Some mathematical calculations of the dimensions, weight, etc., of Earth, Moon and Sun.

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(continued.)

Earth's form.—

A true globe would not be compatible with rotation. Thus measurements show that degrees of arc of the Earth's circumference along a meridian arc of variable length. In order to find the Earth's ellipticity mathematically, we must first measure the lengths of arcs along a meridian from Equator to pole. Thence we can find their radii of curvature, which are the radii of their osculatory circles, and hence from the evolute of their centres, we can assign by the principles of conic sections, the proportions as well as the actual lengths of all these radii, and hence the Earth's true form.

Thus if XY be any arc, and G be a point on the Evolute where the normals of X and Y meet, and if l' and l represent the latitudes of X and Y respectively.

Then the circular measure of $XGY = \frac{arc\ XY}{YG}$

$$\therefore\ YG = \frac{arc\ XY}{\text{circ. meas. of } XGY} = \frac{\text{arc } XY}{(l'-l) \times \frac{\pi}{180}} = \frac{\text{arc } XY}{(l'-l)} \times \frac{182}{\pi}$$

The Earth's ellipticity $= \frac{1}{298}$

The lengths of degrees of latitude vary between
68·8 miles = 363,000 ft. at the Equator
and 69·4 miles = 368,000 ft. at the Poles.

An ellipse is shorter than its circumambient sphere, by half its ellipticity.

Therefore, a meridian is shorter than the Equator by $\frac{1}{2} \times \frac{1}{298} = \frac{1}{596}$ th

$\therefore$ a meridian $= 24{,}900 \times \frac{595}{596}$ miles in length

$= 24{,}856\frac{1}{2}$ miles in length.

Earth's Ellipticity.—

Let E = equatorial radius, and let P = Polar radius then ellipticity $= \frac{E-P}{E} = \frac{3963-3949\cdot7}{3963}$

$$= \frac{13\frac{1}{2}}{3963} = \frac{1}{298}\text{th.}$$

Earth's Eccentricity $= \sqrt{\frac{E^2-P^2}{E^2}} = \sqrt{\frac{E^2-P^2}{E^2}}$

$$= \frac{(3963)^2-(3949\cdot7)^2}{3963}$$

$$= \cdot0826$$

The Earth's form is the figure produced by the revolution of the Ellipse about its minor axis, which is an oblate spheroid.

The other chief methods of ascertaining the form of the Earth which we can only barely mention here, are the dynamical methods of (1) calculating the value of the irregularities in the Moon's motion that are due to the Earth's elliptical form, (about 8″), which works out at something between $\frac{1}{295}$ and $\frac{1}{311}$; (2) calculations from the variation of gravity at different points on a meridian; and (3) calculating the effects of luni—solar nutation, from which Harkness makes the Earth's ellipticity to be $\frac{1}{297}$th. But this method is not very certain, as the distribution of the Earth's matter is not accurately known.

Earth's mass.—The best method for calculating the Earth's mass is by that of the "Torsion balance." As I warned you, however, I cannot enter into the details as to how the practical experiment is carried out. Any elementary text-book of astronomy will explain that. The mathematical principle of the calculation is simply that we compare the amount which a large ball, of known mass and radius, is able to attract a small body, as compared with the amount the Earth is able to attract it, or in other words its weight. It is a purely dynamical calculation. The ratio will be proportional to their respective masses, and inversely proportional to the square of their radii.

Thus if x = Earth's mass required, and a = amount of attraction of large ball, and $\bar{w}$ = that of the Earth, and r =

radius of large ball, and R = radius of Earth, and m = mass of large ball the formula will be—

$$x = m \times \frac{W}{a} \times \frac{R^2}{r^3}$$

The Earth's mass can also be calculated by means of the common balance.

If, for example, a small body when attracted by a large one placed above it loses ·00000001 of its weight, and the large body is a globe of 1 foot radius, and weighs 3 cwt.

Let x = weight of Earth (required) in tons

$$\text{then } \cdot 00000001 : 1 :: \frac{\cdot 15}{7} : \frac{x}{(20,900,000)^2} \text{ tons}$$

$$\therefore x = \frac{(20,900,000)^2 \times \cdot 15}{\cdot 00000001} \text{ tons}$$

$$= \frac{43,681 \times 3 \times 10^{18}}{20} \text{ tons}$$

$$= 6552 \times 10^{18} \text{ tons, or 6552 trillions of tons}$$

(a more accurate basis of calculations gives 5840 trillions).

