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Electromagnetic field due to a non-axisymmetric current loop around Kerr blackhole

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Abstract. We derive expressions for test electromagnetic field of a non-axisymmetric current loop around a Kerr blackhole. For this, we write complete solution for the 'inside' as well as 'outside' regions of the current loop using vacuum solutions of King (1977). We calculate discontinuities in two ways. First, we integrate radially the inhomogeneous Maxwell equations across a thin shell at r_0 the loop radius, written in Newmann-Penrose (NP) formalism. Second, the discontinuity in the electromagnetic field components is obtained from the complete solution in which the 'inside' solution is characterized by a complex constant α_{lm} and the 'outside' solution by β_{lm}. Now we equate the two discontinuities and determine the complete solution. A particular solution, namely, electromagnetic field of an equatorial current loop is obtained.

Key words : general relativity—blackhole—electromagnetic fields

1. Introduction

Weak electromagnetic fields in the vicinity of a blackhole have been the subject of much current interest on account of possible applications in explaining various astrophysical phenomena. However most of the authors, exploiting the simplicity of Maxwell equations for axisymmetry, assumed that electromagnetic field is stationary and axisymmetric. Here we drop the assumption of axisymmetry and investigate the general relativistic effects when the magnetic field is inclined at an angle γ with the rotation axis of the hole. This problem has been investigated by seveal authors with various motivations.

Chandrasekhar (1976), using the decoupled and separated electromagnetic equations derived by Teukolsky (1972, 1973), obtained the general solution for potentials. Pollock (1976) investigated the interaction between a weak magnetic field and a slowly rotating blackhole. He solved the Maxwell equations for the electromagnetic field tensor by a perturbation expansion, retaining only first order terms. Pollock & Brinkmann (1977) treated the same problem without the assumption of slow rotation. However, King (1977) and Bičák & Dvořák (1977) obtained the general multipole solutions of the Maxwell equations with appropriate boundary conditions.

In this paper we obtain, using the formalism developed by King, the electromagnetic field due to a current loop inclined at a given angle with the rotation axis of the hole. The basic equations and the solutions are presented in section 2. In section 3 we demonstrate that the electromagnetic field due to an equatorial loop can be obtained from the results of previous section. Conclusions are given in the last section.

2. Basic equations and solutions

We assume the electromagnetic field due to an inclined current loop around a Kerr blackhole to be stationary. The Maxwell equations, in view of stationarity and non-axisymmetry, reduce to

$$\left[\left(\frac{\partial}{\partial r}+\frac{\partial}{\Delta}\frac{\partial}{\partial\varphi}\right)+\frac{2}{(r-ia\cos\theta)}\right]\varphi_1-\left[\frac{1}{\sqrt{2}\,(r-ia\cos\theta)}\right.$$

$$\left.\times\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)-\frac{ia\rho^2\sin\theta}{\sqrt{2}}-\frac{\rho\cot\theta}{\sqrt{2}}\right]\varphi_0=-2\pi J_l, \quad \ldots(1)$$

$$\left[\frac{1}{\sqrt{2}\,(r+ia\cos\theta)}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\theta}{\varphi\partial}\right)+\frac{\sqrt{2}\,i\sin\theta}{\Sigma}\right]\varphi_1$$

$$-\left[\frac{1}{2\Sigma}\left(-\Delta\frac{\partial}{\partial r}+a\frac{\partial}{\partial\varphi}\right)-\frac{\rho\Delta}{2\Sigma}-\frac{(r-m)}{\Sigma}\right]\varphi_0=-2\pi J_m, \quad \ldots(2)$$

$$\left[\left(\frac{\partial}{\partial r}+\frac{a}{\Delta}\frac{\partial}{\partial\varphi}\right)+\frac{1}{(r-ia\cos\theta)}\right]\varphi_2-\left[\frac{1}{\sqrt{2}(r-ia\cos\theta)}\right.$$

$$\left.\times\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)+\sqrt{2}\,ia\,\rho^2\sin\theta\right]\varphi_1=-2\pi J_m, \quad \ldots(3)$$

$$\left[\frac{1}{\sqrt{2}\,(r+ia\cos\theta)}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)+\frac{ia\sin\theta}{\sqrt{2}\,\Sigma}-\frac{\bar{\rho}\cot\theta}{\sqrt{2}}\right]\varphi_2$$

$$-\left[\frac{1}{2\Sigma}\left(-\Delta\frac{\partial}{\partial r}+a\frac{\partial}{\partial\varphi}\right)-\frac{\rho\Delta}{\Sigma}\right]\varphi_1=-2\pi J_n. \quad \ldots(4)$$

The symbols in the above equations have their usual meaning.

