wave flux as a function of ξ (figure 3). Alfvén p-waves transfer energy only along the lines of force of the magnetic field. It can be seen from figure 3 that coupling between magneto-gravity waves and Alfvén p-waves indeed takes place (Zhukov 1985). Hence magneto-gravity waves excited by Alfvén p-waves are capable of carrying out partial energy from sunspots. Figure 4 shows the ratio of the module of the energy flux through the lateral surfaces of a cylinder with a radius R ($\int_{-\infty}^{+\infty} | F_x | \, d\xi \, 2\pi R$) to that through the cylindrical cross-section $F_z \, \pi R^2$ at $\xi = -1.4$. It is assumed that $F_z = 2.5 \times 10^{10}$ erg cm^{-2} s^{-1} at that level.

When studying the hypothesis that spots are cooled by their strong emission of Alfvén waves, the value ($V_z \cdot V_z^*$) was determined from observation of sunspots at different layers (Beckers 1976; Beckers & Schneeberger 1977). We have calculated the variation of ($V_z \cdot V_z^*$) with height in a sunspot (figures 5 and 6).

The oscillatory structure in figure 5 is explained by the presence of a reflected wave. It can be shown from equation (5) that at sufficiently deep layers ($-\xi \gg 1$)

$$| V_z |^2 \simeq 1 + r^2 + 2r \cos (2\Lambda\xi / \Omega). \qquad ...(15)$$

When $\Omega^2 \ll 1$ one can easily find the asymptotic solutions of equation (1) (Wasow 1965). For Alfvén p-waves they have the form

$$V_z \sim f(\xi)^{-3/4} \exp (\pm i (\Lambda / \Omega)) \int^{\xi} \sqrt{f} \, d\xi). \qquad ...(16)$$

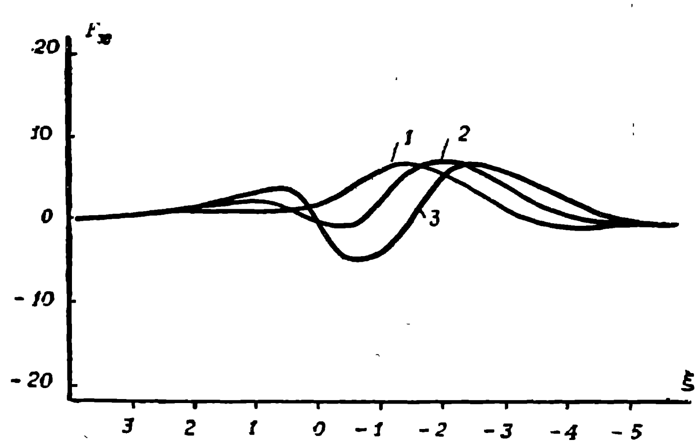

Figure 3. Horizontal component of the wave energy flux F_x (in arbitrary units) versus height for the horizontal wavelength $\lambda = 2 \times 10^8$ cm for the periods 250s, 300s and 350s.

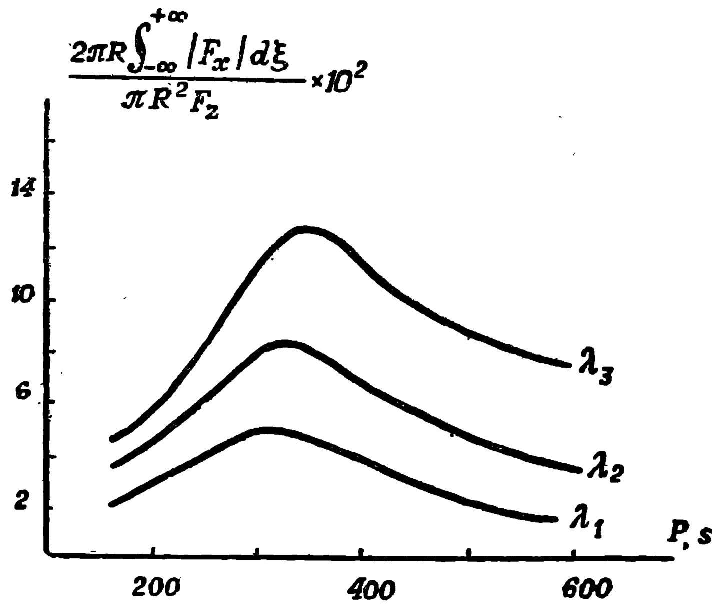

Figure 4. Dependence of wave transmission efficiency on period for horizontal wavelengths $\lambda_1 = 1.5 \times 10^8$ cm, $\lambda_2 = 2 \times 10^8$ cm, $\lambda_3 = 2.5 \times 10^8$ cm and $R = 10^9$ cm.

