Bull. Astr. Soc. India (1985) 13, 372–386

Charged particle trajectories in the field of a charge near a Schwarzschild blackhole

V. P. Sonar*§, S. V. Dhurandhar† and N. Dadhich*
University of Poona, Pune 411 007

Received 1985 May 29; accepted 1985 September 11

Abstract. We consider motion of charged particles in the field of a charge placed near the Schwarzschild blackhole. The electromagnetic field gets modified owing to gravitational field of the blackhole. The system, charge plus the hole, is axisymmetric (no longer spherically symmetric) which poses difficulty in obtaining analytic solutions of equations of motion. However, motion along the axis and circular orbits about the axis of symmetry are discussed. In view of the asymmetry in charge distribution, a particle will have circular orbits only off the equatorial plane.

Key words : blackhole—charged particles—charged particle orbits

1. Introduction

The problem of the electric field generated by a charged object in the vicinity of a nonrotating uncharged blackhole has been discussed by Cohen & Wald (1971) and by Hanni & Ruffini (1972). Maxwell's equations in curved spacetime are treated in the perturbative approximation for obtaining the field. The multipole expansion of the field is worked out and it is shown that field remains well behaved on the horizon. At large distances the multipole moments fall off quickly and only the monopole term remains, indicating the Coloumbic nature of the electric field as for the Reissner-Nordström blackhole. However close to the charged object and the blackhole the field differs. The lines of force emanating from the charged object are no longer radial (Hanni & Ruffini 1972). This is due to the existence of the higher multipole moments than the monopole.

The system of charge and blackhole produces a composite force field, electrostatic and gravitational. We use the orbits of the charge particles to probe the

§Permanent address : K. S. K. College, Beed 431 122.
*Department of Mathematics.
†Department of Physics.

nature of this field. The trajectories in general will not be geodesics but will be determined by the combined relative strengths and the directions of the two fields. In the region close to the blackhole the field is complex in nature and it bears no resemblance to any one field. As one proceeds away from the blackhole the higher multipole moments fade away and the orbits are essentially determined by the leading terms of the multipole expansion.

The complex nature of the fields does not yield to a full treatment of charged particle trajectories. The combined fields do not possess sufficient symmetries to provide the requisite number of constants of motion. However one can analytically deduce the orbits in the following two special cases : (i) The motion along the axis, and (ii) the circular orbits about the axis of symmetry. In section 2 we specialize to the first two terms of the multipole expansion. The equations of motion are considered in section 3 and motion along the axis is discussed in section 4. Circular orbits about the axis of symmetry are considered in section 5 which is followed by the concluding remarks.

2. The Cohen and Wald solution

In this section we briefly discuss the electrostatic field of a point charge at rest near a Schwarzschild blackhole. The background curved spacetime is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad \ldots(2.1)$$

where m is the mass of the blackhole. We use here geometric units, $G = c = 1$. The covariant Maxwell's equations to be solved are

$$F^{ik}_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g}\, F^{ik}) = 4\pi j^k, \quad \ldots(2.2)$$

$$*F^{ik}_{;i} = 0, \quad \ldots(2.3)$$

where

$$F_{ik} = A_{k;i} - A_{i;k} = A_{k,i} - A_{i,k} \quad \ldots(2.4)$$

is the electromagnetic field tensor, and A_i is the 4-potential. $*F^{ik}$ is the dual of F^{ik}. Here a semicolon denotes a covariant derivative and a comma denotes ordinary derivative.

Consider a point charge held at rest at the point $r = b > 2m$, on the axis $\theta = 0$. We choose a sufficiently small charge and neglect the back reaction of the electrostatic field (assumed to be perturbative) on the background metric. The field in this case is static and axially symmetric. Therefore the components of the electric field will not be functions of t and ϕ. Since the charge is at rest the spacelike components of the 4-current vector j^k will be zero. That means $j^\alpha = 0$ and $A_\alpha = 0$ for $\alpha = 1, 2, 3$ indicating absence of magnetic field. Thus we have

$$A_i = (A_t, 0, 0, 0). \quad \ldots(2.5)$$

Then the equation for the vector potential is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_t}{\partial r}\right)+\frac{1}{1-2mr^{-1}}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_t}{\partial\theta}\right)=-4\pi j^0. \quad ...(2.6)$$

Using the axial symmetry of the field and its regularity on the axis of symmetry we can write the solution in terms of the Legendre polynomials in cos θ :

$$A_t(r,\theta)=\sum_{l=0}^{\infty} R_l(r)\,P_l(\cos\theta). \quad ...(2.7)$$

In the source free regions where $j^0 = 0$, the function $R_l(r)$ can be shown to satisfy the second order differential equation

$$\left(1-\frac{2m}{r}\right)\frac{d}{dr}\left(r^2\frac{d}{dr}R_l\right)-l(l+1)\,R_l(r)=0. \quad ...(2.8)$$

The solutions to this equation have been obtained by Israel (1968) and Anderson & Cohen (1970). The two linearly independent solutions of equation (2.8) are

$$g_l(r)=\begin{cases}1 \text{ for } l=0,\\ \dfrac{2^l l!\,(l-1)!\,m^l}{(2l)!}(r-2m)\dfrac{d}{dr}P_l\left(\dfrac{r}{m}-1\right) \text{ for } l\neq 0,\end{cases} \quad ...(2.9)$$

$$f_l(r)=-\frac{(2l+1)!}{2^l(l+1)!\,l!\,m^{l+1}}(r-2m)\frac{d}{dr}Q_l\left(\frac{r}{m}-1\right), \quad ...(2.10)$$

where P_l and Q_l are the Legendre polynomials of first and second type respectively. For a point charge e at rest at $r = b$, $\theta = 0$, $j^0 = e\,\delta(r-b)\,\delta(\cos\theta-1)$ the equation (2.6) is now solved with the source term and after considerable algebra one obtains the solution :

$$A_t=\begin{cases}e\displaystyle\sum_{l=0}^{\infty} g_l(b)\,f_l(r)\,P_l(\cos\theta), & r>b\\ e\displaystyle\sum_{l=0}^{\infty} f_l(b)\,g_l(r)\,P_l(\cos\theta), & r>b.\end{cases} \quad ...(2.11)$$