To obtain the solution of the above equations for a specified source one obtains two linearly independent solutions of the Maxwell equations one of which is regular at the horizon and the other at infinity. The desired solution is then obtained by matching these solutions across the source. Thus, the Maxwell equations without sources can be written as

$$\left[\left(\frac{\partial}{\partial r}+\frac{a}{\Delta}\frac{\partial}{\theta\phi}\right)+\frac{2}{(r-ia\cos\theta)}\right]\varphi_1=\left[\frac{1}{\sqrt{2}\,(r-ia\cos\theta)}\right.$$

$$\left.\times\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)-\frac{ia\,\rho^2\sin\theta}{\sqrt{2}}-\frac{\rho\cot\theta}{\sqrt{2}}\right]\varphi_0, \quad \ldots(5)$$

$$\left[\frac{1}{\sqrt{2}\,(r+ia\cos\theta)}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)+\frac{\sqrt{2}\,i\sin\theta}{\Sigma}\right]\varphi_1$$
$$=\left[\frac{1}{2\Sigma}\left(-\Delta\frac{\partial}{\partial r}+a\frac{\partial}{\partial\varphi}\right)-\frac{\rho\Delta}{\Sigma}-\frac{(r-M)}{\Sigma}\right]\varphi_0, \quad \ldots(6)$$

$$\left[\left(\frac{\partial}{\partial r}+\frac{a}{\Delta}\frac{\partial}{\partial\varphi}\right)+\frac{1}{(r-ia\cos\theta)}\right]\varphi_2$$
$$=\left[\frac{1}{\sqrt{2}\,(r-ia\cos\theta)}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)+\sqrt{2}\,ia\,\rho^2\sin\theta\right]\varphi_1, \quad \ldots(7)$$

$$\left[\frac{1}{\sqrt{2}\,(r+ia\cos\theta)}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)+\frac{ia\sin\theta}{\sqrt{2}\,\Sigma}-\frac{\bar{\rho}\cot\theta}{\sqrt{2}}\right]\varphi_2$$
$$=\left[\frac{1}{2\Sigma}\left(-\Delta\frac{\partial}{\partial r}+a\frac{\partial}{\partial\varphi}\right)-\frac{\rho\Delta}{\Sigma}\right]\varphi_1. \quad \ldots(8)$$

The above equations can be decoupled and separated (Teukolsky 1972, 1973). The separation of variables for φ_0 and φ_2 may be expressed in the following form :

$$\varphi_0^{(lm)} = R_{lm}(r)\,\eth\, Y_{lm}(\theta,\varphi),$$
$$\varphi_0^{(l,m)}\,\rho^{-2} = S_{lm}(r)\,\eth'\,Y_{lm}(\theta,\varphi) \quad \ldots(9)$$

for an (l, m) multipole, where $Y_{lm}(\theta,\varphi)$ is the usual spherical harmonic function. Here l is an integer $\geqslant 1$ and m an integer with $|m| \leqslant l$. $\eth$ and $\eth'$ are the spin-raising and spin-lowering operators defined as (Goldberg *et al.* 1967)

$$\eth = -(\sin\theta)^s\left[\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right](\sin\theta)^{-s}$$
$$\eth' = -(\sin\theta)^{-s}\left[\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right](\sin\theta)^{s} \quad \ldots(10)$$

when acting over the fields of spin-weight s. $R_{lm}(r)$ and $S_{lm}(r)$ obey the equations

$$\left.\begin{aligned}&\frac{d}{dr}\left(\Delta^2\frac{dR_{lm}}{dr}\right)+[a^2m^2+2iam(r-M)\\&\qquad\qquad-(l-1)(l+1)]\,R_{lm}=0,\\&\Delta^2\frac{d^2S_{lm}}{dr^2}+[a^2m^2-2iam(r-M)-l(l+1)\,\Delta]\,S_{lm}=0.\end{aligned}\right\} \quad \ldots(11)$$

