

An Introduction to  
**Radiative Transfer**

Methods and applications in astrophysics

Annamaria Perrotta



Springer

## An Introduction to Radiative Transfer

Methods and applications in astrophysics

Astrophysicists have developed several very different methodologies for solving the radiative transfer equation. *An Introduction to Radiative Transfer* presents these techniques as applied to stellar atmospheres, planetary nebulae, supernovae and other objects with similar geometrical and physical conditions. Accurate methods, fast methods, probabilistic methods and approximate methods are all explained, including the latest and most advanced techniques. The book includes the different methods used for computing line profiles, polarization due to resonance line scattering, polarization in magnetic media and similar phenomena. Exercises at the end of each chapter enable these methods to be put into practice, and enhance understanding of the subject. This textbook will be of great value to graduates, postgraduates and researchers in astrophysics.

ANNAMANENI PERAIAH obtained his doctorate in radiative transfer from Oxford University. He was formerly a Senior Professor at the Indian Institute of Astrophysics, Bangalore, India. He has held positions in India, Canada, Germany and the Netherlands. His research interests include developing solutions to the radiative transfer equation in stellar atmospheres and line formation in expanding atmospheres with different physical and geometrical conditions.

# An Introduction to Radiative Transfer

## Methods and applications in astrophysics

Annamaneni Peraiah



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## Contents

Preface xi

### Chapter 1 Definitions of fundamental quantities of the radiation field 1

- 1.1 Specific intensity 1
- 1.2 Net flux 2
- 1.2.1 *Specific luminosity* 4
- 1.3 Density of radiation and mean intensity 5
- 1.4 Radiation pressure 7
- 1.5 Moments of the radiation field 8
- 1.6 Pressure tensor 8
- 1.7 Extinction coefficient: true absorption and scattering 9
- 1.8 Emission coefficient 10
- 1.9 The source function 12
- 1.10 Local thermodynamic equilibrium 12
- 1.11 Non-LTE conditions in stellar atmospheres 13
- 1.12 Line source function for a two-level atom 15
- 1.13 Redistribution functions 16
- 1.14 Variable Eddington factor 25
- Exercises 25
- References 27

### Chapter 2 The equation of radiative transfer 29

- 2.1 General derivation of the radiative transfer equation 29
- 2.2 The time-independent transfer equation in spherical symmetry 30
- 2.3 Cylindrical symmetry 32

2.4	The transfer equation in three-dimensional geometries	33
2.5	Optical depth	38
2.6	Source function in the transfer equation	39
2.7	Boundary conditions	40
2.8	Media with only either absorption or emission	41
2.9	Formal solution of the transfer equation	42
2.10	Scattering atmospheres	44
2.11	The $K$ -integral	46
2.12	Schwarzschild–Milne equations and $\Lambda$ , $\Phi$ , $X$ operators	47
2.13	Eddington–Barbier relation	51
2.14	Moments of the transfer equation	52
2.15	Condition of radiative equilibrium	53
2.16	The diffusion approximations	53
2.17	The grey approximation	55
2.18	Eddington’s approximation	56
	Exercises	58
	References	63

**Chapter 3 Methods of solution of the transfer equation** 64

3.1	Chandrasekhar’s solution	64
3.2	The $H$ -function	70
3.2.1	<i>The first approximation</i>	72
3.2.2	<i>The second approximation</i>	73
3.3	Radiative equilibrium of a planetary nebula	74
3.4	Incident radiation from an outside source	75
3.5	Diffuse reflection when $\omega = 1$ (conservative case)	78
3.6	Iteration of the integral equation	79
3.7	Integral equation method. Solution by linear equations	82
	Exercises	83
	References	86

**Chapter 4 Two-point boundary problems** 88

4.1	Boundary conditions	88
4.2	Differential equation method. Riccati transformation	90
4.3	Feautrier method for plane parallel and stationary media	92
4.4	Boundary conditions	93
4.5	The difference equation	94
4.6	Rybicki method	99
4.7	Solution in spherically symmetric media	101