In the orthonormal frame

$$w^0=(1-2mr^{-1})^{1/2}\,dt,$$

$$w^1=(1-2mr^{-1})^{-1/2}\,dr, \quad w^2=r\,d\theta, \quad w^3=r\sin\theta\,d\phi.$$

The nonvanishing components of the field tensor F_{ij} are

$$\left.\begin{aligned}\tilde F_{01}&=-\tilde F_{10}=-\frac{\partial A_t}{\partial r},\\ \tilde F_{02}&=-\tilde F_{20}=-\frac{1}{r}\left(1-\frac{2m}{r}\right)^{-1/2}\frac{\partial A_t}{\partial\theta};\end{aligned}\right\} \quad ...(2.12)$$

It follows from equation (2.12) that for $r \sim 2m$, $\tilde{F}_{20} \to 0$, while $\tilde{F}_{10}$ remains finite. So for a stationary observer at r with $b > r \sim 2m$ the field is radial. With r and b both near $2m$ and $l \neq 0$, contribution to the electrostatic field for $r > b$ is mostly tangential because $\tilde{F}_{10}/\tilde{F}_{20} \sim \left(1 - \frac{2m}{r}\right)^{1/2} \ln \left(1 - \frac{2m}{r}\right)$ which tends to zero as $r \to 2m$. However the dominant contribution to the field in this case comes from the $l = 0$ term because the coefficients $g_l(b)$ of the higher multipole terms vanish. Hence the total electrostatic field is mostly radial in this case.

The complete solution of the field due to a point charge on the axis is an infinite series given by equation (2.11) of which we include here only the modes $l = 0$ and $l = 1$. Hence the source of our field will be different from a point charge. As $b \to 2$, $A_t \to e/r$ as in the case of Reissner-Nordström solution. The field becomes radial in this limit. This is in agreement with the no hair theorem for blackholes (Misner *et al.* 1973). One may say that the sources are present at $r = b$, since discontinuity in the field occurs at this radial coordinate. Hence dropping of the terms is equivalent to having a distribution of charge on the sphere of radius b. However at large distance from the blackhole the electrostatic field is the same as that of the point charge source placed on the axis. Therefore one may expect some similarity between the structure of the two types of sources.

Corresponding to the first two terms in equation (2.11), we have

$$A_t = \begin{cases} e\{f_0(b)\, g_0(r)\, P_0 (\cos\theta) + f_1(b)\, g_1(r)\, P_1 (\cos\theta)\}, & r < b \\ e\{f_0(r)\, g_0(b)\, P_0 (\cos\theta) + f_1(r)\, g_1(b)\, P_1 (\cos\theta)\}, & r > b \end{cases} \quad \ldots(2.13)$$

which in view of equations (2.9) and (2.10) finally give :
For $r < b$ (interior field)

$$A_t = e\left\{\frac{1}{b} + \frac{3(r-2m)}{2m^2}\left[1 - \frac{m}{b} + \left(\frac{b}{2m} - 1\right)\ln\left(1 - \frac{2m}{b}\right)\right]\cos\theta\right\}; \quad \ldots(2.14)$$

For $r > b$ (exterior field)

$$A_t = e\left\{\frac{1}{r} + \frac{3(b-2m)}{2m^2}\left[1 - \frac{m}{r} + \left(\frac{r}{2m} - 1\right)\ln\left(1 - \frac{2m}{r}\right)\right]\cos\theta\right\}. \quad \ldots(2.15)$$

The potential is continuous at $r = b$. The tangential component of the field is continuous for $0 \leqslant \theta \leqslant \pi$, whereas the radial component of the field is discontinuous. The above solution corresponds to an axially symmetric nonuniform distribution of charge on the sphere of radius b. Discontinuity in the field falls off as we go away from the charge (figure 1). The charge density falls off but the amount of charge remains the same.

To examine the charge distribution on the sphere, we compute the discontinuity

$$\Delta A_{t,\rho} = (-A_{t,\rho})_{\rho=a+} - (-A_{t,\rho})_{\rho=a-}$$

for $e/m = 1$, $\rho = r/m$ and $a = b/m$. From equations (2.14) and (2.15), it turns out to be

$$\Delta A_{t,\rho} = \frac{1 + 3\cos\theta}{a^2}. \qquad \text{...(2.16)}$$

This shows that discontinuity in the radial component of the field is maximum at $\theta = 0$, and the maximum value is $4/a^2$. The discontinuity is minimum at $\theta = \pi$ and is zero at $\theta \simeq 109.47°$. Thus we have axially symmetric nonuniform distribution of charge on the sphere with radius a. Figure 1 gives the plots of $\Delta A_{t,\rho}$ against θ for some values of a.

3. Equations of motion

In this section we consider the motion of a test charged particle in the Schwarzschild geometry with the superposed electrostatic field due to a point charge at rest at $r = b$, $\theta = 0$. The motion of a charged particle will not only be governed by the background gravitational field of the blackhole but also by the Lorentz force acting on the particle due to the perturbative electrostatic field.