Equations (11) have been solved by King (1977) and independently by Bičák & Dvořák (1977). Thus, $R_{lm}^{(H)}(r)$ and $S_{lm}^{(H)}(r)$, the radial functions regular on the horizon, are

$$R_{lm}^{(H)} = \delta^{l+1}\frac{2p\,(2p+1)\ldots\ldots(2p+l)}{l(l+1)\ldots\ldots(2l)}$$
$$\times\,(r-r_+)^{-1+p}(r-r_-)^{-1-p}\,F\left[-l-1;\,l,\,2p,\,\frac{(r-r_+)}{(-\delta)}\right], \quad \ldots(12)$$

$$S^{(H)}_{lm} = -\delta^{l-1}\frac{(2p+2)\ldots\ldots(2p+2l)}{2(2l+2)\ldots\ldots(2l)} \times (r-r_+)^{1+p}(r-r_-)^{1-p} F\left[-l+1, l+2, 2+2p, \frac{(r-r_+)}{(-\delta)}\right].$$

Similarly, $R^{(\infty)}_{lm}(r)$ and $S^{(\infty)}_{lm}(r)$, the radial functions regular at infinity, are

$$R^{(\infty)}_{lm} = (r-r_+)^{-l-1-p}(r-r_-)^{-1-p} F\left[l, l+1-2p, 2l+2, \frac{(-\delta)}{(r+r_+)}\right], \qquad \ldots(13)$$

$$S^{(\infty)}_{lm} = -\tfrac{1}{2}(r-r_+)^{-l-1-p}(r-r_-)^{1-p} F\left[l+2, l+1-2p, 2l+2, \frac{(-\delta)}{(r-r_+)}\right].$$

In the above equations the function F is the hypergeometric function and $p = iam/\delta$; $\delta = 2\sqrt{(M^2 - a^2)}$. r_+ and r_- are the larger and smaller roots of the equation $\Delta = 0$.

The general solution of the electromagnetic quantities φ_0 and φ_2 is a sum of two independent solutions given above. Thus

$$\left.\begin{aligned}\varphi_0 &= \sum_{l,m}[\alpha_{lm} R^{(H)}_{lm} + \beta_{lm} R^{(\infty)}_{lm}]\,\eth Y_{lm},\\ \varphi_2 &= \sum_{l,m}\rho^2[\alpha_{lm} S^{(H)}_{lm} + \beta_{lm} S^{(\infty)}_{lm}]\,\eth' Y_{lm}.\end{aligned}\right\} \qquad \ldots(14)$$

Here α_{lm} and β_{lm} are the complex constants given by

$$\left.\begin{aligned}\alpha_{lm} &= \alpha^{r}_{lm} + i\,\alpha^{i}_{lm},\\ \beta_{lm} &= \beta^{r}_{lm} + i\,\beta^{i}_{lm}.\end{aligned}\right\} \qquad \ldots(15)$$

The general solution for φ_1 can also be obtained as

$$\varphi_1 = \sum_{l,m}\rho^2\Big[-ia\,2^{-3/2}\alpha_{lm}\sin\theta\,\eth Y_{lm}\int_{r_+}^{r} R^{(H)}_{lm}\,dr + ia\,2^{-1/2}\alpha_{lm}\sin\theta \times \eth' Y_{lm}\int_{r^+}^{r}\frac{S^{(H)}_{lm}}{\Delta}\,dr + 2^{-3/2}\,l(l+1)\,\alpha_{lm}Y_{lm} \times \int_{r_+}^{r}\rho^{-1}\left(R^{(H)}_{lm} - \frac{2S^{(H)}_{lm}}{\Delta}\right)dr$$

+ corresponding term for β_{lm} modes with lower limit r_+ replaced by ∞. ...(16)

