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Transmission and reflection operators of radiative transfer equation with aberration and advection terms. II. Line radiation in spherical symmetry

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Abstract. A formal solution of the equation of radiative transfer is presented with aberration and advection terms corresponding to the lines included. The operators of reflection and transmission in a radially expanding spherically symmetric shell are derived. This solution is valid in a medium moving with velocity v such that $v/c \approx 0.0167$. Complete or partial frequency redistribution of photon frequencies can be incorporated if desired.

Key words : radiative transfer—aberration and advection—line transfer

1. Introduction

Mihalas, Kunasz & Hummer (1976) investigated the effects of aberration and advection terms on line radiation for $(v/c) \sim 0.01$. They did not find substantial differences between the radiation field obtained with aberration and advection terms included in the line transfer equation and that calculated without including these terms. They found that the Doppler shifts cause more changes in the line than aberration and advection. Mihalas & Klein (1982) found that time dependent terms are important when advection and aberration terms are included.

Recently we took up an ab initio investigation to see whether or not the terms of the order of $v/c \approx 0.01$ would create any substantial changes in the solution. The study of the order of magnitude of various terms in the transfer equation was undertaken and it was found that the terms of the order $v/c \sim 0.01$ change the solution significantly (Peraiah 1986). Aberration and advection terms were included in a plane parallel medium and a numerical solution was obtained by employing coherent, and isotropic scattering in the radiation field (Peraiah 1987a). We find substantial changes when the total optical depth τ changes from 1 to 50. The changes in the mean intensities and fluxes are of the order of 90–95% when τ is between 40 and 50. We developed a formal solution for a mono chromatic radiation field in a spherically symmetric medium (Peraiah 1987, Paper I). In this paper, we developed the solution of the line transfer equation in the comoving frame with aberration and advection terms included when the matter is moving with velocities of the order of $v/c \approx 0.1$, in a spherically symmetric medium.

2. Development of the solution

The terms that introduce changes in the equation of transfer in the comoving frame are

$$\left\{\frac{v'}{r}(1-\mu_0^2)+\mu_0^2\frac{dv'}{dr}\right\}\frac{\partial U(r,\mu_0,x)}{\partial x}. \quad ...(1)$$

Here $v' = v/v_{th}$ is the velocity of the gas in terms of the mean thermal Doppler velocity v_{th}; μ_0 is given by

$$\mu_0 = \frac{\mu+\beta}{1-\mu\beta}, \quad ...(2)$$

where μ is the cosine of the angle made by the ray with the radius vector

$$\beta = v'/c; \quad ...(3)$$

$$U(r,\mu_0,x) = 4\pi r^2 I(r,\mu_0,x), \quad ...(4)$$

$I(r,\mu_0,x)$ being the specific intensity of the ray making an angle $\cos^{-1}\mu$ with the radius vector. The quantity x is defined as

$$x = (f-f_0)/\Delta_D, \quad ...(5)$$

where f_0 and f are the frequencies at the line centre and at any other point in the line. Δ_D is the standard Doppler width.

The equation of radiative transfer in the comoving frame including aberration and advection terms in spherical symmetry at any point x in the line is (Castor 1972; Mihalas 1978, Munier & Weaver 1986)

$$\begin{aligned}(\mu_0+\beta)\frac{\partial U(r,\mu_0,x)}{\partial r}&+\frac{1-\mu_0^2}{r}\left\{1+\mu_0\beta\left(1-\frac{r}{\beta}\frac{d\beta}{dr}\right)\right\}\frac{\partial U(r,\mu_0,x)}{\partial\mu_0}\\ &=\left\{\frac{v'}{r}(1-\mu_0^2)+\mu_0^2\frac{dv'}{dr}\right\}\frac{\partial U(r,\mu_0,x)}{\partial x}\\ &-3\left\{\frac{\beta}{r}(1-\mu_0^2)+\mu_0^2\frac{d\beta}{dr}\right\}+\frac{2(\mu_0+\beta)}{r}U(r,\mu_0,x)\\ &+K(r)\left[S(r,x)-U(r,\mu_0,x)\right] \quad ...(6)\end{aligned}$$

for $0<\mu\leqslant 1$. For $-1\leqslant\mu<0$, the corresponding equation is

$$\begin{aligned}(-\mu_0+\beta)\frac{\partial U(r,-\mu_0,x)}{\partial r}&-\frac{1-\mu_0^2}{r}\left[1-\mu_0\beta\left(1-\frac{r}{\beta}\frac{d\beta}{dr}\right)\right]\frac{\partial U(r,-\mu_0,x)}{\partial\mu_0}\\ &=\left[\frac{v'}{r}(1-\mu_0^2)+\mu_0^2\frac{dv'}{dr}\right]\frac{\partial U(r,\mu_0,x)}{\partial x}\\ &-3\left[\frac{\beta}{r}(1-\mu_0^2)+\mu_0^2\frac{d\beta}{dr}\right]+\frac{2(-\mu_0+\beta)}{r}U(r,-\mu_0,x)\\ &+K(r)\left[S(r,x)-U(r,-\mu_0,x)\right. \quad ...(7)\end{aligned}$$

Letting

$$U(r, \mu_0, x) = U_0 + U_r\xi + U_{\mu_0}\eta + U_x\chi + U_{r\mu_0}\xi\eta + U_{\mu_0 x}\eta\chi + U_{xr}\chi\xi + U_{r\mu_0 x}\xi\eta\chi \qquad \ldots(8)$$

(*cf.* Peraiah, Rao & Varghese 1987), equations (6) and (7) become

$$\frac{2}{\Delta r}(\mu_0 + \beta)(U_r + U_{r\mu_0}\eta + U_{xr}\chi + U_{r\mu_0 x}\eta\chi) + \frac{2}{\Delta\mu_0}\left(\frac{1-\mu_0^2}{r}\right)\left[1 + \mu_0\beta\left(1 - \frac{r}{\beta}\frac{d\beta}{dr}\right)\right] \times (U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0 x}\xi\chi)$$
$$= K(r)\,S(r) + \left[-K + \frac{1}{r}\{2(\mu_0+\beta) - 3\beta(1-\mu_0^2)\} - 3\mu_0^2\frac{d\beta}{dr}\right]\left[U_0 + U_r\xi + U_{\mu_0}\eta + U_x\chi + U_{r\mu_0}\xi\eta + U_{\mu_0 x}\eta\chi + U_{xr}\chi\xi + U_{r\mu_0 x}\xi\eta\chi\right] + \frac{2}{\Delta x}\left[\frac{v'(r)}{r}(1-\mu_0^2) + \mu_0^2\frac{dv'(r)}{dr}\right] \times [U_x + U_{\mu_0 x}\eta + U_{xr}\xi + U_{r\mu_0 x}\xi\eta], \qquad \ldots(9)$$

$$\frac{2}{\Delta r}(-\mu_0 + \beta)(U_r + U_{r\mu_0}\eta + U_{xr}\chi + U_{r\mu_0 x}\eta\chi) - \frac{2}{\Delta\mu_0}\left(\frac{1-\mu_0^2}{r}\right) \times \left[1 - \mu_0\beta\left(1 - \frac{r}{\beta}\frac{d\beta}{dr}\right)\right](U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0 x}\xi\chi)$$
$$= K(r)\,S(r) + \left[-K(r) + \frac{1}{r}\{2(-\mu_0+\beta)\} - 3\beta(1-\mu_0^2) - 3\mu_0^2\frac{d\beta}{dr}\right]\left[U_0 + U_r\xi + U_{\mu_0}\eta + U_x\chi + U_{r\mu_0}\xi\eta + U_{\mu_0 x}\eta\chi + U_{xr}\xi\chi + U_{r\mu_0 x}\xi\eta\chi\right] + \frac{2}{\Delta x}\left[\frac{v'(R)}{r}(1-\mu_0^2) + \mu_0^2\frac{dv'(r)}{dr}\right] \times (U_x + U_{\mu_0 x}\eta + U_{xr}\xi + U_{r\mu_0 x}\xi\eta). \qquad \ldots(10)$$