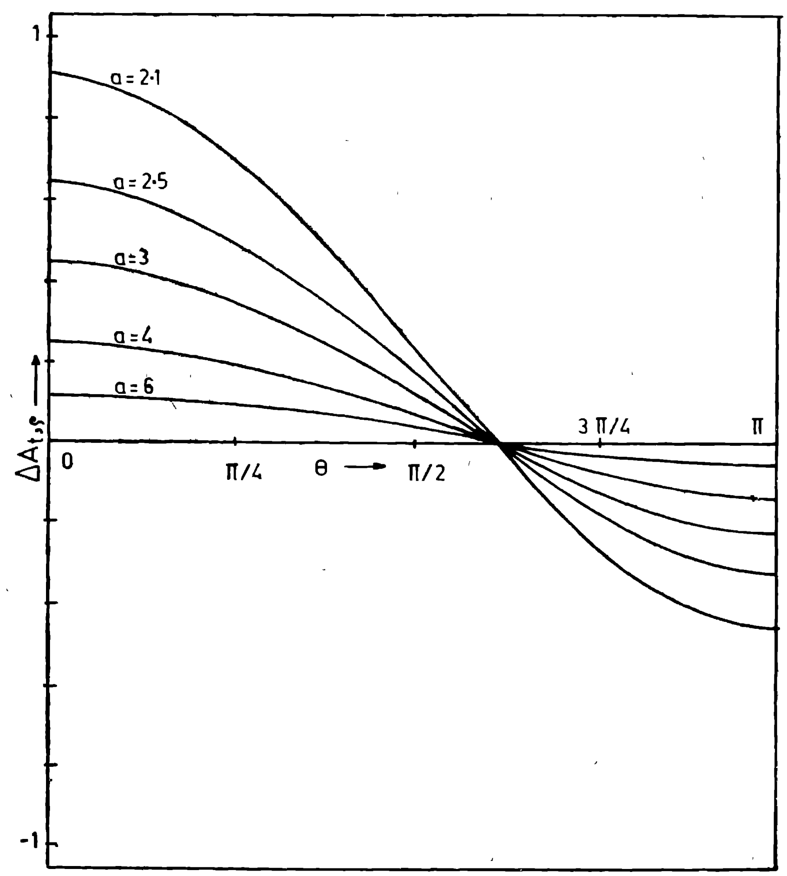

Figure 1. $\Delta A_{t,\rho}$ *versus* θ. It reflects the variation of the charge distribution on the sphere of radius a (charge e is located at $\rho = a$, $\theta = 0$) with respect to θ. The density of charge is maximum at $\theta = 0$, it goes to zero at $\theta \approx 110°$ and attains minimum at $\theta = \pi$.

The equations of motion of a charged particle of rest mass μ and charge q in an electromagnetic field are given by

$$\frac{du^i}{ds} + \Gamma^i_{jk}\, u^j u^k = \frac{q}{\mu} F^i_{\ m}\, u^m, \qquad \ldots(3.1)$$

where u^i is the four-velocity of the charged particle. The right hand side of equation (3.1) represents the Lorentz force indicating the non-geodesic character of the motion. These equations can be derived from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{ij}\dot{x}^i\dot{x}^j - \frac{q}{\mu} A_i \dot{x}^i, \qquad \ldots(3.2)$$

where a dot denotes the ordinary differentiation with respect to the proper time s. Equations (2.1) and (2.5) lead to

$$\mathcal{L} = \tfrac{1}{2}\left\{-\left(1-\frac{2m}{r}\right)\dot{t}^2 + \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right\} - \frac{q}{\mu} A_t \dot{t}. \qquad \ldots(3.3)$$

Since the electrostatic field is axisymmetric and the background metric is spherically symmetric, the Lagrangian will be independent of the azimuthal coordinate ϕ. Further, as both the fields considered are static, the Lagrangian is independent of t. These two symmetries give rise to the two integrals of motion, the canonical angular momentum and the total energy.

Thus we have

$$\left(1-\frac{2m}{r}\right)\dot{t} = E - \frac{q}{\mu} A_t \qquad \ldots(3.4)$$

and

$$r^2\sin^2\theta\dot{\phi} = l, \qquad \ldots(3.5)$$

where E and l are the energy and angular momentum per unit rest mass of the particle as measured by the observer at rest at infinity.

The equations of motion corresponding to r and θ coordinates are given by

$$\ddot{r} - \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 = -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)\dot{t}^2 + r\left(1-\frac{2m}{r}\right)\{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\} - \frac{q}{\mu} A_{t,r}\left(1-\frac{2m}{r}\right)\dot{t} \qquad \ldots(3.6)$$

and

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 - \frac{q}{\mu r^2} A_{t,\theta}\dot{t}. \qquad \ldots(3.7)$$

For convenience we introduce the dimensionless quantities

$$\left.\begin{aligned} &\rho = \frac{r}{m},\ a = \frac{b}{m},\ \bar{s} = \frac{s}{m},\ L = \frac{l}{m},\ \bar{t} = \frac{t}{m},\\ &A_t = \frac{e}{m}\bar{A}_t,\ \lambda = \frac{eq}{m\mu}. \end{aligned}\right\} \qquad \ldots(3.8)$$

Dropping the bars over the dimensionless quantities we can rewrite the equations of motion as

$$\ddot{\rho} - \frac{1}{\rho^2}\left(1 - \frac{2}{\rho}\right)^{-1}\dot{\rho}^2 = -\frac{1}{\rho^2}\left(1 - \frac{2}{\rho}\right)^{-1}(E - \lambda\bar{A}_t)^2 - \bar{A}_{t,\rho}(E - \lambda\bar{A}_t)$$
$$+ \rho\left(1 - \frac{2}{\rho}\right)\left\{\dot{\theta}^2 + \frac{L^2}{\rho^4 \sin^2\theta}\right\}, \quad \ldots(3.9)$$

$$\ddot{\theta} + \frac{2}{\rho}\dot{\rho}\dot{\theta} = \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{\rho^4} - \frac{\lambda}{\rho^2}\bar{A}_{t,\theta}(E - \lambda\bar{A}_t)\left(1 - \frac{2}{\rho}\right)^{-1}. \quad \ldots(3.10)$$

where we have substituted for $\dot{t}$ and $\dot{\theta}$ from equations (3.4) and (3.5).

The line element (2.1) itself provides the first integral given by

$$1 = \left(1 - \frac{2}{\rho}\right)\dot{t}^2 - \left(1 - \frac{2}{\rho}\right)^{-1}\dot{\rho}^2 - \rho^2\dot{\theta}^2 - \rho^2\sin^2\theta\dot{\phi}^2. \quad \ldots(3.11)$$

After using the earlier equations, it becomes

$$1 = \left(1 - \frac{2}{\rho}\right)^{-1}(E - \lambda\bar{A}_t)^2 - \left(1 - \frac{2}{\rho}\right)^{-1}\dot{\rho}^2 - \rho^2\dot{\theta}^2 - \frac{L^2}{\rho^2\sin^2\theta} \ldots(3.12)$$

The first integrals of equations (3.9) and (3.10) seem very difficult to obtain owing to the absence of any obvious symmetry in the Lagrangian. Also $\bar{A}_t$ involves the transcendental function which makes it almost untractable. Since we are unable to obtain all the first integrals we cannot do the orbit analysis in general. We therefore resort to some interesting special cases. We consider the following two cases : (a) Motion along the axis, and (b) circular orbits about the axis of symmetry.

