The Forces which go to determine the Motions of the Moon in Space.

By the Revd. A. C. Ridsdale, M.A.

The problem of the forces which go to determine the Moon's motions in space involves many very elaborate and difficult mathematical calculations. The unique genius of Newton, who was the first to open up this new path of enquiry and the greatest mathematicians since his time, such as Laplace, Lagrange, Euler, Clairant, Adams, Airy and Leverrier, have all been at work upon the "Lunar Theory," and yet the problem is not even to this day completely solved. I will try and make my paper as clear and as little technical as possible, but the subject-matter is a difficult one, and in the nature of things it cannot be made so easy and obvious as, for example, a treatise upon purely pictorial or observational astronomy. I may remind you that the study of the celestial motions, involving as it does the study of so many of the mightiest and most universal forces in God's creation, is by far the most important branch of our science. An earnest amateur astronomer will scarcely be able to satisfy all his curiosity regarding the celestial sphere by merely gazing through his telescope again and again at the well-worn craters of the Moon, or the mere dozen or so objects in the skies, that have any interest for the possessor of only a moderate sized instrument. He will want to know, not only what the celestial objects look like through a telescope, but how they really move, and how fast and at what relative distance, etc., and above all the reason why. To possess a clear understanding and thorough grasp of the main principles which determine the motions of the Moon is of the utmost value to the earnest student of Astronomy, inasmuch as the Moon's orbital motions are, so far as we know at present, typical of all the other celestial motions throughout this Universe. Moreover the study of the various motions of the Moon has incidentally yielded most valuable information, such as, for example, the form of the Earth, the vicissitudes of the tides, the distance of the Sun, and consequently the magnitude of the whole solar system. But above all the Moon's motions and still more her irregularities have taught us the universality of the law of gravitational attraction. The motions of the Moon are moreover of the very highest importance practically to the navigator and geographer, since measurements of lunar distances and occultations of stars afford the most accurate determinations of longitude. I will begin by reminding you of two very important laws in regard to bodies moving in space. The first is, that in the case of a body moving undisturbed along a straight line, the radius vectors joined from any point arbitrarily chosen outside that line to points along it, will always describe equal triangular areas in equal periods of time. The second law which should be remembered, is that any force applied along the line of the radius vector so as to deflect that body from its original motion in a straight line (such, for example, as the Earth's attraction on the Moon, regarded as the sole occupants of space), will not interfere with its sweeping out these equal areas in equal periods, and retaining the same orbital plane. Hence in all cases in which a body is moving under the influence of a central force, and under no other, we can deduce the following laws of motion :—

Firstly.—The areal velocity, or square miles per second, swept through by the radius vector, will always be constant at all parts of the orbit, or in other words, the radius vector will describe areas proportional to the time.

Secondly.—The linear velocity, or miles per second, will vary inversely as the distance at which the body will happen to be at any given moment from the central attracting force (the angular velocity varying inversely as the square of that distance).

These all important laws in regard to celestial motions were discovered as plain facts by Kepler, from his examination of Tycho's observational records. But it was Newton who first proved them to be the mathematically necessary and universal laws of motion. Newton further proved that if a body be moving in an ellipse, having a centre of force at one of its foci, then the force of attraction at different points in the orbit will vary inversely as the square of the distance from that centre. And this was an epoch-making discovery of vast importance in the science of Astronomy, as being the basis of the universal principle of gravitation. Newton was able to prove that it is the attraction of the Earth, which determines the main motion of the Moon in her elliptical orbit, and that this attraction is comparable with the amount of attraction or gravity at the Earth's surface. For at the Earth's surface, that is at the distance of the Earth's circumference from its centre, a body falls a little more than 16 feet, or 193 inches in one second. And since the Moon is sixty times further away from the Earth's centre than is a body at the Earth's surface, therefore a body at the distance of the Moon should fall according to this law only one-sixtieth squared as far per second as it would do at the Earth's surface. It ought then to fall 193 inches diminished in the ratio of 1 to 3,600 or $\frac{1}{19}$ inch per second. And $\frac{1}{19}$ inch is just about the amount which the Moon is actually deflected towards the Earth in each second. It was from the satisfactory proof of the Earth's attraction on the Moon that Newton was led on to his great discovery of the universality of the law of reciprocal attraction between all the bodies in space. Hence each body is a centre of attraction extending infinitely into space, and hence results the almost infinite complexity of the celestial motions. He further proved by very subtle and beautiful calculations, that any body moving under the influence of a central mass, must describe some kind of conic. A conic section is the curve traced out by a point which moves in such a manner that its distance from a given fixed point called the focus continually bears the same ratio to its distance from a given imaginary fixed line called the directrix. When this ratio is unity the curve will be a parabola, when more than unity an hyperbola, when less than unity an ellipse. As then the curve must be a conic, it must be either an ellipse (a circle is only one form of an ellipse) or a parabola or else a hyperbola. What the particular conic would actually be in any given case, would depend upon the original or primitive velocity and direction imparted to the circulating body. It may be as well to remind you of the principal practical characteristics of these three kinds of curves called conic sections which are traced out by celestial bodies. And these curves are called conic sections, because, when a right circular cone (not any cone) is intersected by a plane surface, the boundary of the section so formed will be one or other of these curves.