Applying the operator

$$X_{\mu_0} = \frac{1}{\Delta\mu_0}\int_{\Delta\mu_0} \ldots\ldots d\mu \qquad \ldots(11)$$

on equation (9), we obtain

$$\frac{2}{\Delta r}\{(\overline{\mu_0} + \beta)(U_r + U_{xr}\chi) + \tfrac{1}{6}\Delta\mu_0(U_{r\mu_0} + U_{r\mu_0 x}\chi)\} + \frac{2}{r\cdot\Delta\mu_0}\{(1 - \overline{\mu_0^2}) + \overline{\mu_0}(1 - \langle\mu_0^2\rangle)\beta R\} \times$$

(*equation continued*)

$$\times (U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0 x}\xi\chi)$$

$$= K[(S_0 + S_r\xi + S_x\chi + S_{xr}\chi\xi) - (U_0 + U_r\xi$$

$$+ U_x\chi + U_{xr}\chi\xi)] + \left\{\frac{2\overline{\mu_0} + \beta(3\overline{\mu_0^2} - 1)}{r} - 3\overline{\mu_0^2}\,\frac{d\beta}{dr}\right\}$$

$$\times (U_0 + U_r\xi + U_x\chi + U_{xr}\chi\xi) + \Delta\mu_0 \cdot \overline{\mu_0}\left\{\frac{1}{r}\left(\beta + \frac{1}{3\overline{\mu_0}}\right) - \frac{d\beta}{dr}\right\}$$

$$\times (U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0 x}\xi\chi) + \frac{2}{\Delta x}\Big[(U_x + U_{xr}\xi)$$

$$\times \left\{(1 - \overline{\mu_0^2})\,\frac{v'(r)}{r} + \overline{\mu_0^2}\,\frac{dv'(r)}{dr}\right\} + \tfrac{1}{3}(U_{\mu_0 x} + U_{r\mu_0 x}\xi)\,\overline{\mu_0}\cdot\Delta\mu_0$$

$$\times \left\{\frac{dv'(r)}{dr} - \frac{v'(r)}{r}\right\}\Big], \qquad \text{...(12)}$$

where $$R = 1 - \frac{r}{\beta}\,\frac{d\beta}{dr} \qquad \text{...(13)}$$

and $$\langle\mu_0^2\rangle = \tfrac{1}{2}(\mu_{0,j}^2 + \mu_{0,j-1}^2). \qquad \text{...(14)}$$

Similarly applying X_{μ_0} on equation (10) we get

$$\frac{2}{\Delta r}\{(\beta - \overline{\mu_0})(U_r + U_{xr}\chi) - \tfrac{1}{6}\Delta\mu_0(U_{r\mu_0} + U_{r\mu_0 x})\}$$

$$-\frac{2}{\Delta\mu_0}\cdot\frac{1}{r}\left\{(1 - \overline{\mu_0^2}) - \overline{\mu_0}\,(1 - \langle\mu_0^2\rangle)\,\beta\left(1 - \frac{r}{\beta}\,\frac{d\beta}{dr}\right)\right\}$$

$$\times (U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0 x}\xi\chi)$$

$$= K[(S_0 + S_r\xi + S_x\chi + S_{xr}\chi\xi) - (U_0 + U_r\xi + U_x\chi + U_{xr}\chi\xi)]$$

$$-\left\{\frac{2\overline{\mu_0} + \beta - 3\beta\overline{\mu_0^2}}{r} + 3\overline{\mu_0^2}\,\frac{d\beta}{dr}\right\}(U_0 + U_r\xi + U_x\chi + U_{xr}\chi\xi)$$

$$-\frac{\Delta\mu_0\cdot\overline{\mu_0}}{r}\left(r\,\frac{d\beta}{dr} - \beta + \frac{1}{3\overline{\mu_0}}\right)(U_{\mu_0} + U_{r\mu_0}\xi + U_{\mu_0 x}\chi + U_{r\mu_0 x}\xi\chi)$$

$$+\frac{2}{\Delta x}\left[(U_x + U_{xr}\xi)\left\{(1 - \overline{\mu_0^2})\,\frac{v'(r)}{r} + \overline{\mu_0^2}\,\frac{dv'(r)}{dr}\right\}\right.$$

$$\left.+ \tfrac{1}{3}(U_{\mu_0 x} + U_{r\mu_0 x}\xi)\,\overline{\mu_0}\cdot\Delta\mu_0\left\{\frac{dv'(r)}{dr} - \frac{v'(r)}{r}\right\}\right]. \qquad \text{...(15)}$$

Applying the operator

$$Y_V = \frac{1}{V}\int_{\Delta r} \ldots\ldots 4\pi r^2 dr \qquad \text{...(16)}$$

on equation (12) we obtain

$$\frac{2}{\Delta r}\{(\overline{\mu}+\beta)(U_{\mathrm{r}}+U_{\mathrm{xr}}\chi)+\tfrac{1}{6}\Delta\mu_0(U_{\mathrm{r}\mu_0}+U_{\mathrm{r}\mu_0\mathrm{x}})\}$$

$$+(U_{\mu_0}+U_{\mu_0\mathrm{x}}\chi)\left\{G\left(\tfrac{1}{2}\frac{\Delta A}{V}-\frac{1}{\beta}\frac{d\beta}{dr}\right)+\frac{\Delta A}{V}\left(\frac{1-\mu_0^2}{\Delta\mu_0}\right)\right\}$$

$$+(U_{\mathrm{r}\mu_0}+U_{\mathrm{r}\mu_0\mathrm{x}}\chi)\,G\left\{\left(\frac{2}{\Delta r}-\frac{r}{\Delta r}\cdot\frac{\Delta A}{V}\right)-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{1}{\beta}\frac{d\beta}{dr}\right.$$

$$\left.+\frac{2}{\Delta r}\frac{1-\overline{\mu_0^2}}{\Delta\mu_0}\left(2-\frac{\bar{r}\Delta A}{V}\right)\right\}$$

$$=K\left\{\left[(S_0+S_{\mathrm{x}}\chi)+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}(S_{\mathrm{r}}+S_{\mathrm{xr}}\chi)\right]-\left[(U_0+U_{\mathrm{x}}\chi)\right.\right.$$

$$\left.\left.+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}(U_{\mathrm{r}}+U_{\mathrm{xr}}\chi)\right]\right\}+\{2\overline{\mu_0}+\beta(3\overline{\mu_0^2}-1)\}$$

$$\times\left\{\tfrac{1}{2}\frac{\Delta A}{\bar{A}}(U_0+U_{\mathrm{x}}\chi)+\frac{1}{\Delta r}\left(2-\frac{\bar{r}\Delta A}{V}\right)(U_{\mathrm{r}}+U_{\mathrm{rx}}\chi)\right\}$$

$$-3\overline{\mu_0^2}\frac{d\beta}{dr}\left\{(U_0+U_{\mathrm{x}}\chi)+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}(U_{\mathrm{r}}+U_{\mathrm{xr}}\chi)\right\}$$

$$+\Delta\mu_0\cdot\overline{\mu_0}\left[\left(\beta+\frac{1}{3\overline{\mu_0}}\right)\left\{\tfrac{1}{2}\frac{\Delta A}{V}(U_{\mu_0}+U_{\mu_0\mathrm{x}}\chi)\right.\right.$$

$$\left.+\frac{1}{\Delta r}\left(2-\frac{\bar{r}\Delta A}{V}\right)(U_{\mathrm{r}\mu_0}+U_{\mathrm{r}\mu_0\mathrm{x}}\chi)\right\}-\frac{d\beta}{dr}\left\{(U_{\mu_0}+U_{\mu_0\mathrm{x}}\chi)\right.$$