The result is indeed a little too large a quantity, because we had supposed too large a loss of weight in the experiment. The mass of the Earth can also be calculated by the dynamical methods of—

(1) the position of the centre of gravity of Earth and Moon, being = ½ the Sun's angular displacement at first and last quarters; their combined mass being known from the orbital motion round the Sun.

(2) by calculating the attraction of the superposed masses of water which we call the tides.

The old mountain or "Schehallien method" (as also the mine experiment) is too inaccurate to yield results of any value. The mathematical principle involved, is the measurement of the difference between the geographical and astronomical latitudes of two places, due to the attraction of the known mass of a mountain between them. I need hardly remind you that Mass is not strictly equivalent to weight. Whilst Mass is a constant quantity, weight is a variable one, because weight unlike mass is proportional to the variations of gravity, which is itself proportional to the square of the distance from the centre.

The Density of the Earth—

$= \frac{\text{mass}}{\text{volume}} = \frac{6 \times 10^{21}}{259,850,000}$ tons in 1 cubic mile.

$= \frac{8,960 \times 6,000,000,000,000,000,000,000}{259,850,000 \times (1760 \times 3)^3}$ lbs. in 1 cubic foot.

= 342 lbs. in 1 cubic foot.

Specific gravity of Earth—

= number of lbs. in 1 cubic foot of Earth's matter divided by the number of lbs. in 1 cubic foot of water.

$= \frac{342}{62\frac{1}{2}}$ (since 1 cubic foot of water weighs 62½ lbs.)

= 5·472 (where water is unity).

I may add that, whilst 5·5 is the density of the Earth as a whole, or in other words its average density, only 2·6 is the density of the Earth's surface. Laplace calculates that 10·74 is the density of that portion of our globe which lies within a sphere of 3,000 miles radius, and that is more than four times the surface density. The pressure towards the centre of the Earth can thus be calculated as rather over three million tons per square foot.

The surface gravity of Earth—

$$= \frac{\text{Earth's mass}}{(\text{Earth's radius})^2}$$

Weight of bodies on the Earth's surface

$$= \frac{\text{Earth's mass} \times \text{body's mass}}{(\text{Earth's radius})^2}$$

Owing to the so-called centrifugal force (which would I think be more accurately styled "resistance to deviation") surface gravity on the Earth is not a constant quantity. The loss of gravity at the Equator, is the fraction represented by the number of feet in the Earth's radius, multiplied by the square of the radius per mean Solar second (in feet), and divided by the number of feet per second represented by gravity.

Thus $C = \frac{V^2}{R} = \frac{4 \pi^2 R^2}{t^2} \times \frac{1}{R} = \frac{4 \pi^2 R}{t^2}$ hence loss of gravity $= (3960 \times 5280) \times \frac{4 \pi^2}{(86400)^2} \times \frac{1}{32\frac{1}{6}}$

$$= \frac{\cdot 11127}{32\frac{1}{6}} = \frac{1}{289.}$$

and since $C = \frac{V^2}{R}$; if velocity be ×17 at the Earth's equator, then $C = (17)^2 \times$ present centrifugal force $= 289 \times$ present centrifugal force, thus $C = 289 \times \frac{1}{289} g = 1g$. In this case bodies at the Equator would weigh zero.

The weight then of a body is less at the Equator than at the poles (owing to centrifugal force) by $\frac{1}{289}$th. At other latitudes than at the Equator the centrifugal force $C^1 = C \cos^2 \phi$; ϕ representing the angle of latitude. The tangential component is given by the formula $C^1 = C \cos \phi \sin \phi$.

There is another reason, however, why the surface gravity at the Equator should be less than at the Poles, *viz.*, because the radius is greater at the Equator than at the poles. And gravity is reduced in the ratio of the square of the radius. This works out to a loss of 2⅝ oz. on 100 lbs., represented by the fraction $\left\{ 1 - \frac{(\text{Polar } r)^2}{(\text{equatorial } r)^2} \right\}$ 100 lbs.

That is, a body weighing 100 lbs. 2⅝ oz. at the poles will weigh 100 lbs. at the Equator. In order to find the total diminution of gravity at the Equator, the amount due to centrifugal force and that due to increased radius have of course to be added together, thus :—

$$\left(\frac{1}{289} + \frac{1}{595 \cdot 5} \right) g = (\text{about}) \frac{1}{194} g.$$

Variations of the gravitational force can also be dynamically calculated in terms of a second's pendulum.

$$\text{Since } t = \pi \sqrt{\frac{l}{g}}, \text{ and } t^2 = \frac{\pi^2 l}{g}$$

therefore $g = \pi^2 \times$ length of pendulum (in feet) $\div t^2$ (= unity). In the latitude of Greenwich, such a pendulum is 3¼ feet in length.