4. The motion along the axis

Along the axis $\theta = 0$, equation (3.7) implies that $\ddot{\theta} = 0$ if $\dot{\theta} = 0$ initially (for the expression for $A_{t,\theta}$ contains $\sin\theta$). Thus a particle starting on the axis remains along the axis. Since 4-velocity has unit norm we obtain from equation (2.1) in terms of dimensionless quantities

$$-1 = -\left(1 - \frac{2}{\rho}\right)\dot{t}^2 + \left(1 - \frac{2}{\rho}\right)^{-1}\dot{\rho}^2. \quad \ldots(4.1)$$

Using equation (3.4) in (4.1) we obtain

$$-1 = -\left(1 - \frac{2}{\rho}\right)^{-1}(E - \lambda\bar{A}_t)^2 + \left(1 - \frac{2}{\rho}\right)^{-1}\dot{\rho}^2. \quad \ldots(4.2)$$

The effective potential is obtained by solving for E the equation $\dot{\rho} = 0$ and so we obtain

$$V = \lambda\bar{A}_t + \left(1 - \frac{2}{\rho}\right)^{1/2}. \quad \ldots(4.3)$$

We choose positive sign for the radical, because $\dot{t} = E - \bar{A}_t$ should be positive for the future moving particle. Setting $\dot{\theta} = 0$ in equations (2.15) and (2.16) and using equation (4.3) we get :

(i) For interior field ($\rho < a$)

$$V = \lambda \left\{ \frac{1}{a} + \frac{3(\rho - 2)}{2} \left[1 - \frac{1}{a} + \left(\frac{a}{2} - 1 \right) \ln \left(1 - \frac{2}{a} \right) \right] \right\} + \left(1 - \frac{2}{\rho} \right)^{1/2}. \quad \ldots(4.4)$$

(ii) For exterior field ($\rho > a$)

$$V = \lambda \left\{ \frac{1}{\rho} + \frac{3(a - 2)}{2} \left[1 - \frac{1}{\rho} + \left(\frac{\rho}{2} - 1 \right) \ln \left(1 - \frac{2}{\rho} \right) \right] \right\} + \left(1 - \frac{2}{a} \right)^{1/2}. \quad \ldots(4.5)$$

It can be seen that the effective potential V is continuous across $\rho = a$. However, at $\rho = a$ the derivative of V with respect to ρ does not exist. This is the reflection of the fact that the electrostatic field is discontinuous at $\rho = a$, indicating the presence of a source. The 4-momentum of the particle is continuous at $\rho = a$ because it depends on the effective potential, although the Lorentz force will have a discontinuity.

In figure 2, V has extrema at $\rho = a$. These extrema are of finite extent because of approximation of including only the first two terms in the multipole expansion. When all the terms are taken into account, $V \to \pm\infty$ as $\rho \to a$ indicating the seat of a point charge there. The Coloumbic force will obviously diverge. We recall that our approximation is only good at reasonable distance from the charge and not so good in close vicinity. However, we may take the view that we are essentially considering a nonuniform spherical charge distribution (figure 1) as is truly described by the first two terms in the solution. The following discussion refers to such a charge distribution.

Figure 2 gives the plots of V effective against ρ. We observe that for $\lambda > 0$, $V > 0$ always and has a maximum at $\rho = a$. This is due to repulsive Coloumbic force between the charges. Here for $E > V_{max}$ particle plunges into the hole while for $E < V_{max}$ the particle will be bounced back as it will meet the potential barrier.

For $\lambda < 0$ we always have $V < 0$ near the hole, which is due to the attractive electric field. All particles with $\lambda < 0$ coming in from infinity or having $E > V_{(\rho)}$ do not meet any barrier and directly fall into the hole. If $E < V_{(\rho)}$ then the particle is captured into the potential well and can exhibit the oscillatory motion about the source charge.

It should be noted that the total energy of the particle with $\lambda < 0$ can be negative (figure 2). Hence the Penrose (1969) process of energy extraction can be set up to extract electrostatic energy (for details see Dhurandhar & Dadhich 1984a, b). The extent of negative energy region is proportional to λ. It is similar to the Reissner-Nordström blackhole where electrostatic energy can be extracted out (Denardo & Ruffini 1973).