$$\left.\left.+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}(U_{\mathrm{r}\mu_0}+U_{\mathrm{r}\mu_0\mathrm{x}}\chi)\right\}\right]+\frac{2}{\Delta x}\left[(1-\overline{\mu_0^2})\,\nu'(r)\left\{\tfrac{1}{2}\frac{\Delta A}{V}U_{\mathrm{x}}\right.\right.$$

$$\left.+\frac{1}{\Delta r}\left(2-\frac{\bar{r}\Delta A}{V}\right)U_{\mathrm{xr}}\right\}+\overline{\mu_0^2}\frac{d\nu'(r)}{dr}\left\{U_{\mathrm{x}}+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}U_{\mathrm{xr}}\right\}$$

$$+\tfrac{1}{3}\overline{\mu_0}\Delta\mu_0\left\{\frac{d\nu'(r)}{dr}\left(U_{\mu_0\mathrm{x}}+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}U_{\mathrm{r}\mu_0\mathrm{x}}\right)\right\}-\nu'(r)\cdot\tfrac{1}{3}\overline{\mu_0}\cdot\Delta\mu_0$$

$$\left.\times\left\{\tfrac{1}{2}\frac{\Delta A}{V}U_{\mu_0\mathrm{x}}+\frac{1}{\Delta r}\left(2-\frac{r\Delta A}{V}\right)U_{\mathrm{r}\mu_0\mathrm{x}}\right\}\right], \qquad \text{...(17)}$$

where

$$G=\frac{2\overline{\mu_0}}{\Delta\mu_0}(1-\langle\mu_0^2\rangle)\,\beta. \qquad \text{...(18)}$$

Application of Y_V on equation (15) gives

$$\frac{2}{\Delta r}\Big\{(\beta - \overline{\mu_0})\,(U_r + U_{xr}\chi) - \tfrac{1}{6}\Delta\mu_0(U_{r\mu_0} + U_{r\mu_0 x}\chi)$$

$$+ (U_{\mu_0} + U_{\mu_0 x}\chi)\left\{G\left(\tfrac{1}{2}\frac{\Delta A}{V} - \frac{1}{\beta}\frac{d\beta}{dr}\right) - \frac{\Delta A}{V}\,\frac{1-\mu_0^2}{\Delta\mu_0}\right.$$

$$\left.+ (U_{r\mu_0} + U_{r\mu_0 x}\chi)\left\{G\left(p - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{1}{\beta}\frac{d\beta}{dr}\right) - \frac{2}{\Delta\mu_0}(1-\overline{\mu_0^2})\,p\right\}\right.$$

$$= K\Big\{\Big[(S_0 + S_x\chi) + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}(S_r + S_{xr}\chi)\Big] - \Big[(U_0 + U_x\chi)$$

$$+ \tfrac{1}{6}\frac{\Delta A}{\bar{A}}(U_r + U_{xr}\chi)\Big]\Big\} - (U_0 + U_x\chi)\Big\{\tfrac{1}{2}\frac{\Delta A}{V}(2\overline{\mu_0} + \beta - 3\beta\overline{\mu_0^2})$$

$$+ 3\overline{\mu_0^2}\frac{d\beta}{dr}\Big\} - (U_r + U_{xr}\chi)\Big\{p(2\overline{\mu_0} + \beta - 3\beta\overline{\mu_0^2})$$

$$+ 3\mu_0^2\frac{d\beta}{dr}\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\Big\} - (U_{\mu_0} + U_{\mu_0 x}\chi)\Big\{\Delta\overline{\mu_0}\mu_0\Big[\frac{d\beta}{dr} + \tfrac{1}{2}\frac{\Delta A}{V}$$

$$\times\left(\frac{1}{3\overline{\mu_0}} - \beta\right)\Big]\Big\} - (U_{r\mu_0} + U_{r\mu_0 x}\chi)\Big\{\Delta\mu_0\overline{\mu_0}\Big[\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}$$

$$+ p\left(\frac{1}{3\overline{\mu_0}} - \beta\right)\Big]\Big\} + \frac{2}{\Delta x}\Big[(1 - \overline{\mu_0^2})\,v'(r)\Big\{\tfrac{1}{2}\frac{\Delta A}{V}U_x + pU_{xr}\Big\}$$

$$+ \overline{\mu_0^2}\frac{dv'(r)}{dr}\left(U_x + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}U_{xr}\right) + \tfrac{1}{3}\mu_0\Delta\mu_0\Big\{\frac{dv'(r)}{dr}\Big(U_{\mu_0 x}$$

$$+ \tfrac{1}{6}\frac{\Delta A}{\bar{A}}U_{r\mu_0 x}\Big)\Big\} - \tfrac{1}{3}\mu_0\Delta\mu_0 v'(r)\Big\{\tfrac{1}{2}\frac{\Delta A}{V}U_{\mu_0 x} + pU_{r\mu_0 x}\Big\}\Big], \quad \ldots(19)$$

where $$p = \frac{1}{\Delta r}\left(2 - \frac{\bar{r}}{V}\Delta A\right). \quad \ldots(20)$$

Applying the operator

$$Z_x = \frac{1}{\Delta x}\int_{\Delta x} \ldots\ldots dx \quad \ldots(21)$$

on equation (17) we obtain

$$\frac{2}{\Delta r}\{(\bar{\mu} + \beta)\,U_r + \tfrac{1}{6}\Delta\mu_0\cdot U_{r\mu_0}\}) + U_{\mu_0}\Big\{\left(\tfrac{1}{2}\frac{\Delta A}{V}\frac{G}{\beta}\frac{d\beta}{dr}\right)$$

$$+ \frac{\Delta A}{V}\left(\frac{1-\overline{\mu_0^2}}{\Delta\mu}\right)\Big\} + U_{r\mu_0}\Big\{\frac{G}{\Delta r}\left(2 - \frac{\bar{r}\Delta A}{V}\right) - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr}$$

$$+ \frac{2}{\Delta r}\frac{1-\overline{\mu_0^2}}{\Delta\mu_0}\left(2 - \frac{\bar{r}\Delta A}{V}\right)\Big\} = K\Big\{\Big[S_0 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}S_r\Big] -$$

(*equation continued*)

$$-\left[U_0 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_r\right]\Big\} + \left\{2\overline{\mu_0} + \beta(3\overline{\mu_0^2} - 1)\right\}\left\{\tfrac{1}{2}\frac{\Delta A}{V} U_0 + \frac{1}{\Delta r}\right.$$

$$\times\left(2 - \frac{\bar{r}\Delta A}{V}\right) U_r\Big\} - 3\overline{\mu_0^2}\frac{d\beta}{dr}\left(U_0 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_r\right)$$

$$+ \Delta\mu_0\cdot\overline{\mu_0}\left[\left(\beta + \frac{1}{3\overline{\mu_0}}\right)\left\{\tfrac{1}{2}\frac{\Delta A}{V} U_{\mu_0} + \frac{1}{\Delta r}\left(2 - \frac{\bar{r}\Delta A}{V}\right) U_{r\mu_0}\right\}\right.$$

$$\left.- \frac{d\beta}{dr}\left(U_{\mu_0} + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_{r\mu_0}\right)\right] + \frac{2}{\Delta x}\left[(1 - \overline{\mu_0^2})\, v'(r)\left\{\tfrac{1}{2}\frac{\Delta A}{V} U_x\right.\right.$$

$$\left.+ \frac{1}{\Delta r}\left(2 - \frac{\bar{r}\cdot\Delta A}{V}\right) U_{xr}\right\} + \overline{\mu_0^2}\frac{dv'(r)}{dr}\left(U_x + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_{xr}\right)$$

$$+ \tfrac{1}{3}\overline{\mu_0^2}\cdot\Delta\mu_0\left\{\frac{dv'(r)}{dr}\left(U_{\mu_0 x} + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_{r\mu_0 x}\right) - v'(r)\cdot\tfrac{1}{3}\overline{\mu_0}\cdot\Delta\mu_0\right.$$

$$\left.\left.\times\left\{\tfrac{1}{2}\frac{\Delta A}{V} U_{\mu_0 x} + \frac{1}{\Delta r}\left(2 - \frac{\bar{r}\cdot\Delta A}{V}\right) U_{r\mu_0 x}\right\}\right]. \qquad \ldots(22)$$

