Forces that go to determine the Moon's Motion in Space.

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In all mathematical computations of the inequalities produced in the Moon's motion by the disturbing force of the Sun, it is convenient to resolve this disturbing force into its three rectangular components, known as the Radial, Transversal, and Orthogonal forces.

The Radial acts in the direction of the radius-vector (either towards or away from the focal centre). *The Transversal* acts along the line of the Moon's orbital path, so as to either accelerate or retard her motion. *The Orthogonal force* interferes with the plane of the Moon's orbit, generally tending to bring her into the same plane as the ecliptic. It chiefly results in the retrogressive motion of the Moon's nodes. The two opposite kinds of radial force are called the *Differential* and the *Constrictive*. In other words the radial force at one time tends to separate the Moon from the Earth, and at another time to pull them together. The radial force is differential or separative at Syzygies, or new and full Moon, when Earth and Moon are in a line with the Sun. The result is that the Earth's attraction is then lessened by about 1/89th.

At Quadratures or half Moons, when Earth and Moon are at right angles to the Sun, the radial force is constrictive, adding to or reinforcing the Earth's attraction by about 1/178th. The constrictive force is only about half as powerful as the differential. The radial forces vanish entirely at four points on the Moon's orbit, which are situated at about 36° on either side of the line of Quadratures. The Moon at these points is drawn neither away from nor towards the Earth.

The Transversal force is zero at both Syzygies and Quadratures, but at its maximum at the points 45° from Syzygies and Quadratures. Whilst the differential radial force always acts away from the line of Quadratures and the constrictive force always acts towards the line of Syzygies, the transversal force combines the two in affecting the Moon's velocity. Thus from new Moon to half Moon the transversal force is against or retarding the Moon's forward motion. During the second quarter, it accelerates her, during the third quarter, or from full Moon to the waning half Moon, the transversal force is again retarding her, and for the last quarter, back again to new Moon, accelerating her.

The Orthogonal component which produces irregularities in the Moon's latitude, tends in general to draw the Moon towards the plane of the ecliptic. It neither influences the form of the Moon's orbit (this is done by the two kinds of radial force), nor her velocity (this is done by the transversal force), but it disturbs the plane of her orbit, chiefly by way of making the Moon's nodes revolve backwards. The nodes regress rather more than $1\frac{1}{2}°$ in a month, accomplishing the entire circle in about 19 years. So much for the rectangular components of the Sun's disturbing influence on the Moon's orbital motions. We will now enquire into what are the principal effects thereof. Now, in spite of the immense labour that has been expended by the ablest mathematicians upon the so-called "Lunar Theory," it is still, as we have said, incomplete and even slightly incorrect. Thus after a few years the lunar tables begin to get wrong, and have to be made out afresh, the Moon being frequently behind or before her predicted place by as much as $4''$ of arc which is equivalent to about 4 miles along her orbital path. The number of "perturbations" or "inequalities" in the Moon's motions is indeed countless, but such small inequalities as do not disturb the Moon is her orbit by more than, say, 1/20th of a second of arc or 80 yards, can be safely ignored without any practical loss. It will be sufficient, for the purposes of my paper, to point out only the largest and most important effects of the Sun's disturbing action on the Moon's motions.