Now we specify the source. We take a circular loop of radius r_0 carrying a current I placed at the angle (θ_0, φ_0) to the rotation axis of the hole. Then, components of the current density vector J^i are given by

$$\left.\begin{aligned} J^t &= J^r = J^\theta = 0, \\ J^\varphi &= \frac{I}{\Sigma}\,\delta(r - r_0)\,\delta(\cos\theta - \cos\theta_0)\,\delta(\varphi - \varphi_0), \end{aligned}\right\} \qquad \ldots(17)$$

where the delta function introduced satisfy the following normalization conditions:

$$\int \delta(r - r_0)\,dr = 1, \quad \int \delta(\cos\theta - \cos\theta_0)\sin\theta\,d\theta = 1,$$

$$\int \delta(\varphi - \varphi_0)\,d\varphi = 1.$$

By making use of equations (17) one obtains the NP current density components as

$$J_l = -\frac{a\sin^2\theta\,I}{\Sigma}\,\delta(r - r_0)\,\delta(\cos\theta - \cos\theta_0)\,\delta(\varphi - \varphi_0)$$

$$J_m = -\frac{i\sin\theta\,(r^2 + a^2)\,I}{\sqrt{2}\,(r + ia\cos\theta)\,\Sigma}\,\delta(r - r_0)\,\delta(\cos\theta - \cos\theta_0)\,\delta(\varphi - \varphi_0),$$

$$J_{\bar{m}} = \frac{i\sin\theta\,(r^2 + a^2)\,I}{\sqrt{2}\,(r - ia\cos\theta)\,\Sigma}\,\delta(r - r_0)\,\delta(\cos\theta - \cos\theta_0)\,\delta(\varphi - \varphi_0),$$

$$J_n = -\frac{\Delta a\sin^2\theta\,I}{2\Sigma}\,\delta(r - r_0)\,\delta(\cos\theta - \cos\theta_0)\,\delta(\varphi - \varphi_0). \qquad \ldots(18)$$

In order to obtain the solution of the electromagnetic field due to the current loop we calculate the discontinuities in the electromagnetic field quantities φ_0 and φ_2 by two methods. First, we integrate radially the inhomogeneous Maxwell equations (2) and (3) across a thin shell at r_0. Second, discontinuities in the field quantities are also obtained directly from equations (14). Now equating the discontinuities from the two methods yields the values of the constants which characterize the electromagnetic fields due to the source under discussion.

Discontinuities from Maxwell equations (2) and (3) are given by

$$[\varphi_0] = \frac{2\sqrt{2}\,\pi i\,(r_0^2 + a^2)\,I}{\Delta_0\,(r_0 + ia\cos\theta)}\,\delta(\cos\theta - \cos\theta_0)\,\delta(\varphi - \varphi_0),$$

$$[\varphi_2] = -\frac{\sqrt{2}\,\pi i\,(r^2 + a^2)\,I}{(r_0 - ia\cos\theta)\,(r_0^2 + a^2\cos^2\theta)}\,\delta(\cos\theta - \cos\theta_0)\,\delta(\varphi - \varphi_0),$$

where Δ_0 stands for the value of Δ at $r = r_0$.

Similarly from equations (14) discontinuities at $r = r_0$ are given by

$$\left.\begin{aligned} [\varphi_0] &= \sum_{l,m}\,[\beta_{lm}\,R^{(\infty)}_{lm} - \alpha_{lm}\,R^{(H)}_{lm}]_{r=r_0}\,\eth\,Y_{lm}, \\ [\varphi_3] &= \sum_{l,m}\frac{1}{(r - ia\cos\theta)^2}\,[\beta_{lm}\,S^{(\infty)}_{lm} - \alpha_{lm}\,S^{(H)}_{lm}]_{r=r_0}\,\eth'\,Y_{lm}. \end{aligned}\right\} \qquad \ldots(20)$$