Thus firstly, the ellipse is a plane of section which cuts completely across a right cone, coming out at both slanting sides, but lower down on one side than on the other. A parabola is a plane which cuts a right cone parallel to its opposite slant-side (that is at an angle equal to the constant angle which the generating line forms with the axis) but does not come out at both sides, and is such that its two extremities or legs continually approach each other but never meet, whereas the legs of an hyperbola (which cuts a cone otherwise than parallel to one of its slant-sides) diverge practically to infinity.

Now as the Moon does not fly away from, but goes round and round the Earth, it is obvious from the definitions we have given of the parabola and the hyperbola that she can not be moving in a parabolic nor an hyperbolic curve. She must then move in an ellipse. To be more precise, however, both the Earth and the Moon describe similar ellipses (the Moon's path being eighty times greater than that of the Earth, because its mass is eighty times less) around their common centre of gravity. However, in treating of the motion of the Moon around the Earth, it is convenient in all mathematical calculations to reduce the motion of the Earth to zero, and the mass of the Moon to zero, ascribing the whole mass of the two bodies to the Earth, and all the motion to the Moon. Thus we can place the centre of gravity of the two bodies, not as it really is at about $\frac{1}{80}$ of the Moon's distance, or about 3,000 miles from the Earth's centre, but immediately at the Earth's centre.

Now we will first examine what would be the motion of the Moon around the Earth regarded as its fixed centre of gravity, if there were no Sun or planets to disturb her in her orbit. She would move round and round the Earth, for ever describing exactly the same ellipse, exactly obeying the mathematical laws of motion of two bodies in space, which I have mentioned. The Moon's motion in this ellipse is brought about in the following manner.

When the Moon is at apogee or at the point which is farthest from the Earth, the Earth's attraction then overcomes her velocity, and brings her towards itself with such an accelerated motion that she at length overcomes the Earth's attraction and shoots past the Earth as it were, her velocity at perigee prevailing over the Earth's attraction. She then gradually decreases in velocity until she again arrives at apogee, where the Earth's attraction again prevails over her velocity. This process, if the Moon were undisturbed in her orbit, would repeat itself indefinitely. Thus the radii vectores would for ever sweep out equal areas in equal periods of time; her lineal velocity would be always proportional to her momentary distance from the Earth, her angular velocity being proportional to the square of that distance; or stating the case in less mathematical language, if we compare the Moon at perigee and apogee, then at perigee the radius vector would sweep out a precisely equal area at perigee as at apogee, but the Moon's velocity would be greater than at apogee exactly in proportion as its radius vector would be shorter, and her angular velocity would be greater as the square of this proportion. Thus, supposing for simplicity's sake, that the Moon were twice as near the Earth at perigee as at apogee, then her linear velocity would be twice as great and her angular velocity four times as great at perigee as at apogee. The true amount of her ellipticity can be calculated from the variation of her apparent diameter, which ranges from $29\frac{1}{2}$ minutes to $33\frac{1}{2}$ minutes of arc, which points to her ellipticity being about $\frac{1}{18}$th—over three times as great as the ellipticity of the Earth's present orbit. In order to predict the Moon's position in her ellipse, or in other words to form lunar tables, we must be acquainted with what are called the "elements" of an elliptical orbit. We must know, that is to say, the greater axis of the orbit, the ratio of eccentricity, which is the ratio of half the lesser axis to half the greater axis, the longitude of her perigee, and that of the ascending node, the inclination or the angular projection of her orbit to the plane of the ecliptic, and lastly the longitude of her epoch, or the starting point as it were for our calculations. The first two "elements" determine the nature of the Moon's orbit, the three following its position in space, and the last is the relation of her present position to what it was at a given point of time. The average distance of the Moon, found from its parallax, being about 60 × radius of the Earth, or 239,000 miles, and her ellipticity being known to be $\frac{1}{18}$th, it can easily be calculated that her distance from the Earth must vary from about 221,000 miles at perigee to about 253,000 miles at apogee, a difference of 32,000 miles. The number of miles which the Moon has to travel in each lunation, being π × mean radius (regarding her orbit as circular) is therefore 6·2832 × 239,000 miles, or about $1\frac{1}{2}$ million of miles, or on the average 55,000 miles a day, or 2,300 miles or rather more than her own diameter in one hour, or 1,133 yards in one second. From the Moon's mean velocity per second (or what her velocity would be if she moved in a circle instead of an ellipse) can easily be found what her true velocity really is at any given moment by applying what is called the "Equation of the centre," thereby reducing her imaginary circular motion to her true motion in an orbit of ·055, or about $\frac{1}{18}$th ellipticity. Incidentally I may remind you that the true form of the Moon's orbit with reference to the Sun, is not any series of ellipses nor looped spirals nor cycloids nor even trochoids, but it is nothing else than the orbit of the Earth, with very slight depressions and elevations of its concavity towards the Sun at each New and Full Moon. The Moon's orbit (contrary to what is often imagined) is always concave towards the Sun, even when at the point nearest to the Sun. She will then only be about $\frac{1}{2}$ × distance from the Earth towards the centre of the chord, which joins the two points where she crosses the Earth's path at Quadratures.