And first we will take *The Advance of the Moon's Apsides.* This movement of the line of the Moon's apsides, or advance of her perigee, is caused chiefly by the radial component of the Sun's disturbing force. The effect is brought about in the following manner. Whenever the Moon's perigee or apogee, or in other words the Moon's apsides or major axis, are at Syzygy, or in line with the Sun and Earth, the radial force being then differential, lessons the power of the Earth to pull her round so to speak. The Moon therefore goes on further in her course than she would otherwise have done before she turns the corner, hence the line of the Moon's apsides will advance in the direction of the Moon's motion. When, however, the line of apsides are in Quadrature, or at right angles to the direction of the Sun, the opposite effect takes place. The Sun's disturbing force is then constrictive, and hence the Moon turns the corner so to speak earlier, and consequently the line of apsides regresses. But as the differential force is double that of the constrictive force, therefore the differential force prevails, and therefore the Moon's apsides progress twice as much as they regress. And not only so, but it must also be taken into consideration, that the Sun goes round the same way as the apses whenever they advance, staying in Company with them, whereas when they recede the Sun only meets them—thus indirectly augmenting their tendency to advance. The result of all this is that the line of apsides advances 6 the Moon's diameter, or about $3\frac{1}{2}°$ per month, or nearly 41° per annum, or accomplishes a complete, direct revolution in about $3{,}232\frac{1}{2}$ days, or in a period of rather less than 9 years. The motion of the Moon's perigee appears to be getting slower and slower as time goes on, being now 8″ in a lunation slower than in the time of Hipparchus. We will next consider the *Retrogression of the Moon's nodes*; as we have shown above, the orthogonal component tends in general to identify the plane of the Moon's orbit with that of the ecliptic. The orthogonal force is analogous to the precessional force, in that it affects the Moon's orbit much in the same manner as the precessional force affects the Earth's equatorial plane, causing thereby the retrogression of the first point of Aries. The orthogonal force causes the Moon's nodes on the whole to recede. Because when the Moon's nodes are before quadrature and after Syzygy, the node will in that particular lunation advance. Yet in all the rest of the orbit, the orthogonal force being towards the ecliptic, the nodes will recede. For as the Moon rises from the ecliptic the orthogonal force will cause her to rise at a less angle and descend at a greater angle, and therefore she will come down to her next node on the ecliptic a little sooner than she otherwise would have done, and thus the node recedes. And *vice versa*, as the Moon descends below the ecliptic, she will, owing to the effect of the orthogonal force, descend at a less angle, and will again ascend at a greater angle than otherwise. Hence she will rise to her next node earlier, and hence again her node will recede. Whilst then her nodes can sometimes advance, yet their retrogression greatly preponderates on the whole. They recede on an average about $1\frac{1}{2}°$ in the lunation, $19\frac{1}{3}°$ per annum, or revolve through the whole 360° in 6793·39 days or a little more than $18\frac{1}{2}$ years' period. When the Sun is in a line with the Moon's nodes, that is, when the nodes are in Syzygy (since then the Moon's orbit is in the same plane with the ecliptic) and also twice each month, when the Moon is at Quadratures, the orthogonal force vanishes altogether. And consequently at those times there is no force to disturb the plane of her orbit, so as to make her nodes either advance or recede. I should add that the inclination of her orbit as well as the position of her nodes is affected by the orthogonal force. But the effect in this direction is very slight and nearly compensated for in each lunation, and entirely so in a whole revolution of her nodes. The secular inequality in the notion of the Moon's nodes depends upon the variation of the eccentricity of the Earth's orbit; as her velocity is accelerated the motion of her nodes is retarded and *vice versa.*

We will next notice the effect of the radial force upon the *lengthening of the Moon's sidereal period*, as we have pointed out above, the force of the negative or differential component is double that of the positive or constrictive force. And besides that, the differential force preponderates for 216° out of the 360° of the Moon's monthly orbit. The result is that for every lunation as a whole, the radial force diminishes the Earth's attraction on the Moon by about 1/359th part. It thus enlarges her ellipse, and consequently makes her month nearly an hour longer than it would have been.

Another very important inequality in the Moon's motion is the *change of the Moon's Eccentricity.* This was discovered nearly two centuries before Christ by Hipparchus, being the most considerable of all the Moon's inequalities. Hipparchus was able to discover it, because it very materially affects the times of eclipses, and it was the eclipses that the ancients chiefly studied and understood. The change of eccentricity is caused by the difference in position of the Sun in reference to the Moon's line of apsides. The eccentricity of her orbit varies according as to whether the Sun is towards the line of her major or minor axis, and the variation of her ellipticity due to this cause can be as much as 1/70th. The mean interval between successive conjunctions of Sun and perigee is about 412 days. This change of eccentricity, which is sometimes called Evection, causes the Moon to be displaced in her orbit by more than 1⅓° backwards or forwards, or more than twice her diameter or about 4,500 miles as measured along her orbital path. Its period is the time the Sun takes in going round from perigee to perigee or about 1¼ years. *The Moon's variation* is another very considerable inequality. It is the transversal component of the Sun's disturbing force, which is mainly responsible for this perturbation. The transversal force tends, as we have said, to retard the Moon's motion from new Moon to first quadrature, and from full Moon to second quadrature, and *vice versa* in the other two quarters. Hence the result of this inequality is, that the Moon is behind her undisturbed place at first and third quarters, and ahead at second and fourth quarters. She is most ahead and behind at the octants, her maximum amount being nearly 36′ of arc, or about an hour and twenty minutes measured in time, or about 2,400 miles measured along her path, or rather more than her diameter as seen from the Earth. This inequality called variation does not affect the times of eclipses, because it is zero at the Syzygies, and hence it was not known to the ancients. To Tycho Brane falls the honour of having first noticed this inequality.