Now, making use of equations (19), (20) and (15) for φ_0 and separating the real and imaginary parts, we get

$$\sum_{l,m} [\beta^r_{lm} R^{(\infty)}_{lm} - \alpha^r_{lm} R^{(H)}_{lm}]_{r=r_0} \eth Y_{lm}$$

$$= \frac{2\sqrt{2}\,\pi a (r_0^2 + a^2) I \sin\theta \cos\theta}{\Delta_0 (r_0^2 + a^2 \cos^2\theta)} \delta(\cos\theta - \cos\theta_0)\, \delta(\varphi - \varphi_0), \quad \ldots(21)$$

$$\sum_{l,m} [\beta^i_{lm} R^{(\infty)}_{lm} - \alpha^i_{lm} R^{(H)}_{lm}]_{r=r_0} \eth Y_{lm}$$

$$= \frac{2\sqrt{2}\,\pi r_0 (r_0^2 + a^2) I \sin\theta}{\Delta_0 (r_0^2 + a^2 \cos^2\theta)} \delta(\cos\theta - \cos\theta_0)\, \delta(\varphi - \varphi_0). \quad \ldots(22)$$

In a like manner, equations (19) and (20) give the discontinuities in φ_2.

$$\sum_{l,m} [\beta^r_{lm} S^{(\infty)}_{lm} - \alpha^r_{lm} S^{(H)}_{lm}]_{r=r_0} \eth' Y_{lm}$$

$$= -\frac{\sqrt{2}\,\pi a (r_0^2 + a^2) I \sin\theta \cos\theta}{(r_0^2 + a^2 \cos^2\theta)} \delta(\cos\theta - \cos\theta_0)\, \delta(\varphi - \varphi_0), \quad \ldots(23)$$

$$\sum_{l,m} [\beta^i_{lm} S^{(\infty)}_{lm} - \alpha^i_{lm} S^{(H)}_{lm}]_{r=r_0} \eth' Y_{lm}$$

$$= -\frac{\sqrt{2}\,\pi r_0 (r_0^2 + a^2) I \sin\theta}{(r_0^2 + a^2 \cos^2\theta)} \delta(\cos\theta - \cos\theta_0)\, \delta(\varphi_0 - \varphi_0). \quad \ldots(24)$$

To solve equations (21) and (22), we multiply them by $\bar{\eth}\, \bar{Y}_{l'm'}(\theta, \varphi) \sin\theta\, d\theta d\varphi$ and integrate within appropriate limits to obtain

$$[\beta^r_{lm} R^{(\infty)}_{lm} - \alpha^r_{lm} R^{(H)}_{lm}]_{r=r_0}$$

$$= \frac{2\sqrt{2}\,\pi a (r_0^2 + a^2) I \sin\theta_0 \cos\theta_0}{\Delta_0 (r_0^2 + a^2 \cos^2\theta_0)\, l(l+1)} [\bar{\eth}\, \bar{Y}_{lm}(\theta, \varphi)]_{\theta_0, \varphi_0}, \quad \ldots(25)$$

$$[\beta^i_{lm} R^{(\infty)}_{lm} - \alpha^i_{lm} R^{(H)}_{lm}]_{r=r_0}$$

$$= \frac{2\sqrt{2}\,\pi r_0 (r_0^2 + a^2) I \sin\theta_0}{\Delta_0 (r_0^2 + a^2 \cos^2\theta_0)\, l(l+1)} [\bar{\eth}\, \bar{Y}_{lm}(\theta, \varphi)]_{\theta_0, \varphi_0}, \quad \ldots(26)$$

where we have used the identity

$$\int_0^{2\pi} \int_{-1}^{+1} \bar{\eth}\, \bar{Y}_{l'm'}(\theta, \varphi)\, \eth Y_{lm}(\theta, \varphi) \sin\theta\, d\theta d\varphi = l(+1).$$

Likewise, if we multiply equations (23) and (24) by $\bar{\eth}\, \bar{Y}_{l'm'}(\theta, \varphi) \sin\theta\, d\theta d\varphi$ and integrate, we obtain

$$[\beta^r_{lm} S^{(\infty)}_{lm} - \alpha^r_{lm} S^{(H)}_{lm}]_{r=r_0}$$

$$= -\frac{\sqrt{2}\,\pi a (r_0^2 + a^2) I \sin\theta_0 \cos\theta_0}{(r_0^2 + a^2 \cos^2\theta_0)\, l(l+1)} [\bar{\eth}\, \bar{Y}_{lm}(\theta, \varphi)]_{\theta_0, \varphi_0}, \quad \ldots(27)$$