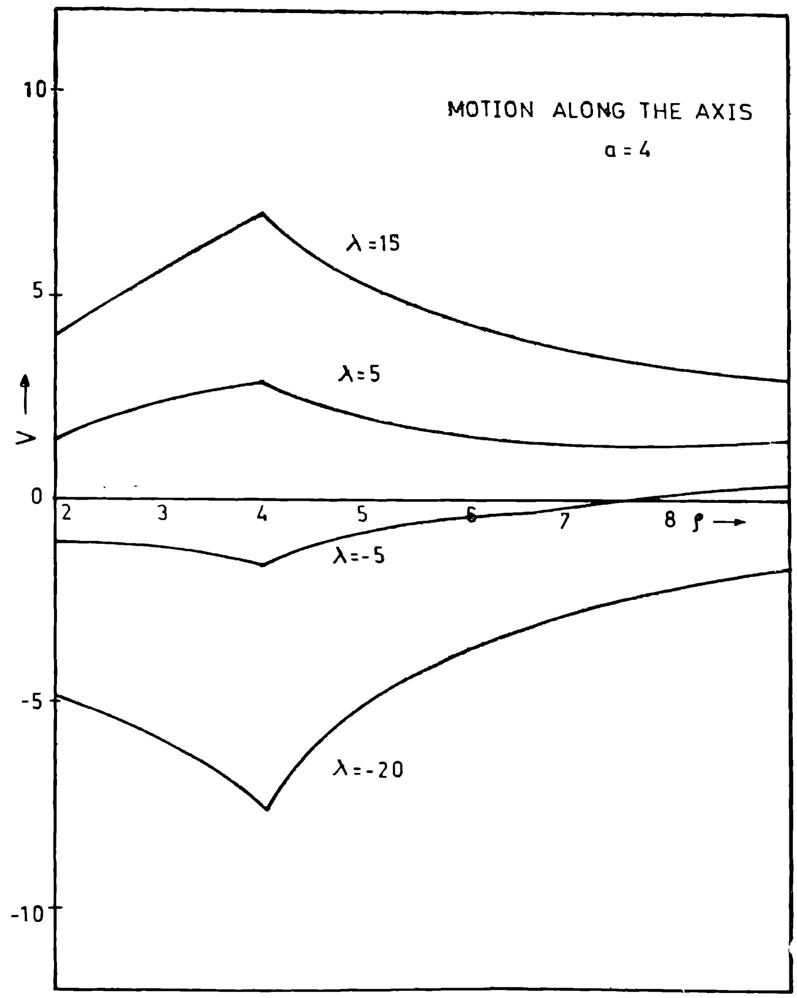

Figure 2. The effective potential V for motion along the axis is plotted for $\lambda = -5, -20, 5, 15$.

5. Circular orbits about the axis of symmetry

In general it follows from equation (3.10) that the orbits do not remain in the θ = constant planes. This is because $\ddot{\theta} = 0$ and $\dot{\theta} = 0$ can only be satisfied by discrete values of ρ. It then suggests that ρ = constant (circular) orbit can occur in the θ = constant plane. For such orbits we should simultaneously have $\dot{\theta} = \ddot{\theta} = 0$ and $\dot{\rho} = \ddot{\rho} = 0$. The three equations (3.9), (3.10) and (3.12) will determine the circular orbits when these conditions are imposed. Setting $\dot{\theta} = 0$ and $\ddot{\theta} = 0$ in equation (3.10) we obtain the relation

$$\frac{L^2}{\rho^2} = \frac{\lambda \sin^3 \theta}{\cos \theta} (E - \lambda \bar{A}_t) \left(1 - \frac{2}{\rho}\right)^{-1} \bar{A}_{t,\theta}. \qquad \ldots(5.1)$$

First of all we note that no circular orbit can occur in the equatorial plane $\theta = \pi/2$, since $L^2/\rho^2 \rightarrow \infty$. From the physical view point, the consideration of force acting on the particle clearly shows that circular orbits cannot occur in the equatorial plane. In figure 3 are shown the different forces to be balanced for a circular

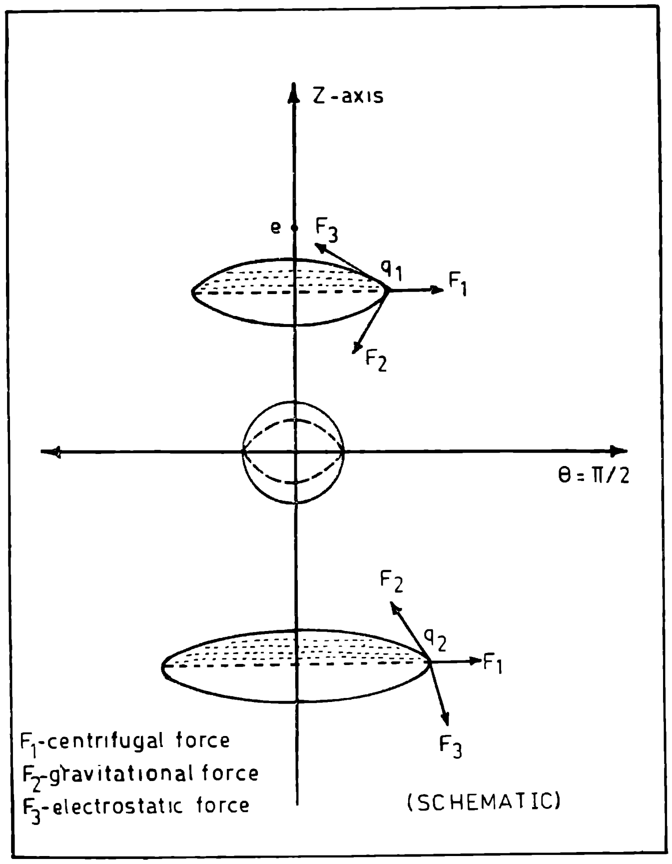

Figure 3. The various forces (gravitational, electric and centrifugal) acting on a particle executing circular orbit are indicated. All those forces have to be properly balanced such that the particle executes a circular orbit about the axis.

orbit. It follows that the electric force should point towards the source charge for the range $0 < \theta < \pi/2$ while it should point away for $\pi/2 < \theta < \pi$. That means $\lambda < 0$ for $0 < \theta < \pi/2$ and $\lambda > 0$ for $\pi/2 < \theta < \pi$.

Setting $\dot{\rho} = 0$ and $\dot{\theta} = 0$ in the equation (3.12) we obtain

$$1 = \left(1 - \frac{2}{\rho}\right)^{-1} (E - \lambda \bar{A}_t)^2 - \frac{L^2}{\rho^2 \sin^2 \theta}. \quad \ldots(5.2)$$

By using equations (5.1) and (5.2) we get

$$E = \lambda \bar{A}_t + \tfrac{1}{2} \lambda \bar{A}_{t,\theta} \tan \theta + \tfrac{1}{2} \left\{\lambda^2 \tan^2 \theta \bar{A}^2_{t,\theta} + 4\left(1 - \frac{2}{\rho}\right)\right\}^{1/2}. \quad \ldots(5.3)$$

The positive sign for the radical has been chosen to ensure that the 4-momentum vector of the test particle is future pointing. Another relation for E follows from the requirement $\ddot{\rho} = 0$ with $\dot{\rho} = \dot{\theta} = 0$, *viz*.