We will next consider the *Moon's Parallactic Inequality*. This is an important disturbance, since it alone of all the disturbances yields a data for computing that all important quantity, the Sun's parallax. It was by observing that this inequality was greater than what it should have been on the old computation of the Sun's distance, that Hansen was enabled to correct the Sun's distance, reducing it by three million miles. The parallactic inequality is due to the fact that the ratio between the distances of Earth and Moon and Sun varies according to whether the Moon is at new Moon or full. The Moon being about 1/400th of the distance of the Sun away from us, this fraction is therefore the ratio of the parallax of the Sun to that of the Moon. The difference of the perturbing force at these positions is in the ratio very nearly of 200 and 203. The last great inequality that is perceptible even without telescopic aid is the *Moon's Annual Equation*. This may be regarded rather as an indirect perturbation by the Sun. It depends upon the fact that when Earth and Moon are nearer than the mean distance from the Sun, the Sun's disturbing influence will then be greater upon the Moon's motion, than when they are farther off and since the Sun's differential or separative force prevails in each lunation, therefore during the summer-half of the year (when Earth and Moon are farther away from the Sun) the Moon will be less disturbed and therefore will approach the Earth, her orbit then contracting and therefore the month will be shorter, and *vice versa* in the winter-half of the year, the lunar orbit will be dilated, and the month will be longer. The maximum amount of this hurrying up and slowing down of the Moon's motion due to the annual equation is rather more than 11′ of arc or about 700 miles backwards and forwards in the Moon's orbital path. The annual equation is a periodic inequality, wholly compensating itself in the period of one anomalistic year, or the time from perihelion to perihelion again. The Moon is most before or behind her mean place in April and October, that is to say, after half a year's excess of acceleration and retardation respectively. It may perhaps be as well to point out that the Moon's "mean" place always means her mean elliptical place. And her mean elliptical place means the mean place she would have if she moved in a circle when corrected by the "equation of the centre." We must not forget to make mention of a certain inequality in the Moon's orbital motions, which is sometimes overlooked. I mean the *inequality (chiefly in latitude) due to the elliptical shape of the Earth.* The Earth is an oblate spheroid or an ellipsoid of revolution. The mutual attraction between all the particles of the Moon's mass and all the particles composing the prominent mass at the Earth's equator (the Earth's ellipticity is about 1/305th) causes a considerable disturbance in the motions both of the Earth and the Moon. This inequality in the Moon's latitudinal position is the reaction of the Earth's axial nutation. Since the plane of the Moon's orbit does not coincide with that of the Earth's equator (where the excess of matter exists) it tends as we know to be drawn by the Moon's attraction into her orbital plane, the consequence of which is the nutation of the Earth's rotational axis. Hence *per contra* the Moon tends to be drawn into the plane of the Earth's equator. It must be remembered that the attraction of oblate spheroids differs from that of spheres in that spheroids do not attract, as though their whole masses were gathered at their centre, but they attract a distant body in the plane of their equator more than if that body were in the plane of their poles. The constant effect on the Moon's latitude due to this inequality is about 8″ of arc or about $7\frac{3}{4}$ miles above or below the path she would otherwise have pursued. The motion of the Moon's nodes and perigee are also affected by the Earth's elliptical shape but only to very small extent. I may add that the elliptical shape of the Moon herself has no sensible effect upon her motions. So far then I have spoken firstly of the Earth and Moon as two bodies alone in space; secondly, of the Earth and Moon as disturbed by a third body the Sun in various ways; and thirdly, as disturbed by the Earth's ellipticity; and now before I close my paper, I must just mention one other very interesting inequality, produced not by the Sun or the Earth, but by the planets. It is called the *Secular Acceleration* of the Moon's mean motion. This action of the planets is somewhat similar to that of the Sun which causes the Annual Equation. The planets are at present indirectly accelerating the Moon's motion by directly affecting the Earth's orbit. It must be remembered that the planets' direct action on the Moon must be practically nil owing to their exceedingly small mass and the great distance of most of them compared with that of the Sun. Two centuries ago the then Astronomer Royal discovered that the month must be getting shorter. He was led to this conclusion by comparing the periods of ancient and modern eclipses. By calculating back to ancient times what the dates of eclipses ought to have been according to the modern lunar tables, he discovered considerable discrepancies between the theoretical dates and the dates as given by Ptolemy, proving that the month was gradually shortening. Laplace later showed that this shortening of the month was proportional to the square of the time, a fact which was proved by all the known ancient and intermediate eclipses. The Moon was then about 1° ahead of the position she would have occupied, but for this so-called secular acceleration. He further discovered that this increasing velocity was due to the Earth's decreasing eccentricity. Owing to the action of the planets, the Earth's orbit is still getting more circular or less elliptical, or its minor axis is increasing (the major axis and mean motions remaining the same) at the rate of 3,900 miles in a century, and it will continue to do so for about 24,000 years to come, when it will again gradually become more elliptical. If the Earth's orbit were to become a circle it would take 36,300 years to do so. But its eccentricity will never decrease to such an extent at that. So long as the Earth's orbit is decreasing in ellipticity, or in other words, so long as the Sun's average distance from us is increasing, so long also must the Moon's velocity be increasing too. The Moon is at present being accelerated by about 1/400″ of arc every year, which accumulates by arithmetical progression to about 10″ in a century. The result is that our months now are about 1/60th of a second of time shorter than they were 20,000 years ago, and each month is 1/57,000,000th of a second shorter than the last. We need not, however, entertain any fears as to our month ever becoming unduly shortened for our remote descendants, since it is abundantly proved from the fact that the sines and cosines of a circular arc which increase with time can together never be greater than unity, or thus exceed the radius, but must oscillate between zero and unity, however much the time increases, that therefore the major axis of the Moon, and consequently her mean motions are subject only to periodic changes. Thus the length of the month has no tendency in the long run, either to increase or diminish, for the motions of the Moon can do no more than oscillate very slightly from faster to slower, and from slower to faster, in fixed periods of time of about 45,000 years. I may perhaps incidentally remark that the *resistance of ether* (if there be any such) and also the *resistance* *of light* have no discoverable effect whatever upon the motions of the Moon. All the complex and subtle motions and sub-motions of the Moon can be entirely and satisfactorily accounted for solely by the universal law of gravitation or the hypothesis of matter attracting directly as the mass and inversely as the square of the distance. I have now tried to put before you, in as clear and concise a manner as I have been able, the principal forces which go to determine the motions of the Moon in space.