As to the Moon's rotatory motion, I need only remind you, that owing probably to the Earth's attraction on some slight protuberance on the Moon's surface (analogous to a fixed tidal wave) she always presents to the Earth the same face, and therefore she rotates synodically once in rather less than $27\frac{1}{3}$ days. In other words, the Moon rotates absolutely $13\frac{1}{2}$ times in a year, and relatively to the Sun and Earth $12\frac{1}{2}$ times. The actual rotatory motion therefore of a point on her equator would be about 10 miles an hour or $\frac{1}{104} \times$ the corresponding rotatory velocity at the equator of the Earth. As the Moon's orbital velocity is variable, and her rotatory velocity is invariable, we consequently see from the Earth's surface sometimes a little in front of her so to speak and sometimes a little behind. This "libration in longitude" amounts to about $7\frac{3}{4}°$ either way. Thus we can see about 15° more of her surface longitudinally than if her orbital velocity were invariable, besides another degree in longitude, by reason of other inequalities in her orbital motion, which we are about to mention, as due to the Sun's disturbing influence. And as her polar axis is inclined $1\frac{1}{2}°$ to the plane of her orbit, and that again is inclined about 5° to the plane of the ecliptic, we can therefore see $5° + 1\frac{1}{2}°$ or $6\frac{1}{2}°$ beyond either pole according as the Moon is at one side or the other of her path around the Earth. Thus her "libration in latitude" is about twice $6\frac{1}{2}°$ or 13°. The net result of the Moon's librations in longitude and latitude is that we are enabled to see at one time or another $\frac{4}{7} \times$ her whole surface instead of only $\frac{1}{2}$ of it. So far we have been treating of the Moon's motion as a simple ellipse around the Earth. We must now consider the far more difficult and complicated problems connected with the "disturbances" or "inequalities" of this original elliptical motion produced by the Sun's action. These perturbations or inequalities are in the nature of a superposition of small motions upon the main or normal elliptical motion of the Moon regarded simply as revolving about the Earth as its fixed focus. By acting unequally upon Earth and Moon the Sun destroys the mathematical exactness of the Moon's elliptical motion. Thus owing to the Sun's disturbances the Moon does not in fact move in any known or symmetrical curve, but in a path which sometimes approaches to and sometimes recedes from the true elliptical form, and her radii vectores do not sweep out equal areas in equal times. And the amount of the disturbing forces upon the Moon's orbit can be judged from the extent of her deviation from true elliptical motion. Although many of these perturbations are very small in themselves in each lunation, yet in the lapse of ages some of them accumulate so as to become very considerable, and may so modify the Moon's motions after long periods of time, as to render the original elements of her orbit quite inadequate. It is utterly beyond the scope of a short paper like this, to describe any but the largest and most important of these inequalities. Over fifty of them are taken into account in astronomical ephemerides in longitude, and over twenty in latitude. To account for all the Moon's inequalities which are almost infinite in number, is even beyond the present power of mathematics to accomplish. We will content ourselves with trying to get a clear idea of some of the most important (because the largest) of these inequalities in the motions of the Moon. Now when two bodies revolve round their common centre of gravity, and a third body is present to modify or disturb their motions by its attraction, if this third body is very far away (or very small), its action upon the two former is called a "disturbance" or "perturbation," and this third body is called the "disturbing force." Thus in the case of the Earth, Moon and Sun, the Sun is the far away disturbing force, the Earth is the fixed central body and the Moon is the disturbed body. Now it must be thoroughly grasped and understood at the