$$E = \lambda \bar{A}_t + \lambda \rho^2 \left(1 - \frac{2}{\rho}\right)\left(\frac{1}{\rho} \bar{A}_{t,\theta} \tan \theta - \bar{A}_{t,\rho}\right). \quad \ldots(5.4)$$

The consistency will now require equating the two expressions for E. Hence we have

$$\lambda \bar{A}_{t,\theta}\left\{2\rho\left(1-\frac{2}{\rho}\right)-1\right\}\tan\theta-2\lambda\rho^2\left(1-\frac{2}{\rho}\right)\bar{A}_{t,\rho}$$
$$-\left\{\lambda^2\bar{A}^2_{t,\theta}\tan^2\theta+4\left(1-\frac{2}{\rho}\right)\right\}^{1/2}=0.$$

Rearranging the above equation and denoting its left hand side by S we write

$$S(\rho,\theta,a,\lambda)=\lambda(2\rho-5)\bar{A}_{t,\theta}\tan\theta-2\lambda\rho(\rho-2)\bar{A}_{t,\rho}$$
$$-\left\{\lambda^2\bar{A}^2_{t,\theta}\tan^2\theta+4\left(1-\frac{2}{\rho}\right)\right\}^{1/2}=0. \quad \ldots(5.5)$$

The above equation expresses the fact that for given values of θ, λ, and a, only a finite set of values of ρ can satisfy the above consistency relation. It should be noted that particle orbits in general are not confined to a plane (since $\dot{\theta}=0 \not\Leftrightarrow \ddot{\theta}=0$) as in the spherical symmetric field. We thus have an additional relation to keep the orbit confined to $\theta=$ constant $\neq \pi/2$ plane. This condition yields equation (5.1), whereas $\dot{\rho}=0$, $\ddot{\rho}=0$ together with equation (5.1) lead to the two expressions for E in equations (5.3) and (5.4).

In equation (5.5), the roots of $S=0$ give the radii of circular orbits for fixed values of λ and θ. They are plotted in figure 4. It shows that radius of the

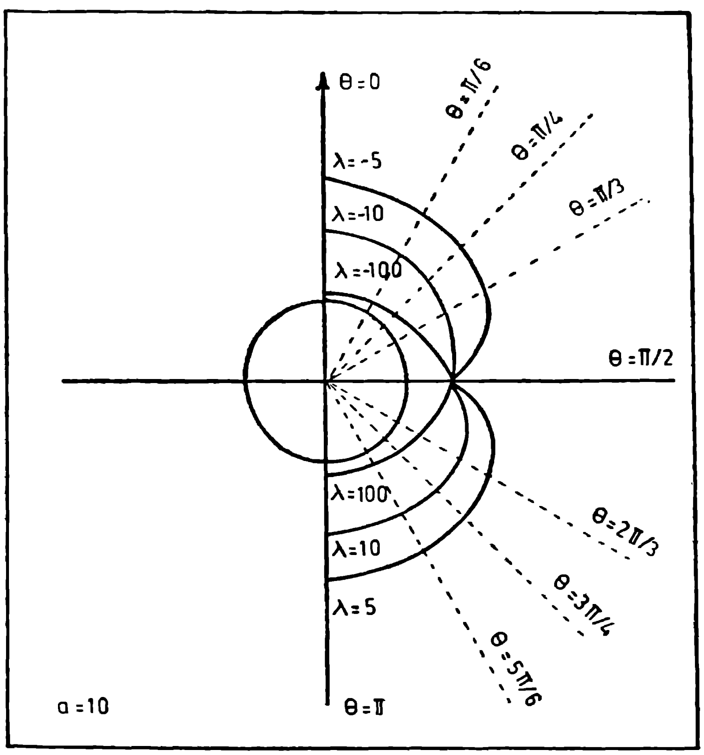

Figure 4. The radii of circular orbits for various values of $\lambda=\pm 5$, ± 10, ± 100 as function of θ. The sphere $\rho=2$ represents the blackhole.

orbits is inversely proportional to $|\lambda|$ but as $\theta \to \pi/2$, it approaches $\rho = 3$ for all λ. However no circular orbit can exist in the plane $\theta = \pi/2$. The θ-variation of the radius is shown for a fixed λ. The curves showing locii of the radius are symmetrical about the $\theta = \pi/2$ plane for $\lambda \to -\lambda$.

In the following we consider some special cases for limiting values of the parameters. They will help us in getting physical understanding of the motion. Let us first compute the derivatives of $\bar{A}_t$:

For $\rho < a$

$$\bar{A}_{t,\rho} = \tfrac{3}{2}\left[1 - \frac{1}{a} + \left(\frac{a}{2} - 1\right)\ln\left(1 - \frac{2}{a}\right)\right]\cos\theta, \quad \ldots(5.6)$$

$$\bar{A}_{t,\theta} = -\tfrac{3}{2}(\rho - 2)\left[1 - \frac{1}{a} + \left(\frac{a}{2} - 1\right)\ln\left(1 - \frac{2}{a}\right)\right]\sin\theta. \quad \ldots(5.7)$$

For $\rho > a$

$$\bar{A}_{t,\rho} = -\frac{1}{\rho^2} + \frac{3(a-2)}{2}\left[\frac{1}{\rho^2} + \tfrac{1}{2}\ln\left(1 - \frac{2}{\rho}\right) + \frac{1}{\rho}\right]\cos\theta, \quad \ldots(5.8)$$

$$\bar{A}_{t,\theta} = -\tfrac{3}{2}(a-2)\left[1 - \rho^{-1} + (\tfrac{1}{2}\rho - 1)\ln(1 - 2\rho^{-1})\right]\sin\theta. \quad \ldots(5.9)$$

We analyse equation (5.5) as follows :