Effecting Z_x on equation (19) we obtain

$$\frac{2}{\Delta r}\{(\beta - \overline{\mu_0})\, U_r - \tfrac{1}{6}\Delta\mu_0\cdot U_{r\mu_0}\}\, U_{\mu_0}\left\{\tfrac{1}{2}\frac{\Delta A}{V} G - \frac{G}{\beta}\frac{d\beta}{dr}\right.$$

$$\left.- \frac{\Delta A}{V}\frac{1 - \overline{\mu_0^2}}{\Delta\mu_0}\right\} + U_{r\mu_0}\left\{Gp - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr} - \frac{2}{\Delta\mu_0}(1 - \overline{\mu_0^2})\, p\right\}$$

$$= K\left\{\left(S_0 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} S_r\right) - \left(U_0 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_r\right)\right\} - U_0 A_1 - U_r A_2$$

$$- U_{\mu_0} A_3 - U_{r\mu_0} A_4 + \frac{2}{\Delta x}\left[(1 - \overline{\mu_0^2})\, v'(r)\left\{\tfrac{1}{2}\frac{\Delta A}{V} U_x + pU_{xr}\right\}\right.$$

$$+ \overline{\mu_0^2}\frac{dv'(r)}{dr}\left(U_x + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_{xr}\right) + \tfrac{1}{3}\overline{\mu_0}\cdot\Delta\mu_0\left\{\frac{dv'(r)}{dr}\left(U_{\mu_0 x}\right.\right.$$

$$\left.\left.\left. + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} U_{r\mu_0 x}\right)\right\} - \tfrac{1}{3}\overline{\mu_0}\cdot\Delta\mu_0 v'(r)\left\{\tfrac{1}{2}\frac{\Delta A}{V} U_{\mu_0 x} + pU_{r\mu_0 x}\right\}\right], \qquad \ldots(23)$$

where

$$A_1 = \tfrac{1}{2}\frac{\Delta A}{V}\left\{2\overline{\mu_0} + \beta(1 - 3\overline{\mu_0^2}) + 3\overline{\mu_0^2}\frac{d\beta}{dr}\right\}, \qquad \ldots(24)$$

$$A_2 = p\{2\overline{\mu_0} + \beta(1 - 3\overline{\mu_0^2})\} + \tfrac{1}{2}\overline{\mu_0^2}\frac{d\beta}{dr}\frac{\Delta A}{\bar{A}}, \qquad \ldots(25)$$

$$A_3 = \Delta\mu_0\cdot\overline{\mu_0}\left[\frac{d\beta}{dr} + \tfrac{1}{2}\frac{\Delta A}{V}\left(\frac{1}{3\overline{\mu_0}} - \beta\right)\right], \qquad \ldots(26)$$

$$A_4 = \Delta\mu_0\cdot\overline{\mu_0}\left[\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr} + p\left(\frac{1}{3\overline{\mu_0}} - \beta\right)\right]. \qquad \ldots(27)$$

Equations (19) and (23) can be rewritten as

$$U_r\left[\frac{2}{\Delta r}(\overline{\mu_0}+\beta)+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}K_x-p\{2\overline{\mu_0}+\beta(3\overline{\mu_0^2}-1)\}\right.$$
$$\left.+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}3\overline{\mu_0^2}\frac{d\beta}{dr}\right]+U_{\mu_0}\left[G\left(\tfrac{1}{2}\frac{\Delta A}{V}-\frac{1}{\beta}\frac{d\beta}{dr}\right)\right.$$
$$\left.+\frac{\Delta A}{V}\left(\frac{1-\overline{\mu_0^2}}{\Delta\mu_0}\right)-\Delta\mu_0\cdot\overline{\mu_0}\left\{\left(\beta+\frac{1}{3\overline{\mu_0}}\tfrac{1}{2}\frac{\Delta A}{V}-\frac{d\beta}{dr}\right\}\right]$$
$$+U_{r\mu_0}\left[\tfrac{1}{3}\frac{\Delta\mu_0}{\Delta r}+Gp-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr}+2p\frac{1-\overline{\mu_0^2}}{\Delta\mu_0}\right.$$
$$\left.-\Delta\mu_0\cdot\overline{\mu_0}\left\{\left(\beta+\frac{1}{3\overline{\mu_0}}\right)p-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}\right\}\right]$$
$$+U_0\left[K_x-\tfrac{1}{2}\frac{\Delta A}{V}\{2\overline{\mu_0}+\beta(3\overline{\mu_0^2}-1)\}+3\overline{\mu_0^2}\frac{d\beta}{dr}\right]$$
$$=K_x\left[S_0+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right]+U_x\left[\frac{2}{\Delta x}\left\{\tfrac{1}{2}\frac{\Delta A}{V}(1-\overline{\mu_0^2})\,v'(r)\right.\right.$$
$$\left.\left.+\overline{\mu_0^2}\frac{dv'(r)}{dr}\right\}\right]+U_{xr}\left[\frac{2}{\Delta x}\left\{p(1-\overline{\mu_0^2})\,v'(r)+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\overline{\mu_0^2}\frac{dv'(r)}{dr}\right\}\right]$$
$$+U_{\mu_0x}\left[\frac{2}{\Delta x}\left\{\tfrac{1}{3}\mu_0\Delta\mu_0\left(\frac{dv'(r)}{dr}-\tfrac{1}{2}\frac{\Delta A}{V}v'(r)\right)\right\}\right]$$
$$+U_{r\mu_0x}\left[\frac{2}{\Delta x}\left\{\tfrac{1}{3}\Delta\mu_0\overline{\mu_0}\left(\frac{dv'(r)}{dr}\cdot\tfrac{1}{6}\frac{\Delta A}{\bar{A}}-v'(r)\,p\right)\right\}\right], \qquad \ldots(28)$$

and

$$U_1\left[\frac{2}{\Delta r}(\beta-\overline{\mu_0})+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}K_x+A_2\right]+U_{\mu_0}(B_1+A_3)$$
$$+U_{r\mu_0}\left(B_2+A_4-\tfrac{1}{3}\frac{\Delta\mu_0}{\Delta r}\right)+U_0(A_1+K_x)$$
$$=K_x\left(S_0+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right)+U_x\left[\frac{2}{\Delta x}\left\{(1-\overline{\mu_0^2})\,v'(r)\cdot\tfrac{1}{2}\frac{\Delta A}{V}\right.\right.$$
$$\left.\left.+\overline{\mu_0^2}\frac{dv'(r)}{dr}\right\}\right]+U_{xr}\left[\frac{2}{\Delta x}\left\{p(1-\overline{\mu_0^2})\,v'(r)+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\overline{\mu_0^2}\frac{dv'(r)}{dr}\right\}\right]$$
$$+U_{\mu_0x}\left[\frac{2}{\Delta x}\left\{\tfrac{1}{3}\overline{\mu_0}\Delta\mu_0\left(\frac{dv'(r)}{dr}-\tfrac{1}{2}\frac{\Delta A}{V}v'(r)\right)\right\}\right]$$
$$+U_{r\mu_0x}\left[\frac{2}{\Delta x}\left\{\tfrac{1}{3}\mu_0\Delta\mu_0\left(\tfrac{1}{6}\frac{\Delta A}{V}\frac{dv'(r)}{dr}-v'(r)\,p\right)\right\}\right]. \qquad \ldots(29)$$