$$[\beta^{i}_{lm}\ S^{(\infty)}_{lm} - \alpha^{i}_{lm}\ S^{(H)}_{lm}]_{r=r_0}$$
$$= -\frac{\sqrt{2}\ \pi r_0(r^2+a^2)\ I \sin\theta_0}{(r_0^2+a^2\cos^2\theta_0)\ l(l+1)}\ \bar{\eth}'\ \bar{Y}_{lm}\ (\theta,\varphi)]_{\theta_0,\varphi_0}, \qquad \ldots(28)$$

where we have used the identity

$$\int_0^{2\pi}\int_{-1}^{+1} \bar{\eth}'\bar{Y}_{l'm'}\ (\theta,\varphi)\ \eth' Y_{lm}(\theta,\varphi)\sin\theta\ d\theta d\varphi = l(l+1).$$

The two sets of equations (25) and (26), and (27) and (28) are solved to obtain the values of the constants. In order to get the desired result we multiply equation (25) by $[S^{(H)}_{lm}]_{r=r_0}$ and equation (27) by $[R^{(H)}_{lm}]_{r=r_0}$. Thus we obtain

$$[\beta^{r}_{lm}\ R^{(\infty)}_{lm}\ S^{(H)}_{lm} - \alpha^{r}_{lm}\ R^{(H)}_{lm}\ S^{(H)}_{lm}]_{r=r_0}$$
$$= \frac{2\sqrt{2}\ \pi a(r_0^2+a^2)\ I \sin\theta_0\cos\theta_0}{\Delta_0\ (r_0^2+a^2\cos^2\theta_0)\ l(l+1)}\ [\bar{\eth}\ \bar{Y}_{lm}\ (\theta,\varphi)]_{\theta_0,\varphi_0}\ [S^{(H)}_{lm}]_{r=r_0}, \qquad \ldots(29)$$

$$[\beta^{r}_{lm}\ S^{(\infty)}_{lm}\ R^{(H)}_{lm} - \alpha^{r}_{lm}\ S^{(H)}_{lm}\ R^{(H)}_{lm}]_{r=r_0}$$
$$= -\frac{\sqrt{2}\ \pi a(r_0^2+a^2)\ I \sin\theta_0\cos\theta_0}{(r_0^2+a^2\cos^2\theta)\ l(l+1)}\ [\bar{\eth}'\ \bar{Y}_{lm}(\theta,\ \varphi)]_{\theta_0,\varphi_0}\ [R^{(H)}_{lm}]_{r=r_0}. \qquad \ldots(30)$$

Subtracting equation (29) from (30), we get

$$\beta^{r}_{lm} = -\frac{\sqrt{2}\ \pi a(r_0^2+a^2) I \sin\theta_0\cos\theta_0}{(r_0^2+a^2\cos^2\theta_0)\ l(l+1)}$$
$$\frac{\left\{[R^{(H)}_{lm}]_{r=r}\ [\bar{\eth}'\ \bar{Y}_{lm}(\theta,\ \varphi_0]_{\theta_0,\varphi_0} + \frac{2}{\Delta_0}\ [S^{(H)}_{lm}]_{r=r_0}\ [\bar{\eth}\ \bar{Y}_{lm}(\theta,\varphi)]_{\theta_0,\varphi_0}\right\}}{[R^{(H)}_{lm}\ S^{(\infty)}_{lm} - R^{(\infty)}_{lm}\ S^{(H)}_{lm}]_{r=r_0}} \qquad \ldots(31)$$

