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General relativistic effects of rotation on the structure and surface redshift of fast pulsars*

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Abstract. General relativistic effects of rotation on the structure and surface emission of the fast pulsar PSR 1937 + 214 are illustrated using a rotationally perturbed interior spherical metric. The results are found to differ markedly from those derived on the basis of simple spherical models, and are expected to be generally valid for the class of fast pulsars.

Key words : general relativity—neutron star structure—fast pulsars—redshift

1. Introduction

The recently discovered millisecond pulsar PSR 1937+214 (Backer *et al.* 1982) has generated considerable astrophysical interest because its extremely rapid rotation rate (about 640 revolutions every second) implies that it is close to the point of rotational instability. In this essay we illustrate the implications of the short period of PSR 1937 + 214 in relation to the properties of a stable neutron star. Stability requirement for this pulsar allows us to obtain lower limits on the neutron star's mass and moment of inertia and an upper limit on the radius. The condition of hydrostatic equilibrium with respect to radial perturbations provides an upper limit on the mass and moment of inertia of a star. These are also discussed here. In addition, we emphasize that the effect of substantial rotation on surface redshift and Doppler line-broadening will be significant, features which are rather different from those derived on the basis of simple Schwarzschild models.

2. Rotational instability in a star

Neutron stars have essentially flat density profiles and are well approximated by Maclaurin spheroids. Above a certain rotational speed, there is bifurcation from the axisymmetric Maclaurin sequence to nonaxially-symmetric Jacobi ellipsoids, at

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which point the star may become unstable. This secular instability limit corresponds to an angular speed Ω_s given by (Tassoul 1978)

$$\frac{\Omega_s^2}{\pi G\langle\rho\rangle} = 0.36 \qquad \ldots(1)$$

where $\langle\rho\rangle$ is the average density of the Maclaurin spheroid.

The limiting criterion implied by equation (1) is, strictly speaking, semi-Newtonian, valid for homogeneous stars in uniform rotation. Secular instability against radiation of gravitational waves has been shown to be generally true for all rotating stars (Friedman & Schutz 1975). Using a scalar theory of gravitation and a Newtonian homogeneous star model, Papaloizou & Pringle (1978) have shown that such instability will be important, on astronomically interesting timescales, for neutron stars with period $\lesssim$ a few milliseconds. While the secular instability for homogeneous stars in general relativity in arbitrarily fast but uniform rotation is not completely understood and some authors have even questioned if this instability operates for a uniformly rotating, centrally condensed star (see Shapiro *et al.* 1983) we use condition (1) in this paper on the assumption that relativistic analogues of this criterion exist for neutron stars which are nearly uniform in density.

On the assumption that PSR 1937 + 214 is a uniformly rotating neutron star on the verge of secular instability, equation (1) implies, corresponding to its period of 1.5577 ms, $\langle\rho\rangle \geqslant \langle\rho\rangle_{\min} = 2.4 \times 10^{14}$ gm cm^{-3}. Once rotating neutron star configurations are worked out, this gives lower limits on the mass and moment of inertia and an upper limit on the radius of this pulsar.

3. Rotating neutron star configuration

The effect of rotation on the structure of a star (assumed homogeneous and in rigid rotation) is to produce both spherical and quadrupole deformations. This is calculated using the prescription of Hartle & Thorne (1968). This is valid for strong gravitational fields but in the limit of uniform, slow rotation (slow compared to the critical speed $\Omega_c = (M/R^3)^{1/2}$ for centrifugal breakup). Neutron star models rotating at $\Omega = \Omega_s$, relevant in the context of the millisecond pulsar, are within this bound. (Ω = angular velocity of the surface as seen by a distant observer).

For a fixed central density, the fractional changes in the (gravitational) mass $\delta M/M$ and radius $\delta R/R$ due to spherical deformation are proportional to Ω^2, and can be calculated from a knowledge of the radial distributions of the mass and pressure perturbation factors. The nonrotating mass M and radius R are obtained by integrating the relativistic structure equations (Arnett & Bowers 1977).