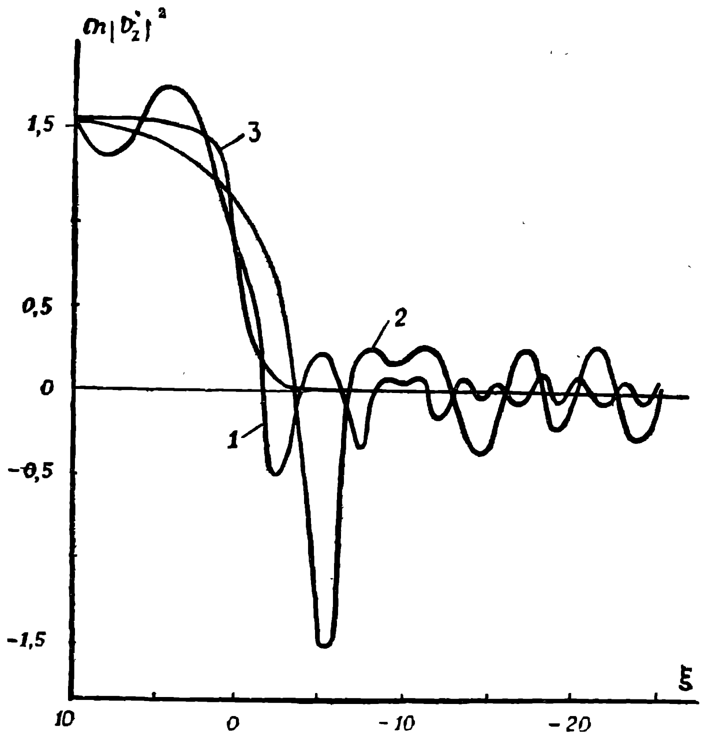

Figure 5. Dependence of the logarithm of the squared vertical velocity component on height for the horizontal wavelength $\lambda = 1 \times 10^9$ cm for the periods 200s and 400s. Curve 3 is calculated by formula (16) for the period 200s.

Magneto-gravity waves are described with a reduced second order equation

$$\frac{d}{d\xi}\left(f(\xi)\, \frac{dV_z}{d\xi}\right) - \Lambda^2 \left(f(\xi) + G\, \frac{df(\xi)}{d\xi}\right) V_z = 0. \qquad ...(17)$$

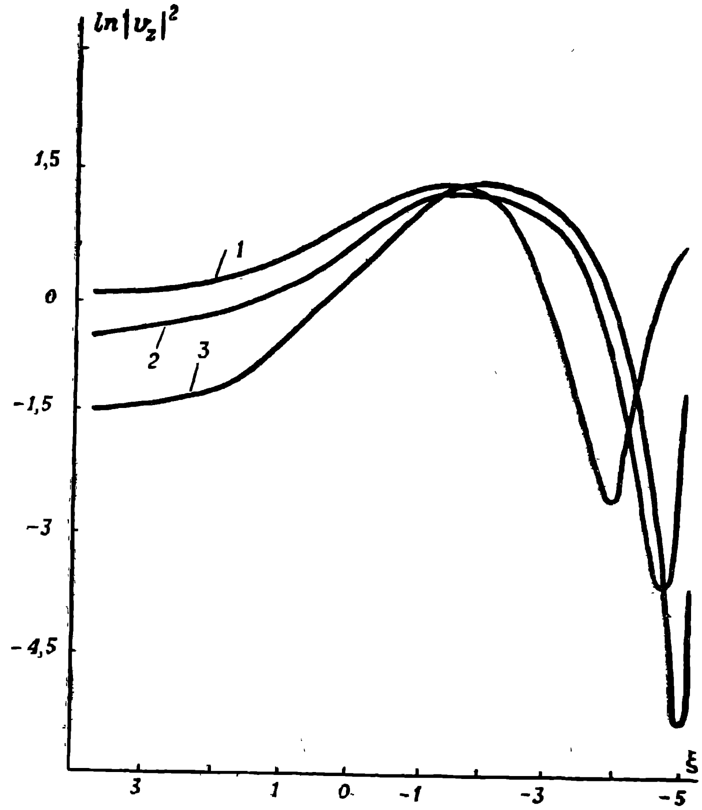

Figure 6. Dependence of the logarithm of the squared vertical velocity component on wavelength for the horizontal wavelength $\lambda = 1.5 \times 10^8$ cm for the periods 350s, 400s, 500s.

