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Vacuum fluctuations in Einstein's elevator*

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Abstract. Vacuum fluctuation in the Einstein's elevator (which mimics a constant gravitational field) is analysed without introducing a plethora of extraneous vacuum states. In the elevator there is no particle creation, though the vacuum appears polarized.

Key words : Einstein's elevator—vacuum fluctuations—particle creation—gravitation

1. Introduction

A constant electric field can produce charged particle pairs from the vacuum (Schwinger 1951). Can a constant gravitational field do the same?

While constant electric field is a 'physical reality', constant gravitational field is not. It exists in Einstein's elevator because of acceleration. Relabelling coordinates will make it vanish. If the concept of the vacuum is coordinate independent, then constant gravitational field cannot create particles. However extensive amount of work in the last decade has shown the vacuum to be a coordinate-dependent concept. Thus only an analysis in the Einstein's elevator can answer the question raised above.

We shall perform such an analysis in this essay and demonstrate the following facts : (i) a constant gravitational field does not create particles, in contrast with an electric field; (ii) the vacuum fluctuation in the Einstein's elevator has a Planckian spectrum ; and (iii) effective action techniques can be used to provide a distinction between nontrivial vacuum fluctuation patterns (which exist in Einstein's elevator) and real particle creation (which does not). We feel that such a distinction will be very helpful in settling many conceptual problems (see *e.g.* Padmanabhan 1982 ; Candelas *et al.* 1983; Unruh & Wald 1984).

We begin by describing briefly the relevant path integral technique as applied to constant electric field (section 2). This method is applied to the constant gravitational field in section 3. Section 4 summarizes the main conclusions of the paper.

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2. Effective Lagrangian in constant electric field

The vacuum persistence amplitude is given in terms of the effective action by the relation

$$\langle\, 0, \text{out} \mid 0, \text{in}\, \rangle = \exp i A_{\text{eff}} = \exp i \int L_{\text{eff}}\, d^4x. \qquad \ldots(1)$$

If the probability for pair creation (via the decay of the vacuum) per unit spacetime volume is p, then we have

$$\mid \langle\, 0, \text{out} \mid 0, \text{in}\, \rangle \mid^2 = \exp(-2 \,\text{Im}\, A_{\text{eff}}) = \exp\left(-\int p\, d^4x\right), \qquad \ldots(2)$$

so that

$$p = 2\, \text{Im}\, L_{\text{eff}}. \qquad \ldots(3)$$

For spinless particles of mass m and charge q, L_{eff} can be computed most conveniently through the relation

$$L_{\text{eff}} = -i \int_0^\infty \frac{d\lambda}{\lambda} \exp(-im^2\lambda)\, K(x^i, \lambda\,; x^i, 0), \qquad \ldots(4)$$

where the kernel $K\,(x_2^i\lambda\,; x_1^i 0)$ has the following path integral representation

$$K(x_2^i\lambda;\, x_1^i 0) = \int \mathcal{D}x^i(s) \exp iA[x_2^i\lambda;\, x_1^i 0]$$

$$= \int \mathcal{D}x^i(s) \exp\left[\, i \int_0^\lambda ds \left\{ \frac{1}{4}\, g_{ik} \frac{dx^i}{ds}\frac{dx^k}{ds} + qA_i \frac{dx^i}{ds} \right\}\right]. \qquad \ldots(5)$$