Here

$$B_1=\tfrac{1}{2}\frac{\Delta A}{V}G-\frac{G}{\beta}\frac{d\beta}{dr}-\frac{\Delta A}{V}\frac{1-\overline{\mu_0^2}}{\Delta\mu_0} \qquad \ldots(30)$$

and

$$B_2 = Gp - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr} - \frac{2}{\Delta\mu_0}(1 - \overline{\mu_0^2})\,p. \qquad \ldots(31)$$

Equations (28) and (29) can be rewritten as

$$\alpha U_r + \beta U_{\mu_0} + \gamma U_{r\mu_0} + \delta U_0$$
$$= \Delta r \cdot K_x\left[S_0 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} S_r\right] + \rho U_x + \sigma U_{xr} + \epsilon U_{\mu_0 x} + \lambda U_{r\mu_0 x} \ldots(32)$$

and

$$\alpha' U_r + \beta' U_{\mu_0} + \gamma' U_{r\mu_0} + \delta' U_0$$
$$= \Delta r \cdot K_x\left[S_0 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} S_r\right] + \rho U_x + \sigma U_{xr} + \epsilon U_{\mu_0 x} + \lambda U_{r\mu_0 x}, \ldots(33)$$

where

$$\alpha = \Delta r\left\{\frac{2}{\Delta r}(\bar{\mu} + g) + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} K_x - p[2\mu_0 + g(3\overline{\mu_0^2} - 1)] + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\, 3\overline{\mu_0^2}\frac{dg}{dr}\right\}, \qquad \ldots(34)$$

$$\beta = \Delta r\left\{G\left(\tfrac{1}{2}\frac{\Delta A}{V} - \frac{1}{g}\frac{dg}{dr}\right) + \frac{\Delta A}{V}\left(\frac{1 - \overline{\mu_0^2}}{\Delta\mu_0}\right) - \Delta\mu_0\overline{\mu_0}\left[\left(g + \frac{1}{3\overline{\mu_0}}\right)\tfrac{1}{2}\frac{\Delta A}{V} - \frac{dg}{dr}\right]\right\}, \qquad \ldots(35)$$

$$\gamma = \Delta r\left\{\tfrac{1}{3}\frac{\Delta\mu_0}{\Delta r} + Gp - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{g}\frac{dg}{dr} + 2p\frac{1 - \overline{\mu_0^2}}{\Delta\mu_0} - \Delta\mu_0 \cdot \overline{\mu_0}\left[\left(g + \frac{1}{3\overline{\mu_0}}\right)p - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{dg}{dr}\right]\right\}, \qquad \ldots(36)$$

$$\delta = \Delta r\left\{K_x - \tfrac{1}{2}\frac{\Delta A}{V}[2\mu_0 + g(3\overline{\mu_0^2} - 1)] + 3\overline{\mu_0^2}\frac{dg}{dr}\right\}, \qquad \ldots(37)$$

$$\rho = \Delta r \cdot \frac{2}{\Delta x}\left\{\tfrac{1}{2}\frac{\Delta A}{V}(1 - \overline{\mu_0^2})\, v'(r) + \overline{\mu_0^2}\frac{dv'(r)}{dr}\right\}, \qquad \ldots(38)$$

$$\sigma = \Delta r \cdot \frac{2}{\Delta x}\left\{p(1 - \overline{\mu_0^2})\, v'(r) + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\overline{\mu_0^2}\frac{dv'(r)}{dr}\right\}, \qquad \ldots(39)$$

$$\epsilon = \Delta r \cdot \frac{2}{\Delta x}\left\{\tfrac{1}{3}\overline{\mu_0}\Delta\mu_0\left(\frac{dv'(r)}{dr} - \tfrac{1}{2}\frac{\Delta A}{V} v'(r)\right)\right\}, \qquad \ldots(40)$$

$$\lambda = \Delta r \cdot \frac{2}{\Delta x}\left\{\tfrac{1}{3}\overline{\mu_0}\Delta\mu_0\frac{dv'(r)}{dr} \cdot \tfrac{1}{6}\frac{\Delta A}{\bar{A}} - pv'(r)\right\}, \qquad \ldots(41)$$

$$\alpha' = \Delta r\left[\frac{2}{\Delta r}(g - \overline{\mu_0}) + \tfrac{1}{6}\frac{\Delta A}{\bar{A}} K_x + A_2\right], \qquad \ldots(42)$$

$$\beta' = \Delta r(B_1 + A_3), \quad ...(43)$$

$$\gamma' = \Delta r\left(A_4 + B_2 - \tfrac{1}{3}\frac{\Delta\mu_0}{\Delta r}\right), \quad ...(44)$$

$$\delta' = \Delta r(K_x + A_1), \quad ...(45)$$

$$g = v'/c. \quad ...(46)$$

Following Peraiah, Rao & Varghese (1987) we replace U_r, U_{μ_0}...by the nodal values U_a, U_b, ..in equations (32) and (33) to get

$$A_a U_a^+ + A_b U_b^+ + A_c U_c^+ + A_d U_d^+ + A_e U_e^+ + A_f U_f^+ + A_g U_g^+ + A_h U_h^+$$
$$= \tau_x^- (S_a^+ + S_b^+ + S_c^+ + S_d^+) + \tau_x^- (S_e^+ + S_f^+ + S_g^+ + S_h^+) \quad ...(47)$$

and

$$A'_a U_b^- + A'_b U_b^- + A'_c U_c^- + A'_d U_d^- + A'_e U_e^- + A'_f U_f^- + A'_g U_g^- + A'_h U_h^-$$
$$= \tau_x^- (S_a^- + S_b^- + S_c^- + S_d^-) + \tau_x^+ (S_e^- + S_f^- + S_g^- + S_h^-), \quad ...(48)$$

where

$$A_a = -\alpha - \beta + \gamma + \delta + \rho - \sigma - \epsilon + \lambda, \quad ...(49)$$

$$A_b = -\alpha + \beta - \gamma + \delta + \rho - \sigma + \epsilon - \lambda, \quad ...(50)$$

$$A_c = -\alpha + \beta - \gamma + \delta - \rho + \sigma - \epsilon + \lambda, \quad ...(51)$$

$$A_d = -\alpha - \beta + \gamma + \delta - \rho + \sigma + \epsilon - \lambda, \quad ...(52)$$

$$A_e = \alpha - \beta - \gamma + \delta - \rho - \sigma + \epsilon + \lambda, \quad ...(53)$$

$$A_f = \alpha - \beta - \gamma + \delta + \rho + \sigma - \epsilon - \lambda, \quad ...(54)$$

$$A_g = \alpha + \beta + \gamma + \delta + \rho + \sigma + \epsilon + \lambda, \quad ...(55)$$

$$A_h = \alpha + \beta + \gamma + \delta - \rho - \sigma - \varepsilon - \lambda, \quad ...(56)$$

$$A'_a = -\alpha' - \beta' + \gamma' + \delta' + \rho - \sigma - \epsilon + \lambda, \quad ...(57)$$

$$A'_b = -\alpha' + \beta' - \gamma' + \delta' + \rho - \sigma + \epsilon - \lambda, \quad ...(58)$$

$$A'_c = -\alpha' + \beta' - \gamma' + \delta' - \rho + \sigma - \epsilon + \lambda, \quad ...(59)$$

$$A'_d = -\alpha' - \beta' + \gamma' + \delta' - \rho + \sigma + \epsilon - \lambda, \quad ...(60)$$

$$A'_e = -\alpha' - \beta' - \gamma' + \delta' - \rho + \sigma + \epsilon + \lambda, \quad ...(61)$$

$$A'_f = \alpha' - \beta' - \gamma' + \delta' + \rho + \sigma - \epsilon - \lambda, \quad ...(62)$$

$$A'_g = \alpha' + \beta' + \gamma' + \delta' + \rho + \sigma + \epsilon + \lambda, \quad ...(63)$$

$$A'_h = \alpha' + \beta' + \gamma' + \delta' - \rho - \sigma - \epsilon - \lambda, \quad ...(64)$$

$$\tau^{+} = \tau\left(1 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right), \qquad \ldots(65)$$

$$\tau^{-} = \tau\left(1 - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right), \qquad \ldots(66)$$

$$\tau = \bar{K}\cdot\Delta r. \qquad \ldots(67)$$

U_a^+, U_b^+, etc. and U_a^-, U_b^-, etc., represent the beams of radiation in the opposite directions. We make the substitutions (see figure 1 of Peraiah, Rao & Varghese 1987).