Case I : $\lambda \to \pm\infty, \quad \rho < a.$

Case II : $\lambda \to -\infty, \quad \rho \to \infty.$

Case III : $\rho \to \infty.$

Case I : For $\lambda \to \pm\infty$, $\rho < a$, equation (5.5) reduces to

$$(2\rho - 5)\bar{A}_{t,\theta}\tan\theta - 2\rho(\rho - 2)\bar{A}_{t,\rho} \mp |\bar{A}_{t,\theta}\tan\theta| = 0, \quad \ldots(5.10)$$

where the upper sign is for $\lambda \to \infty$ while the lower is for $\lambda \to -\infty$. As noted earlier for $\lambda \gtrless 0$ the corresponding range for θ is $\pi/2 < \theta \leqslant \pi$ and $0 < \theta < \pi/2$ respectively. In both cases, equation (5.10) reduces to

$$(\rho - 3)\bar{A}_{t,\theta}\tan\theta - \rho(\rho - 2)\bar{A}_{t,\rho} = 0. \quad \ldots(5.11)$$

Using equations (5.6) and (5.7) in (5.11) we obtain after simplification

$$\rho = 3\sin^2\theta. \quad \ldots(5.12)$$

We note that as $\lambda \to \pm\infty$ equations (5.3) and (5.1) imply that both E and L tend to infinity. Hence equation (5.12) corresponds to photon orbits. However photon orbit must have radius > 2, *i.e.* $3\sin^2\theta > 2$ which restricts $\theta = \theta_0 = \sin^{-1}\sqrt{2/3} \sim 54.7°$. This is for the case $\lambda \to -\infty$ while for $\lambda \to \infty$, $\theta < \pi - \theta_0$. The maximum radius of the photon orbit tends to $\rho = 3$ as $\theta \to \pi/2$ from either side. Figure 5 shows how the orbit radius approaches the horizon as ρ varies on either sides of $\theta = \pi/2$. Outside the given θ range, no circular orbit exterior to the black-hole exists.

Case II : *Lift of orbits over* $\theta = \pi/2$ *plane*—In the following, we consider the interesting case when $\lambda \to -\infty$ together with $\rho \to \infty$. For $\lambda \to -\infty$, $0 < \theta < \pi/2$ we have equation (5.11). Substitution from equations (5.8) and (5.9) in equation (5.11) we get

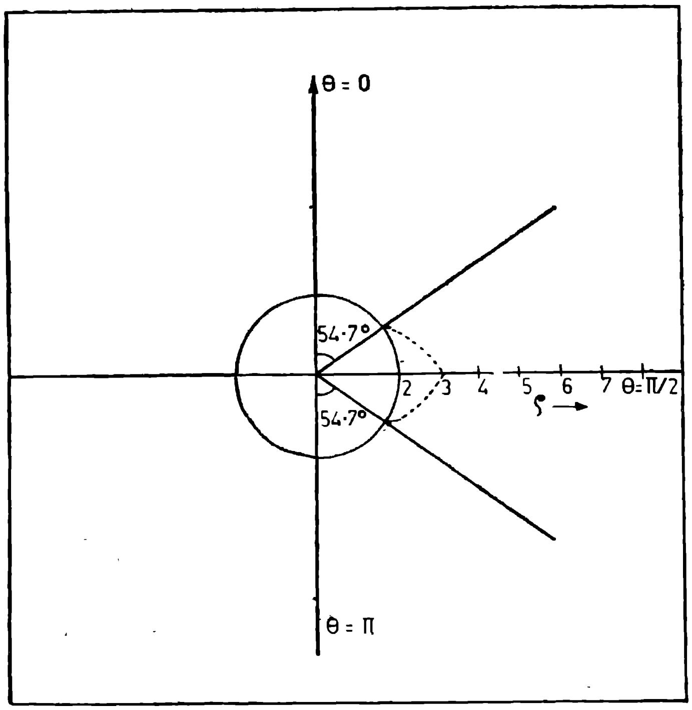

Figure 5. The circular orbit radii for the limiting case of $\lambda \to \pm\infty$ are shown for $\rho < a$. These are limiting photon orbits. The orbits occur in the range $\theta_0 < \theta \leqslant \pi/2$ for $\lambda \to -\infty$ and in the range $\pi/2 < \theta < \pi - \theta_0$ fot $\lambda \to +\infty$ where $\theta_0 = 54.7°$. For $2 < \rho < 3$, θ cannot range upto $\pi/2$ from either side.

$$\Big\{(\rho - 2)\Big[\frac{1}{\rho} + \frac{\rho}{2}\ln\Big(1 - \frac{2}{\rho}\Big) + 1\Big] - (\rho - 3)\Big[1 - \frac{1}{\rho} + \Big(\frac{\rho}{2} - 1\Big)\ln\Big(1 - \frac{2}{\rho}\Big)\Big]\Big\}\cos^2\theta - \frac{2(\rho - 2)}{3\rho(a - 2)}\cos\theta + (\rho - 3)\Big[1 - \frac{1}{\rho} + \Big(\frac{\rho}{2} - 1\Big)\ln\Big(1 - \frac{2}{\rho}\Big)\Big] = 0. \qquad ...(5.13)$$

For large ρ we can expand the logarithmic terms in this equation and keeping the first order terms in $1/\rho$, we obtain

$$\frac{3}{\rho}\cos^2\theta + \frac{(1 - 2/\rho)}{(a - 2)}\cos\theta - \frac{1}{\rho^2} = 0, \qquad ...(5.14)$$

which finally leads to

$$\rho\cos\theta = (a - 2). \qquad ...(5.15)$$

This represents the equation of a plane. This shows that as $\lambda \to -\infty$ and $\rho \to \infty$ plane circular orbits are possible. All these circular orbits lie in a plane lifted above the equatorial plane with height $(a - 2)$. As $a \to 2$ the plane collapses to the equatorial plane and the field goes over to that of the Reissner-Nordström hole.