A relativistic effect of rotation is dragging of inertial frames, so that $\bar{\omega}(r) \neq \Omega$ where $\bar{\omega}(r)$ is the angular velocity of the fluid relative to the local inertial frame and is given by

$$\frac{d}{dr}\left(r^4 j \frac{d\bar{\omega}}{dr}\right) \times 4r^3\bar{\omega}\frac{dj}{dr} = 0, \qquad \ldots(2)$$

where $$j(r) = e^{-\Phi(r)}\left(1 - \frac{2m}{r}\right)^{1/2} \qquad \ldots(3)$$

and $\Phi(r)$ is the potential relating the element of proper time to the element of time at $r = \infty$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{1 - 2m/r} \quad ...(4)$$

Here m is the mass within a radius r and P the pressure at r. Our units are such that $c = 1 = G$. The boundary conditions are

$$\Phi(\infty) = 0,$$

$$\left(\frac{d\bar{\omega}}{dr}\right)_{r=0} = 0,$$

$$\bar{\omega}(\infty) = \Omega.$$

For $r > R$, that is, outside the star,

$$\bar{\omega}(r) = \Omega - 2J/r^3, \quad ...(5)$$

where J is the angular momentum of the star :

$$J = \frac{R^4}{6}\left(\frac{d\bar{\omega}}{dr}\right)_{r=R}. \quad ...(6)$$

The mass perturbation factor $m_0(r)$ and the pressure perturbation factor $P_0(r)$ corresponding to spherical deformation are calculated according to the prescription of Hartle & Thorne (1968). The deformations δM and δR are then

$$\delta M = m_0(R) + J^2/R^3, \quad ...(7)$$

$$\delta R = -\left.\frac{P_0(\rho + P)}{dP/dr}\right|_{r=R}, \quad ...(8)$$

where $\rho(r)$ is the total mass-energy density at the point r.

The metric (signature: $+ - - -$) that describes the above perturbed geometry and matches at the surface to an exterior metric is

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \quad (\alpha, \beta = 0, 1, 2, 3) \\ &= e^{2\nu}dt^2 - e^{2\psi}(d\phi - \omega dt)^2 - e^{2\mu}d\theta^2 \\ &\quad - e^{2\lambda}dr^2 + O\left(\Omega^3/\Omega_c^3\right), \end{aligned} \quad ..(9)$$

where $\Omega_c = (M/R^3)^{1/2}$, and the metric components correspond to the interior with

$$e^{2\nu} = e^{2\Phi}\{1 + 2(h_0 + h_2 P_2)\},$$

$$e^{2\psi} = r^2 \sin^2\theta\{1 + 2(v_2 - h_2) P_2\},$$

$$e^{2\mu} = r^2\{1 + 2(v_2 - h_2) P_2\},$$

$$e^{2\lambda} = \{1 + 2(m_0 + m_2 P_2)(r - 2m)\}/(1 - 2m/r).$$

Here P_2 is the Legendre polynomial of order 2, ω is the angular velocity of the cumulative dragging, and h_0, h_2, m_0, m_2 and v_2 are all functions of r that are proportional to Ω^2. The terms with subscripts 0 and 2 refer to spherical and quadrupole deformations respectively. However, the quadrupole terms will average out to zero over the entire surface of the neutron star. What determines the stability of the star is the average density so that the spherical deformation of the star is the relevant correction to the structure to calculate.

If the star is homogeneous, then

$$\Omega_s = (0.27)^{1/2}\Omega_c. \qquad ...(10)$$

The homogeneity assumption can be justified on the basis that the flatness of the density profiles is not significantly altered by the rotation (Ray & Datta 1984).

4. The surface redshift and line broadening

To determine the effect of rotation on the redshift factor, we use the the geometrical optics approximation that photons are zero rest mass particles which follow null geodesics, and neglect the back-scattering caused by the presence of spacetime curvature. The redshift factor

$$1 + z = \lambda_{obs}/\lambda_{em} \qquad ...(11)$$

requires a knowledge of the quantities which decide the structure of the star. Since we are interested only in the surface radiation, the interior metric equation (9) suffices for our purpose. In the approximations used to obtain the structure of the star, equation (11) then takes the following form (Kapoor & Datta 1984)

$$1 + z = (1 + \Omega q)\, \{e^{2\Phi}(1 + 2h_0) - r^2 \sin^2 \theta_s \bar{\omega}^2\}^{-1/2}_{r=R}, \qquad ...(12)$$

which is valid for general surface emission from the pulsar. Here $\bar{\omega} = (\Omega - \omega)$, q is the photon impact parameter and θ_s is the polar angle made by the line of sight through the centre of the star with the axis of rotation. We take θ_s to be identical to the angle of inclination according to the distant observer, since the polar bending in the trajectory of a photon propagating in the perturbed exterior Schwarzschild spacetime would be small compared to the azimuthal bending.

The surface redshift will correspond to $q = 0$.