Alfvén waves and magneto-gravity waves do not couple in this approximation. Curve 3 in figure 5 shows the relation $|V_z|^2 = [f(\xi)]^{-3/2}$ [equation (16)]. The calculated curve 1 is very similar to curve 3. In this case the reflection coefficient of the Alfvèn p-wave is rather small and the coupling between the magneto-gravity waves and Alfvén p-waves is practically absent.

The calculations (figure 6) show that an essential decrease of $|V_z|^2$ with height takes place at low frequencies, because at these frequencies Alfvèn p-waves are strongly reflected (figure 1). However, this is not sufficient evidence to conclude that sunspots cannot be cooled by waves, since we have seen above that some energy may be transferred by magneto-gravity waves (figure 4).

4. Waveguiding properties of a sunspot's umbra

Calculations of Alfvèn p-wave propagation show that magneto-gravity waves are most effectively excited by Alfvén p-waves at periods close to 300s (figure 4). This indicates that eigenperiods of the oscillations of the magneto-gravity wave resonator in a sunspot's umbra should be close to 300s.

As was noted in section 3, magneto-gravity waves do not, in fact, couple with Alfvèn p-waves at a sufficiently small Ω^2 (for example, at $H_0 \rightarrow 0$). Their propagation can be described by equation (17). For the density profile (15) a rigorous solution of equation (17) can be expressed in terms of a hypergeometric function. This solution describes the propagation of internal gravity waves (Gossard & Hooke, 1975) and has a dispersion relation of the form

$$\omega_{\mathrm{m}}^2 = N^2 \frac{\Lambda^2}{(\Lambda + m - 1)(\Lambda + m)}, \quad N^2 = g \frac{\Delta \rho}{\rho . \Gamma}. \qquad \ldots(18)$$

Figure 7 shows a diagnostic diagram for atmospheric waveguiding eigenmodes for internal gravity waves calculated by equation (18).

When Ω^2 is not small one should use numerical methods to calculate the waveguiding properties of the sunspot umbra for magneto-gravity waves. An important peculiarity should be taken into account that unlike the case $\Omega^2 \ll 1$, at large Ω^2 linear coupling takes place between magneto-gravity waves and Alfvèn p-waves. Hence there is Alfvèn p-wave emission from the region of magneto-gravity wave capture, and this explains why oscillation eigenfrequencies will be complex, i.e. $\omega = \omega_R + i\omega_{\mathrm{I}}$, $\omega_{\mathrm{I}} > 0$.

In order to calculate magneto-gravity wave eigenfrequencies trapped at a given layer of the sunspot umbra, one should solve equation (1) under the following boundary conditions:

$$V_z = a_1 \exp\left(- i \frac{\Lambda}{\Omega} \sqrt{f}.\xi \right) + b_1 \exp(-\Lambda \xi) = X_1 + X_3, \quad \xi = \xi_{\mathrm{up}} \qquad \ldots(19)$$

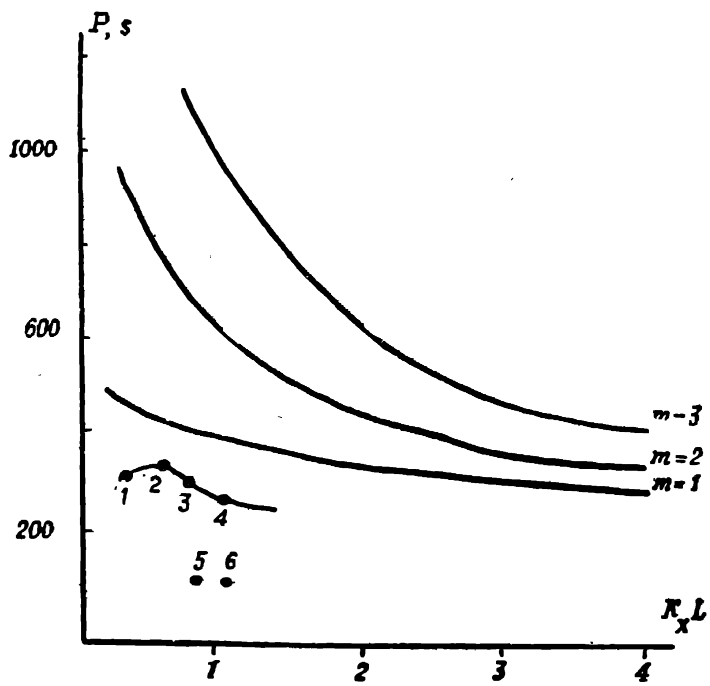

Figure 7. Diagnostic diagram for eigenmodes of the atmospheric waveguide for internal gravity waves ($H_0 = 0$). The points 1–6 are calculated proper periods of the trapped magneto-gravity waves ($H_0 = 2000$G). For them $T_{\mathrm{I}} = \frac{2\pi}{\omega_{\mathrm{I}}} \simeq 100$s.