Case III: We now consider the case $\rho \to \infty$ asymptotically. From equations (5.8) and (5.9) in (5.5) we get

$$\tfrac{3}{2}\lambda(2\rho-5)(a-2)\left[1-\frac{1}{\rho}+\left(\frac{\rho}{2}-1\right)\ln\left(1-\frac{2}{\rho}\right)\right]\tan\theta\sin\theta$$
$$+2\lambda\rho(\rho-2)\left\{-\frac{1}{\rho^2}+\tfrac{3}{2}(a-2)\left[\frac{1}{\rho^2}+\frac{1}{\rho}\right.\right.$$
$$\left.\left.+\tfrac{1}{2}\ln\left(1-\frac{2}{\rho}\right)\right]\cos\theta\right\}+\left\{\tfrac{9}{4}\lambda^2(a-2)^2\left[1-\frac{1}{\rho}\right.\right.$$
$$\left.\left.+\left(\frac{\rho}{2}-1\right)\ln\left(1-\frac{2}{\rho}\right)\right]^2\tan^2\theta\sin^2\theta+4\left(1-\frac{2}{\rho}\right)\right\}^{1/2}=0. \qquad \ldots(5.16)$$

For $\rho\to\infty$ we can expand the logarithmic terms and keeping the second order terms in $1/\rho$ we obtain

$$\lambda(2\rho-5)(a-2)\left(\frac{1}{\rho^2}\right)\sin\theta\tan\theta+2\lambda\rho(\rho-2)\left\{-\frac{1}{\rho^2}-\frac{2(a-2)}{3}\cos\theta\right\}$$
$$+\left\{\frac{\lambda^2}{\rho^4}(a-2)^2\sin^2\theta\tan^2\theta+4\left(1-\frac{2}{\rho}\right)\right\}^{1/2}=0. \quad \ldots(5.17)$$

Proceeding to the limit as $\rho\to\infty$ we obtain

$$\lambda=1. \qquad \ldots(5.18)$$

This shows that in the asymptotic limit as $\rho\to\infty$, circular orbits exist only for the range $\pi/2<\theta<\pi$ for $\lambda\to 1$ implying the balance between electrostatic and gravitational forces.

6. Concluding remarks

In section 2 we have retained the monopole and dipole modes in the multipole expansion of the field. Under this assumption the charge gets distributed non-uniformly over the sphere $\rho=a$ as shown in the figure 1. However this field approximates to the field due to a charge at $\rho=a$ in the two limits (i) $\lambda\to\infty$ and (ii) $a\sim 2$. Outside these two limits the field is not that of a charge at a point but is due to the nonuniform spherical distribution.

Although there is axial symmetry there is no reflection symmetry about the $\theta=\pi/2$ plane. It is this asymmetry which reflects on the nature of the orbits. There are no orbits constrained to the $\theta=\pi/2$ plane as constrasted to Schwarzschild or Reissner-Nordström cases. It is this fact that makes it difficult to analytically treat general orbits other than the ones considered here.

In the asymptotic limit as $\rho\to\infty$ the circular orbits cxist only for the range $\pi/2<\theta<\pi$ for $\lambda\to+1$ implying the balance between electrostatic and gravitational forces. Since for large ρ both forces have inverse square behaviour. For the limit $\lambda\to\pm\infty$ we obtain the analogue of photon threshold orbit for the Schwarzschild blackhole as $\rho=3\sin^2\theta$ in the range $\pi/2<\theta<\pi$ and $0<\theta<\pi/2$. This limit will lie outside the event horizon only for $\theta>54.7°$. For θ just over 54.7° circular orbits can exist very close to event horizon. This is similar to the Reissner-Nordström case where the presence of charge on the blackhole pulls closer the photon orbit threshold (Dadhich & Kale 1977). Another very interesting feature occurs when we consider $\lambda \rightarrow -\infty$ together with $\rho \rightarrow \infty$. In this case all the circular orbits lie in a plane at a height $(a - 2)$ above the equatorial plane. As $a \rightarrow 2$ the plane collapses to the equatorial plane and the field goes over to that of Reissner-Nordström. As noted above these limiting cases distinctly exhibit the asymmetrical features due to the lack of reflection symmetry in the $\theta = \pi/2$ plane.

An astrophysical body is known not to have significant quantity of electric charge on it. However there may occur some kind of a charge distribution in the region around blackholes. The presence of magnetic field around blackhole is now an established fact. The setting of a charged objects in the vicinity of blackhole is rather unlikely to occur. If at all there exists some kind of charge distribution in the vicinity of a blackhole then the nonuniform charge distribution as in our case is more probable. Since the electrostatic field is assumed to be perturbative, the magnitude of the point charge should be small so as not to alter the background Schwarzschild geometry which will however retain its spherically symmetric character for the neutral test particle motion.

It is clear, as in the case of a charged blackhole, negative energy orbits (figure 2) can occur outside the event horizon. The cause for their occurrance is the electrostatic interaction. Hence the electric field energy can be extracted by the Penrose process (Penrose 1969).

It will be interesting to study this problem by restoring the reflection symmetry. That is by putting another charge symmetrically at $\rho = a$, $\theta = \pi$. That will allow orbits in the equatorial plane. Then the results will bear greater resemblance with those of the Reissner-Nordström blackhole.

Acknowledgements

V. P. S. thanks the University Grants Commission, New Delhi, for a teacher fellowship under the faculty improvement program, while SVD thanks the Department of Mathematics, Poona University for a visiting fellowship which facilitated this work. This investigation constitutes the dissertation submitted by V. P. S. towards partial fulfilment of the requirements for the M.Phil. degree of the Poona University.

References

Anderson, J. L. & Cohen, J. M. (1970) *Ap. Sp. Sci.* **9**, 146.
Cohen, J. M. & Wald, Robert, M. (1971) *J. Math. Phys.* **12**, 1845.
Dadhich, N. & Kale, P. P. (1977) *Pramana* **9**, 71.
Denardo, G. & Ruffini, R. (1973) *Phys. Lett.* **45B**, 259.
Dhurandhar, S. V. & Dadhich, N. (1984a, b) *Phys. Rev.* **D29**, 2712; **D 30**, 1625.
Hanni, R. & Ruffini, R. (1972) in *Black Holes* (eds : B. Dewitt & C. Dewitt) Gordon & Breach, App. 2.4.
Israel, W. (1968) *Commun. Math. Phys.* **8**, 245.
Misner, C. W., Thorne, K. S. & Wheeler, J. A. (1973) *Gaavitation*, W. H. Freeman.
Penrose, R. (1969) *Riv. Nuov. Cim.* **1** (Special number), 252.