Bull. Astr. Soc. India (1986) 14, 11–14

Generating solution of the restricted three-body problem in 3-dimension

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Received 1985 May 25; accepted 1985 December 1

Abstract. As it is well known, the KS-transformation is a transformation which regularizes the equations of Keplerian motion in the three dimensional physical space to the four dimensional phase space. Kurcheeva (1977) has given canonical equations of motion for the circular restricted three body problem using KS-regularization.

This paper presents the generating solution of the circular restricted three body problem in four dimensional phase space.

Key words : restricted three body problem—KS-regularization

1. Equations of motion

If q_1, q_2, q_3, q_4 be the generalized coordinates and Q_1, Q_2, Q_3, Q_4 the corresponding generalized components of momentum of the infinitesimal mass, and μ be the ratio of the mass of the smaller primary to the total mass of the primaries, then the regularized canonical equations of motion of the 3-D restricted three body problem, as given by Kurcheeva (1977), are

$$\frac{dq_i}{ds} = \frac{\partial \Omega}{\partial Q_i}, \qquad \frac{dQ_i}{ds} = -\frac{\partial \Omega}{\partial q_i}, \quad (i = 1, 2, 3, 4) \qquad ...(1)$$

where

$$\Omega = \tfrac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) + 2\rho^2(Q_1 q_2 - Q_2 q_1 - Q_3 q_4 + Q_4 q_3) -4(1-\mu) + 4\rho^2[\tfrac{1}{2}c - \tfrac{1}{2}\mu^2 + \mu(q_1^2 - q_2^2 + q_3^2 - q_4^2) -\mu/r_2],$$

c = Jacobi constant,

$$T = Q_1 q_3 - Q_3 q_1 - Q_2 q_4 + Q_4 q_2 = 0.$$

The distance between the infinitesimal mass and the bigger primary is given by

$$r_1 = (q_1^2 + q_2^2 + q_3^2 + q_4^2) = \rho^2.$$

The distance between the infinitesimal mass and the smaller primary is given by

$$r_2^2 = 1 - 2(q_1^2 - q_2^2 + q_3^2 - q_4^2) + \rho^4.$$

The physical time t and the pseudo times s are connected by the relation

$$dt = 4\rho^2\, ds. \quad \ldots(2)$$

2. Limiting case for $\mu = 0$

For $\mu = 0$ we get

$$\Omega = \tfrac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) + 2\rho^2(Q_1 q_2 - Q_2 q_1 - Q_3 q_4 + Q_4 q_3) - 4 + 2c\rho^2.$$

$$\left.\begin{aligned} \therefore\ \dot q_1 &= Q_1 + 2\rho^2 q_2 \Rightarrow Q_1 = \dot q_1 - 2\rho^2 q_2,\\ \dot q_2 &= Q_2 - 2\rho^2 q_1 \Rightarrow Q_2 = \dot q_2 + 2\rho^2 q_1,\\ \dot q_3 &= Q_3 - 2\rho^2 q_4 \Rightarrow Q_3 = \dot q_3 + 2\rho^2 q_4,\\ \text{and}\quad \dot q_4 &= Q_4 + 2\rho^2 q_3 \Rightarrow Q_4 = \dot q_4 - 2\rho^2 q_3, \end{aligned}\right\} \quad \ldots(3)$$

Hence from equations (1), we get

$$\ddot q_1 - 4\rho^2 \dot q_2 - (q_1^2 + q_2^2)\,\dot q_2 - 4\,(q_2 q_3 + q_1 q_4)\,\dot q_3 - 4\,(q_2 q_4 - q_1 q_3)\,\dot q_4 = 12\rho^4 q_1 - 4c q_4,$$

$$\ddot q_2 + 4\rho^2 \dot q_1 + 4\,(q_1^2 + q_2^2)\,\dot q_1 + 4(q_1 q_3 - q_2 q_4)\,\dot q_3 + 4(q_1 q_4 + q_2 q_3)\,\dot q_4 = 12\rho^4 q_2 - 4c q_2,$$

$$\ddot q_3 + 4\rho^2 \dot q_4 + 4(q_1 q_4 + q_2 q_3)\,\dot q_1 + 4\,(q_2 q_4 - q_1 q_3)\,\dot q_2 + 4\,(q_3^2 + q_4^2)\,\dot q_4 = 12\rho^4 q_3 - 4c q_3,$$

$$\ddot q_4 - 4\rho^2 \dot q_3 - 4\,(q_1 q_3 - q_2 q_4)\,\dot q_1 - 4\,(q_2 q_3 + q_1 q_4)\,\dot q_2 - 4\,(q_3^2 + q_4^2)\,\dot q_3 = 12\rho^4 q_4 - 4c q_4. \quad \ldots(4)$$

Now adding equations (4) after multiplying them by $\dot q_1, \dot q_2, \dot q_3, \dot q_4$ respectively and then integrating, we get

$$\dot q_1^2 + \dot q_2^2 + \dot q_3^2 + \dot q_4^2 = 4\rho^6 - 4c\rho^2 + 8. \quad \ldots(5)$$

Here the constant of integration is taken to be 8, the same value as for the two-body problem.