The formal expression in (5) is mathematically ambiguous because of oscillating phases. As usual, we shall assume that the expression (5) is evaluated in the Euclidean sector and is analytically continued to Minkowski spacetime. Denoting the variables in the Euclidean region with a subscript E, we have the analytic continuation

$$K_E(x_{2E}\lambda_E;\, x_{1E}0) = \int \mathcal{D}\, x_E(s) \exp(-A_E), \qquad \ldots(6)$$

where $-A_E$ is obtained from iA by the substitutions

$$t = it_E, \quad s = is_E, \quad \lambda = i\lambda_E. \qquad \ldots(7)$$

In the presence of a constant electric field along the x-axis (*i.e.* $A^i = A_i = (-Ex, 0, 0, 0)$] in the flat Minkowski space [$g_{ik} = \eta_{ik} = \text{dia}\,(1, -1, -1)$], the Euclidean path integral in equation (6) can be exactly evaluated from the classical path. We get ($\Delta t_E \equiv x_{2E}^0 - x_{1E}^0$ *etc.*)

$$K_E(x_2\lambda_E;\, x_1 0) = F(\lambda_E) \exp(-\bar{A}_E) \qquad \ldots(8)$$

with

$$F(\lambda_E) = \frac{1}{(4\pi)^2\lambda_E} \cdot \frac{\mid qE \mid}{\sin(\mid qE \mid \lambda_E)}, \qquad \ldots(9)$$

$$\bar{A}_E = qEx_2\Delta t_E + \frac{1}{4} qE[\Delta t_E)^2 + (\Delta x)^2]\cot(qE\lambda_E) + \frac{1}{4}\frac{1}{\lambda_E}\,[(\Delta y)^2 + (\Delta z)^2].$$

Analytically continuing to Minkowski space and taking $x_1^1 = x_2^1$, we get the coincidence limit

$$K(x\lambda;\, x0) = \frac{1}{i}\,\frac{1}{(4\pi)^2\lambda}\,\frac{|\,qE\,|}{\sinh |\,qE\,|\,\lambda}. \qquad \ldots(10)$$

(The kernel K is normalized for integrations over d^4x while K_E is normalized over d^4x_E; this leads to the extra i^{-1} factor in equation (10) as compared to equation (9). The correctness of expression (10) can also be verified in the limit of $qE \rightarrow 0$). Using equation (10) in equation (4) and combining with equation (3) we get the pair creation probability (spin-0 case) :

$$p = 2\,\mathrm{Im}\, L_{\mathrm{eff}} = -\,2\,\mathrm{Im}\left\{\frac{qE}{(4\pi)^2}\int_0^\infty \frac{d\lambda}{\lambda^2}\,\frac{\exp(-\,im^2\lambda)}{\sinh(qE\lambda)}\right\}$$

$$= \frac{q^2E^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{(-\,1)^n}{n^2}\exp\left\{-\,\frac{m^2\pi}{|qE|}\,n\right\}. \qquad \ldots(11)$$

The vacuum does decay into charged pairs in the presence of constant electric field. The imaginary part to L_{eff} signals $|\,\langle\, 0,\text{ out}\,|\,0,\text{ in}\,\rangle\,|^2 < 1$. Let us now consider the constant gravitational field.

3. Effective Lagrangian in constant gravitational field

The constant gravitational field in Einstein's elevator is represented by the line element,

$$ds^2 = (1 + gx)^2\, dt^2 - dx^2 - dy^2 - dz^2. \qquad \ldots(12)$$

A physicist located at $x = 0$ in the elevator refuses to look at any other frame of reference, and will (justifiably) proceed to evaluate L_{eff} in this frame. To do this one has to evaluate the path integral kernel (5) in the analytically continued ($t = it_E$) region of equation (12). Since we are only interested in the coincidence limit ($x_2 = x_1$) the $\mathcal{D}y_E(s)\,\mathcal{D}z_E(s)$ integrations can be performed leaving one with

$$K_E\left(t_2^E,\, x_2^E,\, 0,\, 0;\, \lambda_E;\, t_1^E,\, x_t^E,\, 0,\, 0;\, 0\right)$$

$$= \int \mathcal{D}^4x_E(s)\,\exp\left[-\,\frac{1}{4}\int_0^{\lambda_E} ds_E\left\{(1 + gx_E)^2\,\dot{t}_E^2 + \dot{r}_E^2\right\}\right]$$

$$= \int \mathcal{D}x_E(s)\,\mathcal{D}t_E(s)\left[\frac{1}{4\pi\lambda_E}\right]\exp\left[-\,\frac{1}{4}\int_0^{\lambda_E} ds_E\left((1 + gx_E)^2\right.\right.$$

$$\left.\left.\times\left[\frac{dt_E}{ds_E}\right]^2 + \left[\frac{dx_E}{ds_E}\right]^2\right)\right]. \qquad \ldots(13)$$

For simplicity we shall consider the $m = q = 0$ case.