outset, that the disturbing power of the third body depends, not upon its force of attraction absolutely but upon the difference (whether this difference be in amount or direction or both) of its attraction upon the two bodies that it disturbs. The mean difference or overplus of attraction by the Sun upon Earth and Moon, does not amount to more than $\frac{1}{640,000} \times$ gravity at the Earth's surface. And this disturbing force is continually varying according to the temporary configuration of the three bodies. As the Sun is about 400 times more remote than the Moon, the Moon is therefore alternately $\frac{1}{400}$ part nearer and $\frac{1}{400}$ part farther from the Sun at New Moon and Full Moon respectively. And it is from these unequal distances and therefore unequal attracting forces that the Sun's disturbing influence is due. If the Sun's attraction on the Earth and on the Moon were always equal and in parallel directions, then this disturbing force would be *nil*. Thus although the Sun's absolute attraction is more than double that which the Earth exerts on the Moon (for his attraction on the Moon + Earth's attraction multiplied by his mass and divided by the square of his distance of $\frac{330,000}{389}$, or more than double that of the Earth), yet his disturbing force is only $\frac{1}{179} \times$ the whole force which keeps the Moon in her orbit. Hence at New Moon the Sun does not deprive the Earth of her satellite, in spite of his attraction being twice as strong as the Earth's because the Sun's attraction on the Moon at Full Moon is only very slightly greater than it is on the Earth. Thus in order to prevent the Moon escaping, the Earth has not to exert an equal pull with that of the Sun, but only a pull equal to the difference of the amount of the Sun's pull upon Earth and Moon at the moment, and this difference of the Sun's pulls is always much less than the whole attraction which the Earth is able to exert upon the Moon. Both Earth and Moon fall towards the Sun together, this falling motion, of course, being combined with any other intrinsic motions which Earth and Moon may possess at the time. When it is New Moon, she is $\frac{1}{389}$th nearer the Sun than is the Earth. The Sun's disturbing influence then makes the Moon fall towards himself slightly faster than the Earth, the Earth's attraction on the Moon is thus diminished for the time being, and the Moon's curvature towards the Earth is diminished, and increased towards the Sun. At half Moon or quadratures, when Earth and Moon are at equal distances from the Sun, the Sun is pulling the Earth and Moon towards himself with equal force indeed but on converging lines, and thereby reinforcing the Earth's attraction on the Moon, rendering the Moon's orbit at quadratures rather more curved towards the Earth, than it would have been if there were no Sun disturbing her true elliptical orbit. The Earth's attraction on the Moon is thus weakened at Syzygies and reinforced at Quadratures, much in the same way as the tides are drawn away from the centre of the Earth, when in a line with the Moon's attraction (disregarding the effects of friction), and pulled towards the Earth's centre when at right angles to the line of the Moon's attraction. The force is directed away from the Earth, or the Earth's attraction is diminished at Full Moon as well as New Moon, because the Sun then attracts the Earth a little more than he attracts the Moon, thereby tending to separate them. Whilst the Sun is the only body which is able sensibly to disturb the Moon's elliptical motion by his direct action the planets do so indirectly, by disturbing the Earth's orbit and therefore slightly modifying the ratios of the distance of Sun, Earth and Moon. But before we enquire into the effects of the planets' attraction upon the Moon's orbital motions, we will first give our attention to the disturbances caused by the Sun alone. [*To be continued.*]

Extracts from Publications.

Planet M. T.

The mean distance of Eros from the Sun is 1,458, that of Mars 1,524, the mean distance of the Earth from the Sun being unity. When the small planet M. T. was discovered,