4.8	Ray-by-ray treatment of Schmid-Burgk	106
4.9	Discrete space representation	108
	Exercises	109
	References	110
<b>Chapter 5</b>	<b>Principle of invariance</b>	<b>112</b>
5.1	Glass plates theory	112
5.2	The principle of invariance	116
5.3	Diffuse reflection and transmission	117
5.4	The invariance of the law of diffuse reflection	119
5.5	Evaluation of the scattering function	120
5.6	An equation connecting $I(0, \mu)$ and $S_0(\mu, \mu')$	123
5.7	The integral for $S$ with $p(\cos \Theta) = \varpi(1 + x \cos \Theta)$	125
5.8	The principle of invariance in a finite medium	126
5.9	Integral equations for the scattering and transmission functions	130
5.10	The $X$ - and the $Y$ -functions	133
5.11	Non-uniqueness of the solution in the conservative case	135
5.12	Particle counting method	137
5.13	The exit function	139
	Exercises	143
	References	144
<b>Chapter 6</b>	<b>Discrete space theory</b>	<b>146</b>
6.1	Introduction	146
6.2	The rod model	147
6.3	The interaction principle for the rod	148
6.4	Multiple rods: star products	150
6.5	The interaction principle for a slab	152
6.6	The star product for the slab	154
6.7	Emergent radiation	157
6.8	The internal radiation field	158
6.9	Reflecting surface	163
6.10	Monochromatic equation of transfer	163
6.11	Non-negativity and flux conservation in cell matrices	168
6.12	Solution of the spherically symmetric equation	171
6.13	Solution of line transfer in spherical symmetry	179
6.14	Integral operator method	185
	Exercises	190
	References	191

**Chapter 7 Transfer equation in moving media: the observer frame 193**

- 7.1 Introduction 193
- 7.2 Observer's frame in plane parallel geometry 194
- 7.3 Wave motion in the observer's frame 199
- 7.4 Observer's frame and spherical symmetry 201
  - 7.4.1 *Ray-by-ray method* 201
  - 7.4.2 *Observer's frame and discrete space theory* 205
  - 7.4.3 *Integral form due to Averett and Loeser* 209
- Exercises 215
- References 215

**Chapter 8 Radiative transfer equation in the comoving frame 217**

- 8.1 Introduction 217
- 8.2 Transfer equation in the comoving frame 218
- 8.3 Impact parameter method 220
- 8.4 Application of discrete space theory to the comoving frame 225
- 8.5 Lorentz transformation and aberration and advection 238
- 8.6 The equation of transfer in the comoving frame 244
- 8.7 Aberration and advection with monochromatic radiation 247
- 8.8 Line formation with aberration and advection 251
- 8.9 Method of adaptive mesh 254
  - Exercises 261
  - References 262

**Chapter 9 Escape probability methods 264**

- 9.1 Surfaces of constant radial velocity 264
- 9.2 Sobolev method of escape probability 266
- 9.3 Generalized Sobolev method 275
- 9.4 Core-saturation method of Rybicki (1972) 282
- 9.5 Scharmer's method 287
- 9.6 Probabilistic equations for line source function 297
  - 9.6.1 *Empirical basis for probabilistic formulations* 297
  - 9.6.2 *Exact equation for  $S/B$*  300
  - 9.6.3 *Approximate probabilistic equations* 301
- 9.7 Probabilistic radiative transfer 303
- 9.8 Mean escape probability for resonance lines 310
- 9.9 Probability of quantum exit 312
  - 9.9.1 *The resolvents and Milne equations* 319



Exercises 324

References 326

**Chapter 10 Operator perturbation methods 330**

10.1 Introduction 330

10.2 Non-local perturbation technique of Cannon 331

10.3 Multi-level calculations using the approximate lambda operator 338

10.4 Complete linearization method 345

10.5 Approximate lambda operator (ALO) 348

10.6 Characteristic rays and ALO-ALI techniques 353

Exercises 359

References 359

**Chapter 11 Polarization 362**

11.1 Elliptically polarized beam 363

11.2 Rayleigh scattering 365

11.3 Rotation of the axes and Stokes parameters 367

11.4 Transfer equation for  $I(\theta, \phi)$  368

11.5 Polarization under the assumption of axial symmetry 373

11.6 Polarization in spherically symmetric media 376

11.7 Rayleigh scattering and scattering using planetary atmospheres 387

11.8 Resonance line polarization 397

Exercises 412

References 413

**Chapter 12 Polarization in magnetic media 416**12.1 Polarized light in terms of  $I, Q, U, V$  416

12.2 Transfer equation for the Stokes vector 418

12.3 Solution of the vector transfer equation with the Milne–Eddington approximation 421

12.4 Zeeman line transfer: the Feautrier method 423

12.5 Lambda operator method for Zeeman line transfer 426

12.6 Solution of the transfer equation for polarized radiation 428

12.7 Polarization approximate lambda iteration (PALI) methods 433

Exercises 438

References 439

**Chapter 13 Multi-dimensional radiative transfer 441**

13.1 Introduction 441

Cambridge University Press

0521770017 - An Introduction to Radiative Transfer: Methods and Applications in Astrophysics - Annamaneni  
Peraiah