The general relativistic line-broadening (due to rotation) of a spectral line of energy E_{em} emitted from the surface of a fast pulsar is given by

$$W = E_{em}\left[\frac{1}{1 + z(q_{min})} - \frac{1}{1 + z(q_{max})}\right] \qquad ...(13)$$

where q_{min} and q_{max} correspond to photons emitted tagentially forwards and backwards respectively.

The line centre which corresponds to $q = 0$ will be redshifted by an amount given by equation (12). The effect of dragging of inertial frames will be to position the broadened spectral line asymmetrically with respect to the line centre. An interesting consequence of rotation will be to make the line profile asymmetrical. From Liouville's theorem, it can be seen that the ratio of intensities at the edges of the broadened line is given by

$$\frac{I_{blue}}{I_{red}} = \left[\frac{1 + z(q_{max})}{1 + z(q_{min})}\right]^3. \qquad ...(14)$$

5. Results and discussion

The structure of a neutron star depends sensitively on the equation of state at high densities, especially around $\rho \sim 10^{15}$ gm cm^{-3}. The results presented in this paper correspond to five equations of state (Pandharipande & Bethe 1973; Friedman & Pandharipande 1981; Canuto *et al.* 1978; Bethe & Johnson 1974; Pandharipande & Smith 1975) chosen on the basis of representative neutron and nuclear matter interactions. The details of the construction of the composite equations of state are given in Datta & Ray (1983).

The results of our calculation regarding the limiting values of the mass M, radius R, and moment of inertia I are summarized in table 1. The values of M_{min}, R_{max} and I_{min} are obtained assuming PSR 1937 + 214 to be on the verge of rotational instability limit given by equation (1), whereas M_{max}, R_{min} and I_{max} are obtained using the condition of hydrostatic equilibrium against gravitational collapse. The upper limits M_{max} are all close to $2\,M_\odot$. It may be noted that masses of neutron stars established through studies of binary x-ray and radio pulsars lie within this limit (Joss & Rappaport 1984). A more interesting result pertains to M_{min}. Until now, the lowest possible neutron star mass has been taken to be $\simeq 0.1\,M_\odot$ (Ruderman 1972). On the basis of rotational stability considerations, we find (for a representative choice of the equation of state) this limit to lie in the range (0.7–1.72) $M_\odot$ for a fast pulsar like PSR 1937 + 214.

Table 1. Bulk properties of rotating neutron stars at the secular instability limit for PSR 1937+214

Equation of state	$M_{min}/M_\odot$	R_{max} (km)	I_{min} (gm cm^2)	$M_{max}/M_\odot$	R_{min} (km)	I_{max} (gm cm^2)
RP	0.70	11.7	4.0×10^{44}	1.70	8.7	1.1×10^{45}
FP	0.76	12.0	5.0×10^{44}	2.08	9.2	1.6×10^{45}
CDK ($f^2 = 2.91$ case)	0.79	12.0	4.7×10^{44}	1.87	11.1	1.6×10^{45}
BJ (Model I)	1.20	13.9	1.1×10^{45}	1.96	10.0	1.6×10^{45}
TI	1.72	15.6	1.56×10^{45}	1.87	12.6	2.4×10^{45}

For the purpose of illustrating the effect of rotation on the redshift and spectral line broadening, we have chosen $(1.4 \pm 0.2)\,M_\odot$ as representative neutron star mass (Joss & Rappaport 1984). The surface redshift, line width and the ratio of intensities at the line edges for a neutron star rotating at $\Omega = \Omega_s$ are illustrated in figures 1–3 for the different equations of state. Our results indicate that the surface redshifts are larger by about 10–30% than those based on the simple Schwarzschild metric considerations which ignore rotation.

To demonstrate the line broadening, we choose the 0.511 MeV electron-positron annihilation line; the formalism, however, is general, and will be valid for any other possible emission line. Our results indicate that while the surface redshift is relatively insensitive to θ_s, the line width and the intensity ratio show a marked dependence on this angle for all equations of state considered here. In fact, depending on the value of this angle, the intensity ratio can be large with the result that the blue end of the line becomes more prominent as $\theta_s \rightarrow \pi/2$. This will suggest more of a blueshifted emission rather than a mere broadened one. Should gamma ray lines be discovered from surfaces of fast pulsars, these results provide a new