We note the well known fact that the metric in (t_E, x_E) plane is analogous to that of polar coordinates, $(dr^2 + r^2 d\theta^2)$. The coordinate singularity at $x_E = - g^{-1}$ implies periodicity of gt_E with a period of 2π. Therefore paths in (t_E, x_E) plane can be separated into various classes labelled by a winding number n which denotes the number of times the projection of an arbitrary path onto (t_E, x_E) plane, 'winds' around $x_E = g^{-1}$. The kernel in equation (13) is the sum of kernels for each n.

The exact calculation is straightforward but lengthy. The final expression is the coincidence limit of $x_2 = x_1 = x$ is however quite simple :

$$K_E(t_{1E} + t_E, x, \lambda_E; t_{1E}, x, 0)$$
$$= \left[\frac{1}{4\pi\lambda_E}\right]^2 \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(1+gx)^2}{4\lambda_E}\left(t_E - \frac{2\pi}{g}n\right)^2\right]. \quad \ldots(14)$$

One can provide a heuristic derivation of equation (14): Consider in equation (13) only the paths with constant x_E (i.e. $dx_E/ds_E = 0$). Since x_E is analogous to r and t_E is analogous to $g^{-1}\theta$ of the (r, θ) polar coordinates, the problem reduces to that of a quantum particle constrained to a circle. Summing over paths that go around the circle n times exactly reproduces equation (14). A more tedious computation, allowing for variations of $x(s)$ as well, is required to fix the correct factor, $(4\pi\lambda_E)^{-2}$, that multiplies the exponent.

Note that as $g \to 0$ we recover the standard expression for K_E. In the limit of $g \to 0$, only $n = 0$ contributes to the sum.

The analytic continuation to Minkowski space gives the kernel

$$K(t, \lambda; 0, 0) \equiv K(t, \lambda)$$
$$= \frac{1}{i}\left(\frac{1}{4\pi\lambda}\right)^2 \sum_{n=-\infty}^{n=\infty} \exp\left[\frac{i}{4\lambda}(1 + gx)^2 \left(t - \frac{2\pi i n}{g}\right)^2\right]. \quad \ldots(15)$$

We can now compute L_{eff} from $K(0, \lambda)$ by straightforward integration. Taking $m = 0$ in equation (4) and using the notation

$$\beta = \frac{2\pi}{g}(1 + gx)^{-1},$$

we get
$$L_{eff} = - i \int_0^\infty \frac{d\lambda}{\lambda} K(0, \lambda)$$
$$= - i \int_0^\infty \frac{d\lambda}{\lambda} \frac{1}{i}\left(\frac{1}{4\pi\lambda}\right)^2 \sum_n \exp\left(-\frac{i\beta^2 n^2}{4\lambda}\right). \quad \ldots(16)$$

The $n = 0$ term is just the L_{eff} which one would have calculated from the kernel in the absence of any gravitational field. Denoting this (formally divergent) Minkowski contribution by L_0, we get

$$L_{\text{ren}} \equiv L_{\text{eff}} - L_0 = +\left(\frac{1}{4\pi}\right) \sum_{n\neq 0} \left[\frac{1}{4}\ \beta^2 n^2\right]^{-2}$$

$$= \frac{1}{\pi^2 \beta^4}\, 2 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{45} \left\{\frac{g}{2\pi} \frac{1}{(1+gx)}\right\}^4. \qquad \text{...(17)}$$

