Bull. Astr. Soc. India (1986) 14, 34-39

A deconvolution method for astronomical photometric and spectroscopic observations

B. G. Anandarao and Suhasini R. Rao
Physical Research Laboratory, Ahmedabad 380 009

Received 1985 September 16; accepted 1985 December 9

Abstract. A numerical method for deconvoluting one-dimensional optical imaging is described along with its algorithm. This method can be very successfully used for achieving image enhancement and restoration with minimized noise, spurious as well as systematic, in the case of spectroscopic and photometric data in optical or infrared astronomy. An example has been given to illustrate the importance of the smoothing and restoration parameters in the method.

Key words : deconvolution—techniques

1. Introduction

In most of the astronomical photometric and spectroscopic work, the observed image of a celestial object under study is inevitably modified by what is known as instrumental function which is essentially defined by (i) aperture function of the instrument or the point-spread function, (ii) filter response, and (iii) the response time of the detector used; and many other limitations in the experimental and observational process. In all these cases, it becomes necessary to retrieve from the noisy image observed the actual signal from the object without the modifications introduced by the extraneous agencies.

Let us suppose that we observed an object with a certain instrument under certain conditions pertaining to various extraneous factors which influence the data. Then, in general, the measured function $s(t)$ does not truly represent the object function $p(t)$; but, on the other hand, it represents the convolution of the object function with a modulation or transfer function $a(t)$:

$$s(t) = p(t) * a(t), \qquad \ldots(1)$$

where the symbol $*$ represents the convolution operation. We can rewrite the above equation as

$$s(t) = \int_{-\infty}^{\infty} p(t')\, a(t - t')\, dt', \qquad \ldots(2)$$

where the integral on the right-side represents the convolution integral. Knowing $s(t)$ and $a(t)$, is is indeed possible to retrieve by direct methods the object $p(t)$ provided the observations are noise-free.

For instance, one can take the Fourier transform on both the sides of equation (1) and then a simple division followed by an inverse Fourier transformation would give the desired object function $p(t)$, in terms of the observed image $s(t)$ and the instrumental function $a(t)$ which is also called as the point-spread function (p.s.f.). However, the p.s.f. in many applications has dominant values in the central part of the data and falls off steeply to near-zero values on either side of central. maximum. This may lead to practical difficulties in obtaining the discrete Fourier transform.

Alternatively, we can write the integral equation (2) as a set of linear equations and can, in principle, solve for the unknown function $p(t)$ by matrix inversion. As already stated, the matrix elements formed by the function $a(t)$ are usually very small and this fact renders the matrix ill-conditioned. Therefore, even though, the errors (or noise) are negligibly small in $s(t)$, they will contribute to large errors in $p(t)$. Thus, it is desirable to approach the problem as if $s(t)$ had errors (however small they may be) and minimize these errors in obtaining $p(t)$.

In this note, we describe a method and its algorithm to find a solution to the problem addressed to here.

2. Mathematics of the method

Writing down the convolution integral and adding the noise term $n(t)$, equation (2) becomes,

$$s(t_m) = \int_{-\infty}^{\infty} p(t_i)\, a(t_m - t_i)\, dt + n(t_m), \qquad \ldots(3)$$

where $m = 1, 2, 3, \ldots, N$; N being the number of data points. Approximating the integral as a finite sum and assuming that there are closely spaced data-points, equation (3) can be written as

$$s(t_m) = \sum_{i=1}^{N} p(t_i)\, a(t_m - t_i)\, w_i + n_m, \qquad \ldots(4)$$

$$m = 1, 2, \ldots, N$$

where w_i are the weightage factors for the numerical method used for integration; $w_i = k_i \Delta t$ with k_i some constant; and Δt the stepsize in t. The summation approximation in equation (4) is valid when $a(|t|)$ is a fast decreasing function of t. This property is satisfied in most of the applications.