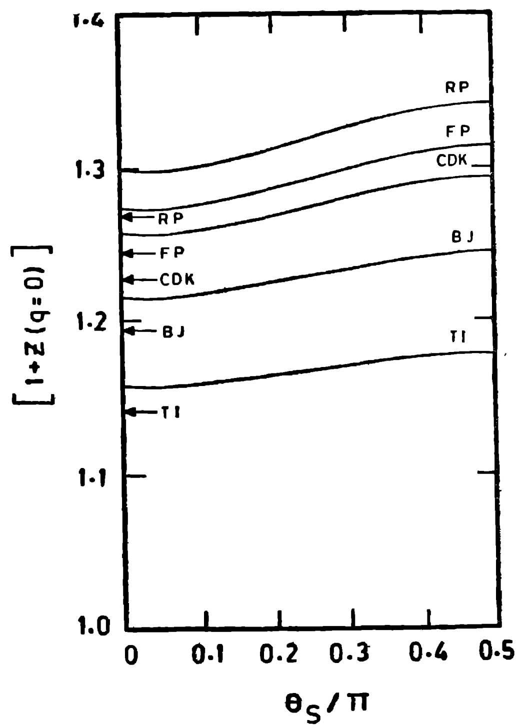

Figure 1. Surface redshift factor versus polar angle θ_s for various equation of state models : RP (Pandharipande & Bethe 1973), FP (Friedman & Pandharipande 1981), CDK Canuto *et al.* 1978), BJ (Bethe & Johnson 1974) and TI (Pandharipande & Smith 1975). Arrows marked on the vertical scale indicate the corresponding Schwarzchild redshift factors.

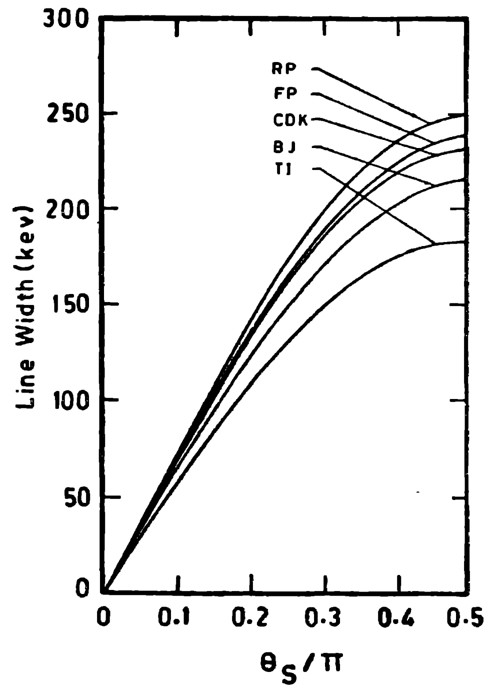

Figure 2. Line width as a function of polar angle for various equations of state.

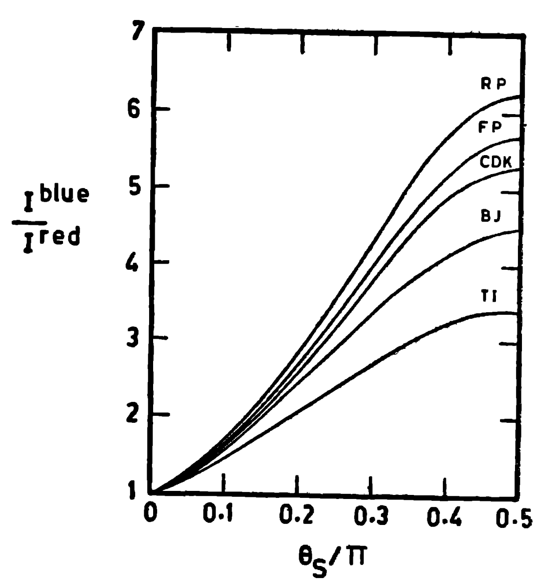

Figure 3. Ratio of intensities at the edges of a rotationally broadened spectral line versus polar angle for various equations of state.

and interesting handle towards constraining the validity of equations of state proposed for neutron star matter. A large intensity ratio, if observed, would lend additional support to the idea that the emission takes place at the surface, and is not predominantly gravitationally broadened.

A new millisecond pulsar (PSR 1953 + 29) in a binary system has recently been discovered (Boriakoff *et al.* 1983) the spin-up scenario for which has been argued to be essentially the same as for PSR 1937 + 214 (Helfand *et al.* 1983). From an analysis of the unique position in $P - \dot{P}$ (period and its time derivative) diagram occupied by PSR 1937 + 214 and several other pulsars, Alpar *et al.* (1982) have suggested that this class of radio pulsars has different origin and evolution from all other long period pulsars. So, if fast pulsars possess similar characteristics, the results presented here will have general validity for at least a class of neutron stars (associated with such pulsars).

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