$$U_a^+ = U_{m-1,n-1}^{i-1,+}, \qquad U_b^+ = U_{m,n-1}^{i-1,+}, \qquad U_c^+ = U_{m,n}^{i-1,+}$$

$$U_d^+ = U_{m-1,n}^{i-1,+}, \qquad U_e^+ = U_{m-1,m}^{i,+}, \qquad U_f^+ = U_{m-1,n-1}^{i,+}$$

$$U_g^+ = U_{m,n-1}^{i,+}, \qquad U_h^+ = U_{m,n}^{i,+} \qquad \ldots(68)$$

together with similar equations for U_a^-, U_b^-, etc. and $S_a^{\pm}$, $S_b^{\pm}$, etc., into equations (47) and (48) and obtain equations for $(m-1)$ th and mth angles :

$$A_{m-1}^{a}\, U_{m-1,n-1}^{i-1,+} + A_m^{b}\, U_{m,n-1}^{i-1,+} + A_m^{c}\, U_{m,n}^{i-1,+} + A_{m-1}^{d}\, U_{m-1,n}^{i-1,+}$$

$$+ A_{m-1}^{e}\, U_{m-1,n}^{i,+} + A_{m-1}^{f}\, U_{m-1,n-1}^{i,+} + A_m^{g}\, U_{m,n-1}^{i,+} + A_m^{h}\, U_{m,n}^{i,+}$$

$$= \tau^{-}\{[\sigma'\phi_{m-1,n-1}^{i-1,+} \sum_{n'=-N}^{N} a_{n'}\,\phi_{n'}^{i-1} \sum_{m'=1}^{M} C_{m'}(U_{m',n-1}^{i-1,+} + U_{m',n-1}^{i-1,-})]$$

$$+ (\epsilon'\phi_{m-1,n-1}^{i-1,+} + \rho'\beta')\, B_{m-1,n-1}^{i-1,+}$$

$$+ [\sigma'\phi_{m,n-1}^{i-1,+} \sum_{n'=-N}^{N} a_{n'}\,\phi_{n'}^{i-1} \sum_{m'=1}^{M} C_{m'}\,(U_{m',n-1}^{i-1,+} + U_{m',n-1}^{i-1,-})]$$

$$+ (\epsilon'\phi_{m,n-1}^{i-1,+} + \rho'\beta')\, B_{m,n-1}^{i-1,+}$$

$$+ [\sigma'\phi_{m-1,n}^{i-1,+} \sum_{n'=-N}^{N} a_{n'}\,\phi_{n'}^{i-1} \sum_{m'=1}^{M} C_{m'}\,(U_{m',n}^{i-1,+} + U_{m',n}^{i-1,-})]$$

$$+ (\epsilon'\phi_{m,n}^{i-1,+} + \rho'\beta')\, B_{m,k}^{i-1,+}$$

$$+ \tau^{+}\{[\sigma'\phi_{m-1,n}^{i,+} \sum_{n'=-N}^{N} a_{n'}\,\phi_{n'}^{i} \sum_{m'=1}^{M} C_{m'}\,(U_{m',n}^{i,+} + U_{m',n}^{i,-})]$$

$$+ (\epsilon'\phi_{m-1,n}^{i,+} + \rho'\beta')\, B_{m-1,n}^{i,+} +$$

(*equation continued*)

$$+ [\sigma'\phi^{i,+}_{m,n-1} \sum_{n'=-N}^{N} a_{n'}\phi^{i}_{n'} \sum_{m'=1}^{M} C_{m'} (U^{i,+}_{m',n-1} + U^{i,-}_{m',n})]$$

$$+ (\epsilon'\phi^{i,+}_{m,n-1} + \rho'\beta')\, B^{i,+}_{m,n-1}$$

$$+ [\sigma'\phi^{i,+}_{m-1,n-1} \sum_{n'=-N}^{N} a_{n'}\phi^{i}_{n'} \sum_{m'=1}^{M} C_{m'} (U^{i,+}_{m',n+1} + U^{i,-}_{m'n-1})]$$

$$+ (\epsilon'\phi^{i,+}_{m-1,n-1} + \rho'\beta')\, B^{i,+}_{m-1,n-1}$$

$$+ [\sigma'\phi^{i,+}_{m,n} \sum_{n'=-N}^{N} a_{n'}\phi^{i}_{n'} \sum_{m'=1}^{M} C_{m'} (U^{i,+}_{m',n} + U^{i,-}_{m',n})]$$

$$+ (\epsilon'\phi^{i,+}_{m,n}\, \rho'\beta')\, B^{i,+}_{m,k}, \qquad \ldots(69)$$

where $\sigma' = \frac{1}{2}(1 - \epsilon')$ i, m, n are the running indices of radial, angle and frequency discretizations; M and N are the total number of angles and frequencies respectively; ϕ is the profile function of the line; ϵ' the probability of thermalisation upon collisional de-excitation; β' the ratio of opacities due to continuum and line centre; ρ' an arbitrary parameter; B is the Planck function. Equation (48) can be written similarly.

Equation (69) and its counterpart can be succinctly written for M angles as

$$(\mathbf{A}^{ab}\mathbf{U}^{i-1,+}_{n-1} - \tau^-\sigma'\mathbf{Q}\boldsymbol{\phi}^{i-1,+}_{n-1}\, \mathbf{Y}^+_{n-1}) + (\mathbf{A}^{dc}\mathbf{U}^{i-1,+}_{n} - \tau^-\sigma'\mathbf{Q}\boldsymbol{\phi}^{i-1,+}_{n}\, \mathbf{Y}^+_{n})$$

$$+ (\mathbf{A}^{fg}\mathbf{U}^{i,+}_{n-1} - \tau^+\sigma'\mathbf{Q}\boldsymbol{\phi}^{i,+}_{n-1}\, \mathbf{Y}^+_{n-1}) + (\mathbf{A}^{eh}\mathbf{U}^{i,+}_{n} - \tau^+\sigma'\mathbf{Q}\boldsymbol{\phi}^{i,+}_{n}\, \mathbf{Y}^+_{n})$$

$$= \tau^-\sigma'\mathbf{Q}(\boldsymbol{\phi}^{i-1,+}_{n-1}\, \mathbf{Y}^-_{n-1} + \boldsymbol{\phi}^{i-1,+}_{n}\, \mathbf{Y}^-_{n}) + \tau^+\sigma'\mathbf{Q}(\boldsymbol{\phi}^{i,+}_{n-1}\, \mathbf{Y}^-_{n-1}$$

$$+ \boldsymbol{\phi}^{i,+}_{n}\, \mathbf{Y}^-_{n}) + \tau^-\sigma'\mathbf{Q}(\rho'\beta' + \epsilon'\boldsymbol{\phi}^{i-1,+}_{n-1})\, \mathbf{B}^{i-1}_{n-1} + \tau^-\sigma'\mathbf{Q}(\rho'\beta'$$

$$+ \epsilon'\boldsymbol{\phi}^{i-1,+}_{n})\, \mathbf{B}^{i-1,+}_{n} + \tau^+\sigma'\mathbf{Q}(\rho'\beta' + \epsilon'\boldsymbol{\phi}^{i,+}_{n-1})\, \mathbf{B}^{i,+}_{n-1}$$

$$+ \tau^+\sigma'\mathbf{Q}(\rho'\beta' + \epsilon'\boldsymbol{\phi}^{i,+}_{n})\, \mathbf{B}^{i,+}_{n}. \qquad \ldots(70)$$