From equation (4) the variance in noise is written as

$$\sigma^2 = \frac{1}{N}\sum_{m=1}^{N}\left(n_m^2\right) = \frac{1}{N}\sum_{m=1}^{N}\left[s(t_m) - \sum_{i=1}^{N} p(t_i)\, a(t_m - t_i)\, w_i\right]^2. \quad ...(5)$$

Now the condition for minimization of σ^2 is

$$\frac{\partial \sigma^2}{\partial p_i} = 0 \qquad \text{for } i = 1, 2, .., N. \quad ...(6)$$

Substituting equation (5) in (6), we obtain the following set of linear equations :

$$\sum_{j=1}^{N} A_{ij}\, p_j = B_i,\ i = 1, 2, ..., N \quad ...(7)$$

where, $A_{ij} = \frac{1}{N}\sum_{m=1}^{N} w_i\, a(t_m - t_i)\, w_j a(t_m - t_j)$

and $$B_i = \frac{1}{N}\sum_{m=1}^{N} w_i a(t_m - t_i) s_m. \quad ...(8)$$

The object function p_j can now be obtained by solving equation (7).

3. Smoothing

The solution obtained by solving equation (7) is known to be contaminated, in general, by spurious ripples (Phillips 1962). This is because of the ill-conditioning of the matrix $[A]$ involving the p.s.f., $a(t)$. To avoid these spurious ripples, in addition to the minimization of the variance of the noise, we seek to impose smoothness constraints on the solution in which we choose an operator C together with a parameter γ and minimize

$$\| ap - s \|^2 + \gamma \| Cp \|^2.$$

In this method, known as regularization (Phillips 1962), the operator C can be I, $\frac{d}{dt}$ or $\frac{d^2}{dt^2}$, and so on. A proper selection of γ removes the ill-posed nature of the problem.

Here in our present discussion, we use C as $\frac{d^2}{dt^2}$ and use the numerically equivalent second differences for minimization.

Let us now define

$$\epsilon^2 = \frac{\gamma}{(N-1)}\sum_{i=2}^{N}\left[p(t_{i+1}) - 2\,p(t_i) + p(t_{i-1})\right]^2. \quad ...(9)$$

Then the condition for minima in equation (6) becomes

$$\frac{\partial(\sigma^2 + \epsilon^2)}{\partial p_1} = 0. \qquad \ldots(10)$$

This leads to the following equation

$$\sum_{j=1}^{N} (A_{1j} + \gamma C_{1j}), p_j = B_1, \qquad i = 1, 2, \ldots, N \qquad \ldots(11)$$

which is a modified version of equation (7) with A_{1j} and B_1 having the same definitions as before, and with

$$C_{1j} = \frac{1}{(N-1)} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \ldots & \ldots \\ -2 & 5 & -4 & 0 & 0 & 0 & \ldots \\ 1 & -4 & 6 & -4 & 1 & 0 & \ldots \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ . & & . & . & . & . & . \end{bmatrix}.$$

The parameter γ is to be so chosen as to give importance to both the observed data and the regularization or smoothing required. In other words, there will be a particular value of γ, say γ_c, above which the smoothing process dominates and we start sacrificing the information or the self-consistency, while below the value of γ_c, the ripples will dominate the solution. In general the value of γ_c has to be chosen in accordance with the signal-to-noise ratio and after a few trials we can fix the value of γ_c for the desired and meaningful solution $p(t)$.

4. Algorithm

The algorithm of the method described above is as follows :

(i) We have the observed data set $s(t)$ at $t = t_1, t_2, \ldots, t_N$.

(ii) We assume the point-spread function $a(t)$ at $t = t_1, t_2, \ldots, t_N$.

(iii) Then we form the constant matrix C.

(iv) We choose the weightage factors $w_i (i = 1, 2, \ldots, N)$ depending upon the numerical method of integration. And we fix the stepsize Δt.

(v) The column matrix elements

$$B_1 = \frac{1}{N} \sum_{m=1}^{N} w_i a(t_m - t_i)\, s_m; \; i = 1, 2, \ldots, N$$

are computed and the matrix B is formed.

(vi) Then the matrix elements

$$A_{ij} = \frac{1}{N} \sum_{m=1}^{N} w_i a(t_m - t_i)\ w_j a(t_m - t_j)$$

for $i = 1, 2, 3, \ldots, N$ and $j = 1, 2, \ldots, N$ are computed and the matrix A is formed.

(vii) We assign a certain value to γ.

(viii) Then the solution $p_i (i = 1, 2, \ldots, N)$ is initialized by assuming, for instance,

$$p_1^0 = 10^{-10} \text{ for all } i.$$

(ix) The matrix equation $(A + \gamma C)\ P = B$ is solved for P

(x) The error, then is computed by $E = -\gamma BP$ and check whether

$$\eta = \max_i |E_i| \leqq \eta_0,$$

a constant value representing the variance of the noise originally present in the data-set. If this inequality is satisfied then the deconvolution procedure ends. If not then we start from the step (vii) all over again by changing the value of γ. Sometimes, one may use the value of η for γ and go to step (vii) — in each iteration substituting the new value of η for γ.