$$V_z = a_2 \exp\left(i \frac{\Lambda}{\Omega}\sqrt{f}.\xi\right) + b_2 \exp(\Lambda\xi) = X_2 + X_4, \ \xi = \xi_{bot}. \quad ...(20)$$

We used a computer program to calculate the coefficient of Alfvèn p-wave transmission (section 3). In fact, it was necessary to find a value of ω at which the amplitude of an incident wave [first term in equation (5)] becomes zero. We used the linear extrapolation method of Betchov & Criminale (1967) and assumed several initial values of ω and $\delta\omega$, and then calculated $X_1(\omega) \mid \xi_{bot}$ and $X_1(\omega + \delta\omega) \mid \xi_{bot}$. Any small variation of ω, say $\Delta\omega$, would lead to

$$X_1(\omega + \Delta\omega) = X_1(\omega) + \frac{X_1(\omega + \delta\omega) - X_1(\omega)}{\delta\omega} \Delta\omega. \quad ...(21)$$

Since we wanted $X_1(\omega + \Delta\omega)$ to be as close to zero as possible, we chose

$$\Delta\omega = -\lambda \left(\frac{X_1(\omega + \delta\omega)}{X_1(\omega)} - 1\right)^{-1} \delta\omega \quad ...(22)$$

where the parameter λ was considered equal to 1. (This is true if X_1 varies slowly with ω, which is the case here).

The results of calculations are shown by a dotted curve in figure 7. It turns out that the eigenmodes of magneto-gravity waves attenuate drastically due to Alfvèn wave emission. Alfvèn p-wave emit predominantly downwards ($a_2^2/a_1^2 \simeq 2/3$). Attenuation is the weakest for eigenmodes with periods near $\sim$ 300s. Thus it is probable that the low frequency part of the umbral oscillation spectrum is not a passive response to excitation by the 5 min oscillations but is caused by proper oscillations in the sunspot.

5. Concluding remarks

Our calculations have shown that some portion of energy can indeed be transferred through lateral boundaries of the sunspot by magneto-gravity waves. However, for a quantitative estimation of the value of this flux one should know Alfvèn p-wave flux generated in the sunspot. Moreover, one should take into account the umbra-penumbra boundary that may cause a strong reflection of the magneto-gravity waves.

The presence of a magneto-gravity wave resonator into the umbra has been shown and oscillation eigenfrequencies calculated. These are in the low frequency part of the observed umbral oscillation spectrum (Lites & Thomas 1985). But our results should be applied to a real sunspot with certain caution, because we used a simplified model. However, it seems to us that our method has good prospects and may be used for explaining the low frequency umbral oscillations (for whose description one can certainly use the approximation of an incompressible medium) and the excitation mechanism of the running penumbral waves.

In conclusion we should observe that the algorithm by Hjalmarson (1967) supplemented with the method of double passes is rather good for calculating wave propagation in a one-dimensional inhomogeneous medium. The advantages of this algorithm should be clear when more complex calculations of wave propagation accounting for all three restoring forces (compressibility, buoyancy, and magnetic restoring force) are made.

Acknowledgements

We thank Dr R. A. McCutcheon from Computer Science Corporation (USA) for improving the language of the paper during his stay at the Central Astronomical Observatory of the USSR Academy of Sciences.

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Appendix

The propagation of harmonic (exp $(i\omega t)$) MHD waves (in the z-direction), in an inhomogeneous ideally conducting medium with a magnetic field is described in the general case by an ordinary sixth order differential equation with variable coefficients

$$\left[\frac{d^n}{dz^n} + \alpha_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \ldots + \alpha_1(z)\frac{d}{dz} + \alpha_0(z)\right]\pi = 0. \qquad \ldots(1A)$$

Here π is a field variable (e.g. a vertical component of the velocity vector), and $n = 6, 4$ or 2,

Equation (1A) can be written in the matrix form

$$\frac{d\hat{\Pi}_n}{dz} = \hat{\hat{M}}_n \hat{\Pi}_n \qquad \text{...(2A)}$$

where $\hat{\Pi}_n$ is a vector-column with the components π, $\frac{d\pi}{dz}$, ..., $\frac{d\pi^{n-1}}{dz^{n-1}}$ and

$$\hat{\hat{M}}_n = \begin{pmatrix} 0 & 1 & 0 & \ldots\ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots\ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \ldots\ldots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix}. \qquad \text{...(3A)}$$