and

$$(\mathbf{D}^{ab}\mathbf{U}^{i-1,-}_{n-1} - \tau^-\sigma'\mathbf{Q}\boldsymbol{\phi}^{i-1,-}_{n-1}\, \mathbf{Y}^-_{n-1}) + (\mathbf{D}^{dc}\mathbf{U}^{i-1,-}_{n-1} - \tau^-\sigma'\mathbf{Q}\boldsymbol{\phi}^{i-1,-}_{n-1}\, \mathbf{Y}^-_{n})$$

$$+ (\mathbf{D}^{fg}\mathbf{U}^{i,-}_{n-1} - \tau^+\sigma'\mathbf{Q}\boldsymbol{\phi}^{i,-}_{n-1}\, \mathbf{Y}^-_{n-1}) + (\mathbf{D}^{eh}\mathbf{U}^{i,-}_{n} - \tau^+\sigma'\mathbf{Q}\boldsymbol{\phi}^{i,-}_{n}\, \mathbf{Y}^-_{n})$$

$$= \tau^-\sigma'\mathbf{Q}(\boldsymbol{\phi}^{i-1,-}_{n-1}\, \mathbf{Y}^+_{n-1} + \boldsymbol{\phi}^{i-1,-}_{n}\, \mathbf{Y}^+_{n}) + \tau^+\sigma'\mathbf{Q}(\boldsymbol{\phi}^{i,-}_{n-1}\, \mathbf{Y}^+_{n-1} +$$

(equation continued)

$$+ \phi_n^{i,-} \mathbf{Y}_n^{+}) + \tau^- \sigma' \mathbf{Q}(\rho'\beta' + \epsilon' \phi_{n-1}^{i-1,-}) \mathbf{B}_{n-1}^{i-1,-} + \tau^- \sigma' \mathbf{Q}(\rho'\beta' + \epsilon' \phi_n^{i-1,-}) \mathbf{B}_n^{i-1,-} + \tau^+ \sigma' \mathbf{Q}(\rho'\beta' + \epsilon' \phi_{n-1}^{i,-}) \mathbf{B}_{n-1}^{i,-} + \tau^+ \sigma' \mathbf{Q}(\rho'\beta' + \epsilon' \phi_n^{i,-}) \mathbf{B}_n^{i,-}, \quad \text{...(71)}$$

where

$$\mathbf{A}^{ab} = \begin{bmatrix} A_{m-1}^{a} & A_m^{b} & & & \\ & A_m^{a} & A_{m+1}^{b} & & \\ & & \ddots & & \\ & & A_{M-1}^{a} & A_m^{b} & \\ & & & \ddots & \\ & & & & A_M^{a} \end{bmatrix} \quad \text{...(72)}$$

$$\mathbf{D}^{ab} = \begin{bmatrix} A'^{a} & A'^{b}_m & & & \\ & A'^{a}_m & A'^{b}_{m+1} & & \\ & & \ddots & & \\ & & A'^{a}_{M-1} & A'^{b}_M & \\ & & & \ddots & \\ & & & & A'^{a}_M \end{bmatrix} \quad \text{...(73)}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 1 \end{bmatrix} \quad \text{...(74)}$$

for M angles (M rows and M columns);

$$\mathbf{U}_{n-1}^{i-1,+} = \begin{bmatrix} U(r_{i-1}, & +\mu_{m-1}, & x_{n-1} \\ U(r_{i-1}, & +\mu_m, & x_{n-1} \\ \vdots & & \\ U(r_{i-1}, & \mu_M, & x_{n-1} \end{bmatrix} \quad \text{...(75)}$$

$$\mathbf{Y}_{n-1}^{i-1,+} = \sum_{n'=-N}^{N} a_{n'} \phi_{n'}^{i-1} \sum_{m'=1}^{M} C_{m'} U_{m',n-1}^{i-1,+}. \quad \text{...(76)}$$

The matrices A^{dc}, A^{fg}, A^{eh}, D^{dg}, D^{eh} are expressed in a similar way.

Defining

$$A_q^{ab} = Q^{-1}A^{ab},\ A_q^{dc} = Q^{-1}A^{dc},\ A_q^{fg} = Q^{-1}A^{fg},\ A_q^{eh} = Q^{-1}A^{eh},$$

$$D_q^{ab} = Q^{-1}D^{ab},\ D_q^{dc} = Q^{-1}D^{dc},\ D_q^{fg} = Q^{-1}D^{fg},\ D_q^{eh} = Q^{-1}D^{eh}, \quad \ldots(77)$$

we rewrite equations (70) and (71) :

$$\begin{aligned}(A_q^{ab} U_{n-1}^{i-1,+} &- \tau^- \sigma' \phi_{n-1}^{i-1,+} Y_{n-1}^{i-1,+}) + (A_q^{dc} U_n^{i-1,+} - \tau^- \sigma' \phi_n^{i-1,+} Y_n^{i-1,+}) \\ &+ (A_q^{fg} U_{n-1}^{i,+} - \tau^+ \sigma' \phi_{n-1}^{i,+} Y_{n-1}^{i,+}) + (A_q^{eh} U_n^{i,+} - \tau^+ \sigma' \phi_n^{i,+} Y_n^{i,+}) \\ &= \tau^- \sigma' (\phi_{n-1}^{i-1,+} Y_{n-1}^{i-1,-} + \phi_n^{i-1,+} Y_n^{i-1}) + \tau^+ \sigma' (\phi_{n-1}^{i,+} Y_{n-1}^{i,-} \\ &+ \phi_n^{i,+} Y_n^{i,-}) + \tau^- \sigma' (\rho' \beta' + \epsilon' \phi_{n-1}^{i-1,+}) B_{n-1}^{i-1,+} \\ &+ \tau^- \sigma' (\rho' \beta' + \epsilon' \phi_n^{i-1,+}) B_n^{i-1,+} + \tau^+ \sigma' (\rho' \beta' + \epsilon' \phi_{n-1}^{i,+}) B_{n-1}^{i,+} \\ &+ \tau^+ \sigma' (\rho' \beta' + \epsilon' \phi_n^{i,+}) B_n^{i,+} \end{aligned} \quad \ldots(78)$$

and

$$\begin{aligned}(D_q^{ab} U_{n-1}^{i-1,+} &- \tau^- \sigma' \phi_{n-1}^{i-1,-} Y_{n-1}^{i-1,-}) + (D_q^{dc} U_n^{i-1,-} - \tau^- \sigma' \phi_n^{i-1,-} Y_n^{i-1,-}) \\ &+ (D_q^{fg} U_{n-1}^{i,-} - \tau^+ \sigma' \phi_{n-1}^{i,-} Y_{n-1}^{i,-}) + (D_q^{eh} U_n^{i,-} - \tau^+ \sigma \phi^{i,-} Y_n^{i,-}) \\ &= \tau^- \sigma' (\sigma_{n-1}^{i-1,-} Y_{n-1}^{i-1,+} + \phi_n^{i-1,-} Y_n^{i-1,+}) \\ &+ \tau^+ \sigma' (\phi_{n-1}^{i,-} Y_{n-1}^{i,-} + \phi_n^{i,-} Y_n^{i,+}) + \tau^+ \sigma' (\rho' \beta' + \epsilon' \phi_{n-1}^{i-1,-}) B_{n-1}^{i-1,-} \\ &+ \tau^- \sigma' (\rho' \beta' + \epsilon' \phi_n^{i-1,-}) B_n^{i-1,-} + \tau^- \sigma' (\rho' \beta' + \epsilon' \phi_{n-1}^{i,-}) B_{n-1}^{i,-} \\ &+ \tau^+ \sigma' (\rho' \beta' + \phi_n^{i,-}) B_n^{i,-}. \end{aligned} \quad \ldots(79)$$