However, if we multiply equation (25) by $[S^{(\infty)}_{lm}]_{r=r_0}$ and (27) by $[R^{(\infty)}_{lm}]_{r=r_0}$ and subtract the former from the latter we obtain

$$\alpha^{r}_{lm} = -\frac{\sqrt{2}\ \pi a(r_0^2+a^2)\ I \sin\theta_0\cos\theta_0}{(r_0^2+a^2\cos^2\theta_0)\ l(l+1)}$$
$$\frac{\left\{[R^{(\infty)}_{lm}]_{r=r_0}\ [\bar{\eth}'\bar{Y}_{lm}\ (\theta,\varphi)]_{\theta_0,\varphi_0} + \frac{2}{\Delta_0}\ [S^{(\infty)}_{lm}]_{r=r_0}\ [\bar{\eth}\ \bar{Y}_{lm}(\theta,\ \varphi)]_{\theta_0,\varphi_0}\right\}}{[R^{(H)}_{lm}\ S^{(\infty)}_{lm} - R^{(\infty)}_{lm}\ S^{(H)}_{lm}]_{r=r_0}}. \qquad \ldots(32)$$

Similarly, we solve equations (26) and (28) for the imaginary parts of the constants and obtain

$$\beta^{i}_{lm} = -\frac{\sqrt{2}\,\pi r_0(r_0^2 + a^2)\, I \sin\theta_0}{(r_0^2 + a^2\cos^2\theta)\, l(l+1)}$$

$$\frac{\left\{[R^{(H)}_{lm}]_{r=r_0}\, \bar{\eth}'\,\bar{Y}_{lm}(\theta,\varphi)]_{\theta_0,\varphi_0} + \frac{2}{\Delta_0}[S^{(H)}_{lm}]_{r=r_0}\,[\bar{\eth}\,\bar{Y}_{lm}(\theta,\varphi)]_{\theta_0,\varphi_0}\right\}}{[R^{(H)}_{lm}\, S^{(\infty)}_{lm} - R^{(\infty)}_{lm}\, S^{(H)}_{lm}]_{r=r_0}}, \quad \ldots(33)$$

$$\alpha^{i}_{lm} = -\frac{\sqrt{2}\,\pi r_0(r_0^2 + a^2)\, I \sin\theta_0}{(r_0^2 + a^2\cos^2\theta_0)\, l(l+1)}$$

$$\frac{\left\{[R^{(\infty)}_{lm}]_{r=r_0}\,[\bar{\eth}'\,\bar{Y}_{lm}(\theta,\varphi)]_{\theta_0,\varphi_0} + \frac{2}{\Delta_0}[S^{(\infty)}_{lm}]_{r=r_0}\,[\bar{\eth}\,\bar{Y}_{lm}(\theta,\varphi)]_{\theta_0,\varphi_0}\right\}}{[R^{(H)}_{lm}\, S^{(\infty)}_{lm} - R^{(\infty)}_{lm}\, S^{(H)}_{lm}]_{r=r_0}} \quad \ldots(34)$$

Now, when we substitute the values of the constants given by equations (31) through (34) into equations (14) and (16) we obtain the electromagnetic field due to the inclined current loop. The expressions are lengthy and tedius and therefore are not written explicitly.

3. Particular solution—equatorial current loop

The electromagnetic field due to an equatorial loop is obtained from the solution presented in the previous section by taking $\theta_0 = \pi/2$ and $m = 0$. The solution inside the loop reads

$$\varphi_0 = \sum_{l=1}^{\infty} (\alpha_{l0} + i\alpha^{i}_{l0})\, R^{(H)}_{l0}\, \eth Y_{l0},$$

$$\varphi_2 = \frac{1}{(r - ia\cos\theta)^2} \sum_{l=1}^{\infty} (\alpha^{r}_{l0} + i\,\alpha^{i}_{l0})\, S^{(H)}_{l0}\, \eth Y_{l0},$$

$$\varphi_1 = \frac{1}{(r - ia\cos\theta)^2} \sum_{l=1}^{\infty} (\alpha^{r}_{l0} + i\alpha^{i}_{l0}) \Bigg[-ia\, 2^{-3/2} \sin\theta\, \eth Y_{l0} \int_{r_+}^{r} R^{(H)}_{l0}\, dr$$

$$+ ia\, 2^{-1/2} \sin\theta\, \eth' Y_{l0} \int_{r_+}^{r} \frac{S^{(H)}_{l0}}{\Delta} dr$$

$$+ 2^{-3/2}\, l(l+1) Y_{l0} \int_{r_+}^{r} (r - ia\cos\theta)\left(R^{(H)}_{l0} - \frac{2S^{(H)}_{l0}}{\Delta}\right) dr\Bigg]. \quad \ldots(35)$$