At least all these inequality that I have now mentioned, must be taken into due account, before we can predict with any accuracy the position in her orbit which the Moon will assume at any required moment, after of course correcting for refraction, parallax, aberration, &c.

I began by reminding you of the fundamental mathematical laws of motion of two bodies in space. I went on to show that the orbits they describe must be one of the conic sections. I further pointed out that the Moon's orbit under the influence of the Earth's attraction is an ellipse with the Earth in the focus, and stated the amount of its ellipticity. I next brought before your notice the principal modifications of this her normal or primitive elliptical motion, due to the Sun's modifying influence, as the third or disturbing body. This disturbing influence I explained as the difference of attraction either in power or direction, that the Sun exerts upon Earth and Moon at the same time. I showed you that the Sun's disturbing influence can be conveniently resolved into its three rectangular components, namely, the Radial (both positive and negative), the Transversal and the Orthogonal. We saw that these disturbing forces had various effects upon the Moon's motion in space, or upon her position in her orbit at any given moment. The chief of these so-called perturbations or inequalities of the Moon's motions I mentioned as—

1. The Advance of the Moon's apsides.
2. The Retrogression of her nodes.
3. The lengthening of her sidereal period.
4. The change of the Moon's eccentricity or evection.
5. The Moon's Variation or inequalities of her velocity.
6. The Parallactic Inequality.
7. The Annual Equation.
8. The Inequality due to the Earth's elliptical shape, and lastly
9. The Moon's Secular Acceleration.