It is easy to show that the characteristic equation for the matrix $\hat{\hat{M}}_n$

$$\det [\hat{\hat{M}}_n - (-ik_q(z))\, \hat{\hat{I}}_n] = 0, \qquad \text{...(4A)}$$

where $\hat{\hat{I}}_n$ is a unit matrix of the dimension $n \times n$, can be reduced to

$$(-ik_q)^n + \alpha_{n-1}(z)\,(-ik_q)^{n-1} + \ldots + \alpha_1(z)\,(-ik_q) + \alpha_0(z) = 0 \ldots \text{(5A)}$$

For a homogeneous medium the characteristic equation (5A) is a dispersion relation which determines the wave numbers of the system. If all the wave numbers of the system differ from one another, wave equation (1A) has n independent solutions of the exp $(-ik_{q_0} z)$ type and the π wave field is given by the following linear combination of these solutions :

$$\pi(z) = \sum_{q_0=1}^{n} A_{q_0} \exp(-ik_{q_0} z), \qquad \text{...(6A)}$$

where A_q are constant values that are derived under the boundary or initial conditions.

In the case of an inhomogeneous medium it is desirable to represent the solution of equation (1) as a linear combination of some X_q modes

$$\pi(z) = \sum_{q=1}^{n} X_q(z), \qquad \text{...(7A)}$$

which should reduce to equation (6A) in a homogeneous medium.

As shown by Hjalmarson (1967) the necessary conditions are satisfied by

$$\hat{\Pi}_n(z) = \hat{\hat{L}}_n(z)\; \hat{X}_n(z), \qquad \text{...(8A)}$$

where

$$\hat{\hat{L}}_n = \begin{pmatrix} 1 & 1 & \ldots\ldots & 1 \\ -ik_1 & -ik_2 & \ldots\ldots & -ik_n \\ . & . & \ldots\ldots & . \\ (-ik_1)^{n-1} & (-ik_2)^{n-1} & \ldots\ldots & (-ik_n)^{n-1} \end{pmatrix} \qquad \text{...(9A)}$$

For the case $n = 4$, which is considered in sections 3 and 4, it is easy to obtain the following expressions for the vector-column $\hat{X}_n$ (Hjalmarson 1967)

$$X_1 = \frac{1}{(k_2 - k_1)(k_3 - k_1)(k_4 - k_1)} \Big[k_2 k_3 k_4 . \pi - i(k_2 k_3 + k_2 k_4 + k_3 k_4) \times \frac{d\pi}{dz} - (k_2 + k_3 + k_4) \frac{d^2\pi}{dz^2} + i \frac{d^3\pi}{dz^3} \Big], \qquad \text{...(10 A)}$$

$$X_2 = \frac{1}{(k_1 - k_2)(k_3 - k_2)(k_4 - k_2)} \Big[k_1 k_3 k_4 . \pi - i(k_1 k_3 + k_1 k_4 + k_3 k_4) \times \frac{d\pi}{dz} - (k_1 + k_3 + k_4) \frac{d^2\pi}{dz^2} + i \frac{d^3\pi}{dz^3} \Big], \qquad \text{...(11A)}$$

$$X_3 = \frac{1}{(k_1 - k_3)(k_2 - k_3)(k_4 - k_3)} \Big[k_1 k_2 k_4 . \pi - i(k_1 k_2 + k_1 k_4 + k_2 k_4) \times \frac{d\pi}{dz} - (k_1 + k_2 + k_4) \frac{d^2\pi}{dz^2} + i \frac{d^3\pi}{dz^3} \Big], \qquad \text{...(12A)}$$

$$X_4 = \frac{1}{(k_1 - k_4)(k_2 - k_4)(k_3 - k_4)} \Big[k_1 k_2 k_3 \pi - i(k_1 k_2 + k_1 k_3 + k_2 k_3) \times \frac{d\pi}{dz} - (k_1 + k_2 + k_3) \frac{d^2\pi}{dz^2} + i \frac{d^3\pi}{dz^3} \Big]. \qquad \text{...(13A)}$$

If the coefficients of equation (1A) tend to a constant with $z \to \pm \infty$ the $\hat{X}_n$ components reduce to solutions of the $A_{q_0} \exp(- ik_{q_0} z)$ type and hence can be easily interpreted from the physical point of view. In this case each component of the vector-column can be considered as a definite wave mode and the coupling of different wave modes can be analysed (see for details Hjalmarson 1967).