Again, multiplying equations (4) by q_2, q_1, q_4, and q_3 respectively and subtracting the sum of the second and third from the sum of the first and fourth and then integrating, we get

$$q_2\dot{q}_1 - q_1\dot{q}_2 - q_4\dot{q}_3 + q_3\dot{q}_4 - 2\rho^4 = b, \qquad ...(6)$$

where b is the constant of integration.
We take

$$q_1 = \frac{\rho}{\sqrt{2}} \sin\left(\frac{\bar{\Omega} - \omega}{2}\right), \quad q_2 = \frac{\rho}{\sqrt{2}} \cos\left(\frac{\bar{\Omega} - \omega}{2}\right),$$

$$q_3 = \frac{\rho}{\sqrt{2}} \cos\left(\frac{\bar{\Omega} + \omega}{2}\right), \text{ and } q_4 = \frac{\rho}{\sqrt{2}} \sin\left(\frac{\bar{\Omega} + \Omega}{2}\right), \qquad ...(7)$$

where $\bar{\Omega}$ and ω are unknown variables.
This will be a solution if equations (5) and (6) are satisfied; *i.e.*, if

$$\frac{\rho^2}{2}\dot{\bar{\Omega}} = 2\rho^4 + b.$$

Following Krasinski (1963), we suppose that $\rho = 0$ when $s = 0$.
Then we find $b = 0$.

Hence $\dot{\bar{\Omega}} = 4\rho^2$. ...(8)

Now from equations (5), (7) and (8), we get

$$\dot{\rho}^2 + \frac{\rho^2}{4}\dot{\omega}^2 = 8 - 4c\rho^2.$$

In the case of plane-restricted three-body problem with $\mu = 0$ we have $\dot{\rho}^2 = 8 - 4c\rho^2$ (Krasinski 1963). This will also be the case if $\dot{\omega} = 0$, *i.e.*, $\omega =$ constant.
Hence, we get

$$\rho = (2/c)^{1/2} \cos(2\sqrt{c}\, s - \phi_0),$$

$$\bar{\Omega} = \frac{1}{n}\ [4\sqrt{c}\, s + \sin(4\sqrt{c}\, s - 2\phi_0)] + \omega_0,$$

where ϕ_0 and ω_0 are constants of integration. Now instead of $\bar{\Omega}$ and ω we can take the variables θ and ϕ which are given by

$$\theta = \frac{\bar{\Omega} - \omega}{2} = \frac{1}{2n}[4\sqrt{c}\, s + \sin(4\sqrt{c}\, s - 2\phi_0)] + \omega_1,$$

$$\phi = \frac{\bar{\Omega} + \omega}{2} = \frac{1}{2n}[4\sqrt{c}\, s + \sin(4\sqrt{c}\, s - 2\phi_0)] + \omega_2, \qquad ...(9)$$

where $\omega_1 = \dfrac{\omega_0 - \omega}{2}$ and $\omega_2 = \dfrac{\omega_0 + \omega}{2}$ and hence ω_1 and ω_2 are constants.

Using relations (3) and (9), the generating solutions can be written as

$$q_1 = \frac{\rho}{\sqrt{2}} \sin\theta,\ q_2 = \frac{\rho}{\sqrt{2}} \cos\theta,\ q_3 = \frac{\rho}{\sqrt{2}} \cos\phi,\ q_4 = \frac{\rho}{\sqrt{2}} \sin\phi,$$

$$Q_1 = \frac{\dot{\rho}}{\sqrt{2}} \sin\theta,\ Q_2 = \frac{\dot{\rho}}{\sqrt{2}} \cos\theta,\ Q_3 = \frac{\dot{\rho}}{\sqrt{2}} \cos\phi,\ Q_4 = \frac{\dot{\rho}}{\sqrt{2}} \sin\phi, \quad \ldots(10)$$

where

$$\rho = (2/c)^{1/2} \cos(2\sqrt{c}\, s - \phi_0)$$

$$\theta = \frac{1}{2n}\left[4\sqrt{c}\, s + \sin(4\sqrt{c}\, s - 2\phi_0)\right] + \omega_1,$$

and

$$\phi = \frac{1}{2n}\left[4\sqrt{c}\, s + \sin(4\sqrt{c}\, s - 2\phi_0)\right] + \omega_2$$

$$(n = c\sqrt{c} = (\sqrt{c})^3 = \text{mean motion}).$$

It is not difficult to see that the solutions (10) will be periodic only in the case when n is a rational number, *i.e.* if it may be represented in the form $n = k/m$, where k and m are mutually prime integers. (k and m are called the characteristic numbers of the periodic solution). The period of such a solution is equal to

$$S = \begin{cases} \dfrac{\pi k}{2\sqrt{c}}, & \text{if } (k + m) \text{ is an even number} \\[2ex] \dfrac{\pi k}{\sqrt{c}}, & \text{if } (k + m) \text{ is an odd number.} \end{cases}$$

Substituting the values of q's from the relation (10) in (2) and then integrating we get the following relations between t and s for $\mu = 0$

$$t = 2(\theta - \omega_1) = 2(\phi - \omega_2) = \frac{1}{n}\left[4\sqrt{c}\, s + \sin(4\sqrt{c}\, s - 2\phi_0)\right]$$

Acknowledgement

We thank Dr R. K. Choudhry for his valuable suggestions.

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