Frontmatter

[More information](#)

## x Contents

13.2	Reflection effect in binary stars	442
13.3	Two-dimensional transfer and discrete space theory	449
13.4	Three-dimensional radiative transfer	452
13.5	Time dependent radiative transfer	455
13.6	Radiative transfer, entropy and local potentials	460
13.7	Radiative transfer in masers	466
	Exercises	466
	References	467
	Symbol index	469
	Index	477

## Preface

Astrophysicists analyse the light coming from stellar atmosphere-like objects with widely differing physical conditions using the solution of the equation of radiative transfer as a tool. A method of obtaining the solution of the transfer equation developed to suit a given physical condition need not necessarily be useful in a situation with different physical conditions. Furthermore, each individual has his/her preferences to a particular type of methodology. These factors necessitated the development of several widely differing methods of solving the transfer equation.

In the second half of the twentieth century several books were written on the subject of radiative transfer: one each by Chandrasekhar, Kourganoff and Sobolev, two books by Mihalas, two by Kalkofen and more recently two books by Sen and Wilson. These books, which describe the developments of the transfer theory, will remain milestones. They will be of great value to the researcher in this field. A beginner needs to understand the basic concepts and the initial development of the subject to proceed to use the latest advances. It is felt that it is necessary to have a book on radiative transfer which presents a comprehensive view of the subject as applied in astrophysics or more particularly in stellar atmospheres and objects with similar geometrical and physical conditions. This book serves such a purpose. Several methods are presented in the book so that the students of radiative transfer can familiarise themselves with the techniques old and new.

It became a daunting task to include all the existing techniques in the book as there is a restriction on its size. This resulted in leaving out a few methods that are of equal interest as those that appear in the book. I apologize to the authors of these methods in advance. The subject matter of the book assumes of the student a knowledge of basic mathematics and physics at the undergraduate level. This book

is intended to be included in the advanced course work of undergraduate students, and the course work of graduate students. Several exercises have been included at the end of each chapter for practising the concepts described in the chapter. These problems are straightforward and can be solved by direct application of the theory. Some of them involve just supplying the intermediate steps in the derivations of the chapter.

The material in the book is largely drawn from the books mentioned earlier and from various other references cited at the end of each chapter. If there are any errors these are mine and I shall be grateful if these are brought to my attention. Any suggestions for improvements and corrections are welcome.

It is a pleasure to thank Dr W. Kalkofen for a brief discussion on the subject matter of the book. I am grateful to Professor K. K. Sen for not only giving a few tips on writing books but also for going through the first draft and pointing out several typographical errors and adding a few conceptual points. This book would not have been possible without the active help from Mr Baba Anthony Varghese who very patiently typed the text. His phenomenal computer expertise enabled the book to rapidly and easily take its present form. It is pleasure to thank him for all this. I thank Drs A. Vagiswari and Christina Louis for their magnanimous and kind help in securing me any reference that I needed. Further, I thank Mr M. Srinivasa Rao, Mr S. Muthukrishnan and Mrs Pramila Kaveriappa for helping me in various ways during the writing of the book.

There is one person whose memory always lingers on in my mind – that of Professor M. K. Vainu Bappu. From him I have learnt several aspects not only of science but also of life. I fondly cherish the memory of my association with him.

I am grateful to my wife Jayalakshmi and my children Rajani (Vaidhyanathan), Chandra (Edith) and Usha (Madhusudan) – spouses in brackets – for the love and affection shown to me.

Finally I thank the staff of Cambridge University Press who have been connected with the publication of the book, especially Dr Simon Mitton and Miss Jacqueline Garget for clearing my doubts from time to time and Ms Maureen Storey, who very patiently went through the manuscript and suggested several corrections.

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