The effective Lagrangian has no imaginary part and thus there is no particle creation, *i.e.* $|\langle 0, \text{out} \mid 0, \text{in} \rangle|^2 = 1$. However, the real part of the L_{eff} is to be interpreted as the potential energy density V, which is the standard practice in all effective potential calculations (see *e.g.* Huang 1982)

Defining $V(x)$ via

$$\frac{\langle 0, \text{out} \mid 0, \text{in}\rangle_{g\neq 0}}{\langle 0, \text{out} \mid 0, \text{in}\rangle_{g=0}} = \exp\left(i \int d^4 x\, (L_{\text{eff}} - L_0)\right)$$

$$= \exp\left(i \int d^4 x\, V(x)\right) \qquad \text{...(18)}$$

we get

$$V = \frac{\pi^2}{45} T^4;\ T = \frac{g}{2\pi} \cdot \frac{1}{(1+gx)}, \qquad \text{...(19)}$$

which is the thermal energy density at the temperature $g/2\pi$ redshifted to the location x by $(1+gx)^{-1}$. Thus vacuum fluctuations in Einstein's elevator contribute a thermal energy density but do not produce any particles.

The energy spectrum of the vacuum fluctuations can be obtained from the Fourier transform of the Green's function

$$\mathcal{G}(t) = \int_0^{\infty} d\lambda K(t, \lambda) = \left(\frac{1}{4\pi}\right)^2 \sum_{n=-\infty}^{\infty} \frac{1}{(t - i\beta n)^2}. \qquad \text{...(20)}$$

We have taken $x = 0$; t is actually $t - i\epsilon$ with $\epsilon \to 0^+$. The Fourier transform gives

$$\mathcal{G}(E) = \frac{1}{2\pi} \left\{\frac{1}{2} E + \frac{E}{e^{\beta |E|} - 1}\right\}, \qquad \text{...(21)}$$

wherein $\frac{1}{2}E$ comes from the $n = 0$ term (representing the vacuum energy density with $g = 0$) and the Planck spectrum arises from $n \neq 0$ part, and represents the contribution of g to the vacuum energy density. Also note that the total energy density from $n \neq 0$, given by V in equation (19), is finite while the $n = 0$ term gives the divergent L_0 term in equation (17).

4. Conclusions

The contrast between electric and gravitational fields highlights the contrast between real particle creation (with $|\langle \text{out} \mid \text{in}\rangle|^2 < 1$) and a change in the energy density of vacuum fluctuations (with $|\langle \text{out} \mid \text{in}\rangle|^2 = 1$). There are no 'real' (based on the above criterion) particles in Einstein's elevator, though the virtual particles have thermal energy density. Two additional points deserve special mention :

(i) The '$|\text{in}\rangle$' and '$|\text{out}\rangle$' vacua in constant gravitational field should be defined in exact analogy with the vacua of constant electric field with suitable adiabatic switching *etc.* We do not require a plethora of vacuum states (inertial vacuum, accelerated vacuum, *etc.*) to study the physics in Einstein's elevator.

(ii) We have shown elsewhere that when an accelerated detector is excited, it absorbs energy from the accelerating source. Thus detector results cannot be construed to prove reality of particles (Padmanabhan 1985).

Einstein's elevator was a powerful heuristic tool in the early days of relativity. It continues to be of value even today.

References

Candelas, P. & Sciama, D. W. (1983) *Phys. Rev.* **D27**, 1715.
Huang, K. (1982) *'Quarks, Leptons and Gauge Fields'*, World Scientific, chap. 10.
Padmanabhan, T. (1972, 1985) *Ap. Sp. Sci.* **83**, 247; *Class. Quant. Grav.* **2**, 117.
Schwinger, J. (1951) *Phys. Rev.* **82**, 664.
Unruh, B. & Wald, R. M. (1984) *Phys. Rev*, **D29**, 1047.