It is noted from equation (32) that for $\theta_0 = \pi/2$

$$\alpha^{r}_{lm} = 0.$$

Now equations (35) take the form

$$\varphi_0 = i \sum_{l=1}^{\infty} \alpha^{i}_{l0} \, R^{(H)}_{l0} \, ð Y_{l0},$$

$$\varphi_2 = \frac{1}{(r - ia\cos\theta)^2} \sum_{l=1}^{\infty} \alpha^{i}_{l0} \, S^{(H)}_{l0} \, ð' Y_{l0},$$

$$\varphi_1 = \frac{1}{(r - ia\cos\theta)^2} \sum_{l=1}^{\infty} \alpha^{i}_{l0} \Big[2^{-3/2} a \sin\theta \, ð Y_{l0} \int_{r_+}^{r} R^{(H)}_{lm} \, dr$$

$$- 2^{-1/2} a \sin\theta \, ð' Y_{l0} \int_{r_+}^{r} \frac{S^{(H)}_{l0}}{\Delta} dr$$

$$+ 2^{-3/2} i \, l(l+1) Y_{l0} \int_{r_+}^{r} (r - ia\cos\theta) \Big(R^{(H)}_{l0} - 2 \frac{S^{(H)}_{l0}}{\Delta} \Big) dr \Big] \quad \ldots(36)$$

Further by substituting the required values, equation (36) becomes

$$\varphi_0 = i \sum_{l=1}^{\infty} \alpha^{i}_{l0} \sqrt{\left(\frac{2l+1}{4\pi}\right) \frac{l! \, (l-1)\,!}{(2l)\,!}} \; \delta^l \frac{d}{dr} P_l(u) \frac{dP_l}{d\theta} (\cos\theta),$$

$$\varphi_2 = - \frac{i\Delta}{(r - ia\cos\theta)^2} \sum_{l=1}^{\infty} \alpha^{i}_{l0} \sqrt{\left(\frac{2l+1}{4\pi}\right)} \frac{l! \, (l-1)\,!}{2(2l)\,!} \; \delta^l \frac{d}{dr}$$

$$\times P_l(u) \frac{d}{d\theta} P_l (\cos\theta),$$

$$\varphi_1 = \frac{1}{\sqrt{2}(r - ia\cos\theta)^2} \sum_{l=1}^{\infty} \alpha^{i}_{l0} \sqrt{\left(\frac{2l+1}{4\pi}\right)} \frac{l! \, (l-1)\,!}{(2l)!} \; \delta^l$$

$$\times \Big[a \sin\theta \, P_l(u) \frac{d}{d\theta} P_l(\cos\theta) + i \frac{l(l+1)}{\rho} P_l(u) \, P_l(\cos)\,\theta$$

$$+ i \Delta \frac{d}{dr} P_l(u) \, P_l(\cos\theta) \Big]. \quad \ldots(37)$$

where we have used the relation

$$l(l+1) P_l(u) = \frac{d}{dr} \left[\Delta \frac{dP_l(u)}{dr} \right]$$

for the evaluation of integrals in φ_1 and the variable $u = (r - M)/\sqrt{(M^2 - a^2)}$. Equations (37) are the electromagnetic fields originally derived by Petterson (1975) with $\beta_l = 0$. In a likewise manner one can demonstrate that the solution which is regular at infinity also agrees with $\alpha_l = 0$.

4. Conclusions

The paper has been devoted to a study of the electromagnetic field due to a current loop inclined at a given angle with the rotation axis of the hole. Further, a particular solution—the electromagnetic field due to an equatorial current loop—has been derived. However, if one applies an approximation procedure, by which the radius of the loop r_0 and the current strength I are allowed to increase simultaneously in such a manner that the ratio I/r_0 stays a finite constant (P_0, say) and impose boundary conditions appropriate to a uniform magnetic field inclined at an angle γ with the rotation axis of the hole, one may obtain the components of the electromagnetic field which is asymptotically uniform and inclined at an angle at infinity. Therefore our results may be of importance in calculating the torque due to the interaction of an uniform magnetic field with a rotating blackhole.

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