We now rewrite equations (78) and (79) for $N(1 \leqslant n \leqslant N)$ frequency points :

$$\begin{aligned}(\bar{A}_{dc}^{ab} - \sigma' \tau^- F_{i-1}^{++}) U_{i-1}^+ &+ (\bar{A}_{eh}^{fg} - \sigma' \tau^+ F_i^{++}) U_i^+ \\ &= \sigma' \tau^- F_{i-1}^{+-} U_{i-1}^- + \sigma' \tau^+ F_i^+ U_i^- + \sigma' \tau^- \Phi_{i-1}^+ B_{i-1}^+ + \sigma' \tau^+ \Phi_i^+ B_i^+ , \end{aligned} \quad \ldots(80)$$

$$\begin{aligned}(\bar{D}_{dc}^{ab} - \sigma' \tau^- F_{i-1}^{--}) U_{i-1}^- &+ (\bar{D}_{eh}^{fg} - \sigma' \tau^+ F_i^{--}) U_i^- \\ &= \sigma' \tau^- F_{i-1}^{-+} U_{i-1}^+ + \sigma' \tau^+ F_i^{-+} U_i^+ + \sigma' \tau^- \Phi_{i-1}^- \bar{B}_{i-1} + \sigma' \tau_i^+ \Phi_i^- B_i^- , \end{aligned} \quad \ldots(81)$$

where $\bar{A}_{dc}^{ab} = Q_F^{-1} A_{dc}^{ab}$. .. (82)

Matrices $\bar{A}^{fg}_{eh}$, $\bar{D}^{ab}_{dc}$ and $\bar{D}^{fg}_{eh}$ are similarly defined.

$$\mathbf{Q}_F = \begin{bmatrix} \mathbf{I} & \mathbf{I} & & & \\ & \mathbf{I} & \mathbf{I} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{I} & \mathbf{I} \\ & & & & \mathbf{I} \end{bmatrix} \quad \ldots(83)$$

where **I** is the identity matrix. Furthermore

$$\mathbf{A}^{ab}_{dc} = \begin{bmatrix} \mathbf{A}^{ab}_{q,n-1} & \mathbf{A}^{dc}_{q,n} & & & \\ & \mathbf{A}^{ab}_{q,n} & \mathbf{A}^{dc}_{q,n+1} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{A}^{ab}_{q,N-1} & \mathbf{A}^{dc}_{q,N} \\ & & & & \mathbf{A}^{ab}_{q,N} \end{bmatrix} \quad \ldots(84)$$

$$\mathbf{\Phi}^{+}_{i-1}\,\mathbf{Y}^{+}_{i-1} = [\mathbf{\Phi}_{+}\mathbf{\Phi}^{T}_{+}\,\mathbf{W}^{++}]\,\mathbf{U}^{+}_{i-1} = \mathbf{F}^{++}_{i-1}\,\mathbf{U}^{+}_{i-1}, \quad \ldots(85)$$

$$\mathbf{W} = [\mathbf{W}_k \delta_{kk'}], \quad \ldots(86)$$

$$\mathbf{W}_k = a_n C_m, \quad k = m + (n-1)M, \quad 1 \leqslant k \leqslant K = NM,$$

$$\mathbf{\Phi}^{+}_{i-1} = [\rho'\beta' + \epsilon'\phi^{+}_{i-1}]\,\delta_{nn'} = [\mathbf{\Phi}_{nm}], \quad \ldots(87)$$

$$\mathbf{B}^{+}_{i,i-1} = \mathbf{B}^{+}_{i,i-1}\,\mathbf{L}', \quad \ldots(88)$$

$$\mathbf{L}' = [1 \;\; 1 \ldots\ldots\ldots 1]^{T}. \quad \ldots(89)$$

$\mathbf{A}^{fg}_{eh}$, $\mathbf{D}^{ab}_{dc}$, D^{fg}_{eh}, $\mathbf{F}^{++}$, $\mathbf{F}^{+-}$, $\mathbf{F}^{--}$, $\mathbf{\Phi}^{+}$, $\mathbf{\Phi}^{-}$ can be expressed similarly.

Equations (80) and (81) can be written as (*cf.* Peraiah 1984)

$$\begin{bmatrix} \mathbf{U}^{+}_{i} \\ \mathbf{U}^{-}_{i-1} \end{bmatrix} = \begin{bmatrix} \mathbf{t}(i, i-1) & \mathbf{r}(i-1, i) \\ \mathbf{r}(i, i-1) & \mathbf{t}(i-1, i) \end{bmatrix} \begin{bmatrix} \mathbf{U}^{+}_{i-1} \\ \mathbf{U}^{-}_{i} \end{bmatrix} + \begin{bmatrix} \mathbf{\Sigma}^{+} \\ \mathbf{\Sigma}^{-} \end{bmatrix}. \quad \ldots(90)$$

Here $\mathbf{t}(i, i-1)$, $\mathbf{t}(i-1, i)$; $\mathbf{r}(i, i-1)$, $\mathbf{r}(i-1, i)$ are the transmission and reflection operators. The qqantities $\mathbf{\Sigma}^{+}$ and $\mathbf{\Sigma}^{-}$ are the source vectors, which represent the contribution to the radiation field from the interval sources.

We now introduce the following quantities :

$$X_1 = \bar{A}^{fg}_{eh} - \sigma'\tau^+F^{++}_i, \quad X_2 = \sigma'\tau^-F^{+-}_{i-1},$$

$$X_3 = \sigma'\tau^+F^{++}_{i-1}, \qquad X_4 = \bar{D}^{ab}_{dc} - \sigma'\tau^+F^{--}_{i-1}, \qquad \text{...(91)}$$

$$Y_1 = A^{ab}_{dc} - \sigma'\tau^-F^{++}_{i-1}, \quad Y_2 = \sigma'\tau^+F^{+-}_i,$$

$$Y_3 = \sigma'\tau^-F^{-+}_{i-1}, \qquad Y_4 = D^{fg}_{eh} - \sigma'\tau^+F^{--}_i, \qquad \text{...(92)}$$

$$t(i, i-1) = R^{+-}X_1^{-1}[X_2X_4^{-1}Y_3 - Y_1], \qquad \text{...(93)}$$

$$t(i-1, i) = R^{-+}X_4^{-1}[X_3X_1^{-1}Y_2 - Y_4], \qquad \text{...(94)}$$

$$r(i, i-1) = R^{-+}X_4^{-1}[Y_3 - X_3X_1^{-1}Y_1], \qquad \text{...(95)}$$

$$r(i-1, i) = R^{+-}X_1^{-1}[Y_2 - X_2X_4^{-1}Y_4], \qquad \text{...(96)}$$

$$R^{+-} = [I - X_1^{-1}X_2X_4^{-1}X_3]^{-1}, \qquad \text{...(97)}$$

$$R^{-+} = [I - X_4^{-1}X_3X_1^{-1}X_2]^{-1}, \qquad \text{...(98)}$$

$$\Sigma^+_{i-1/2} = \sigma'R^{+-}X_1^{-1}[(\tau^-\Phi^+_{i-1}B^+_{i-1} + \tau^+\Phi^+_iB^+_i) + X_2X_4^{-1}(\tau^-\Phi^-_{i-1}B^-_{i-1} + \tau^+\Phi^-_iB^-_i)] \qquad \text{...(99)}$$

$$\Sigma^-_{i-1/2} = \sigma'R^{-+}X_4^{-1}[\tau^-\Phi^-_{i-1}B^-_{i-1} + \tau^+\Phi^-_iB^-_i) + X_3X_1^{-1}(\tau^-\Phi^+_{i-1}B^+_{i-1} + \tau^+\Phi^-_iB^-_i)]. \qquad \text{...(100)}$$

We thus obtain the transmission and reflection operators in a spherical medium expanding radially with aberration and advection effects taken into account. The diffuse radiation field can be obtained by employing the procedure described in Peraiah (1984).

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