Gravity at Greenwich $= (\pi^2 \times 3 \cdot 25)$ feet
$= (9 \cdot 86 \times 3 \cdot 25)$ feet
$= 32 \cdot 045$ feet.

"per second per second," which means, of course, the velocity with which a body would continue to fall in perpetuum, if gravity ceased to operate upon it after the first second. If we take the two causes together, a body weighing 194 lbs. at the Pole will weigh 193 lbs. at the Equator, or an ordinary clock pendulum would lose about 2·25 minutes in a day if transferred from London to Singapore.

Loss of gravity due to tide-raising forces.—

Lastly, it may be of interest to point out the loss of weight incurred by a body on the surface of the Earth, when directly under the Moon, or on the opposite side of the Earth from the Moon. On geometrical principles (which are too elaborate to state here) it can be shown that directly under the Moon, the tide-raising force upon the body is $\frac{1}{30}$ × Moon's whole attraction. And as the Moon's mass is $\frac{1}{81}$ × Earth's mass, and her distance is 60 × Earth's radius, the amount of diminution of gravity, under the Moon can be expressed by the following formula, if d denotes the ratio of the Moon's distance to the Earth's radius, thus—

$$g \times \frac{\text{Moon's mass}}{\text{Earth's mass}} \times \frac{1}{30} \times \frac{1}{d^2}$$

$$= g \times \frac{1}{81} \times \frac{1}{30} \times \frac{1}{3600}$$

$$= \frac{1}{8,748,000} \text{ of gravity.}$$

Thus the Moon's tide-raising force on a body is less than one eight-millionth of the Earth's surface gravity. Similarly the Sun's tide-raising force also reduces the Earth's gravity, when directly overhead or underfoot. And since the tide-raising force varies inversely as the cube of the distance (and not as its square) the tide-raising force of the Sun works out at about $\frac{2}{5}$ of that of the Moon.

Hence the weight of a body when directly under th Sun

$$\text{is diminished by } \frac{1}{8,748,000} \times \tfrac{2}{5}\text{th}$$

$$= \frac{1}{21,870,000} \text{ th.}$$

In other words a body which normally weighs 1,000 tons, loses 1·714275 oz., when the Sun is overhead, or underfoot that is at mid-day or mid-night.

But the tide-raising forces of both Sun and Moon, when overhead, are slightly greater than when underfoot, because, when overhead, the ratio between the Earth's radius and their distances is slightly greater (especially in the case of the Moon) than when underfoot.

Thus the Moon's tide-raising force on a body nearest to her

$$\text{is } g \times \frac{\cdot 0123}{(59)^2}$$

but on a body farthest from her on the Earth's surface is

$$g \times \frac{\cdot 0123}{(61)^2}$$

the difference being $\cdot 00000023 \times g$.

Finally, let me conclude my paper by venturing to remind you of the following mathematical relations and differences; which are of the greatest importance in astronomy:—

That every atom in the Universe attracts every other atom. But that (for convenience sake) it is possible to regard a homogeneous sphere as attracting from its centre of mass.

That the attraction at the Earth's surface $= \frac{\text{Earth's mass}}{(\text{radius})^2}$

But that the weight of bodies at the Earth's surface $=$ the above quantity $\times$ body's mass.

That the accelerating force on

$$M_2 = \frac{M_1 \times M_2}{(\text{distance } M_1 M_2)^2} \div M_2 = \frac{M^1}{d^2}$$

and that the accelerating force on—

$$M^1 = \frac{M_1 \times M_2}{(\text{distance } M_1 M_2)^2} \div M_1 = \frac{M_2}{d^2}$$

That the gravitational force between two bodies

$$= \frac{M_1 \times M_2}{d^2}$$

But that the relative acceleration between two bodies

$$= \frac{M_1 + M_2.}{d^2}$$

That accelerating force is a one-dimensional quantity, thus;

$$\frac{\text{mass (3 dimensional)}}{\text{distance}^2 \text{ (2 dimensional)}}$$

But that weight is a four-dimensional quantity, thus; mass (3 dimensional) $\times$ gravitational force (1 dimensional).

That gravitational force on $M_2 = \frac{M_1}{(r)^2}$

But that tidal force on $M_2 = \frac{M_1}{(r)^3}$