We have used Gauss-Seidel iterative method for solving the matrix equation above, and found that a reasonably fast convergence to a solution is achieved.

5. Example

We have successfully applied the deconvolution method discussed above to far-infrared photometric data. The data were obtained by a far-infrared photometer at the Cassegrain focus of a 32.5 cm telescope on board a Caravelle aircraft. The specific data used refer to the planet Venus in 100 micron photometric band. The planet was observed in a scanning mode with a field aperture of 1.4 arcmin. Since the planet's angular size is about 8 aresec one can consider this as a point source and use it to obtain the instrumental profile or the point-spread function. In order to check this, we have used the actual observations on Venus and deconvolved these with a point-spread function of Gaussian shape with a full width at half-maximum intensity of 1.4 arcmin. The result is shown in figure 1. One can see that there is almost a perfect agreement between the actual data and the deconvolved profile. This proves that a point source like a planet can be used to obtain the point-spread function even at far-infrared wavelengths (Van der Wal *et al.* 1985).

6. Conclusions

The following conclusions emerge regarding the application of the method :

(i) While approximating the convolution integral as a finite sum, in many cases the Simpson's rule is adequate; if the data points are sufficiently closely spaced, even the trapezoidal rule may be used.

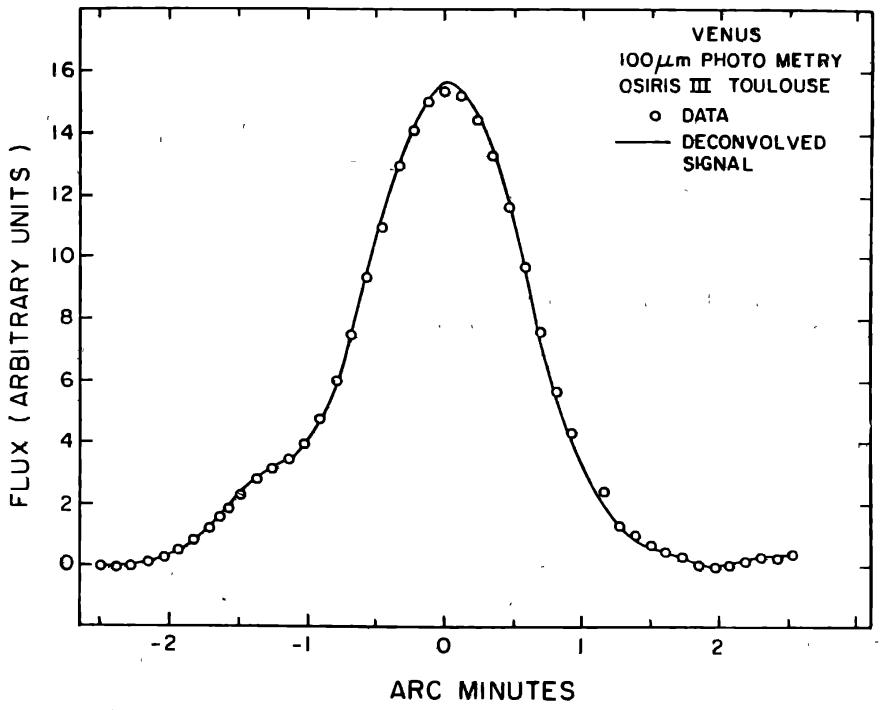

Figure 1. Far-infrared photometry of planet Venus. The beam size is 1.4 arcmin. The curve shows the deconvolved profile while the circles represent the data points (after Van der Wal *et al.* 1985).

(ii) The operator C can, in general, be taken as a second-order difference; however, if necessary, in some cases one may have to go to higher order differences.

(iii) The choice of the critical parameter γ_c depends on the observational errors; generally γ_c can be determined by trial.

Acknowledgement

This work was supported by the Department of Space, Government of India.

References

Phillips, D. L. (1962) *J. Assoc. Comp. Mach.* **9**, 84.
Van der Wal, P. B. *et al.* (1985) *Ap. Sp. Sci.* **117**, 209.