Equilibrium structures in partially ionized rotating plasmas within Hall magnetohydrodynamics

V. Krishan[a] and Z. Yoshida
Graduate School of Frontier Sciences, The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa-shi, Chiba 277-8561, Japan

(Received 9 March 2006; accepted 4 August 2006; published online 11 September 2006)

The formation of equilibrium structures in partially ionized rotating plasmas, consisting of electrons, ions, and neutral molecules, including the Hall effect, is studied in order to diagnose the possible velocity and the magnetic field configurations in a self-consistent manner. A few simple examples show that the linear and the nonlinear force-free magnetic configurations along with essentially nonlinear Beltrami flow field seem to be the general features of plasmas in the special case of the Keplerian rotation relevant for astrophysical plasmas. Thus rotation along with axial bipolar flows emerges as a natural pattern in gravitationally controlled magnetohydrodynamic systems. However, the equilibrium conditions permit more general flow and the magnetic field profiles that can perhaps be fully explored numerically. A special class of equilibria with unit magnetic Prandtl number and equal values of the fractional ion mass density $\alpha=\rho_i/\rho_n$ and the Hall parameter $\epsilon=\lambda_i/L$ exists where ρ's are the uniform mass densities, λ_i is the ion inertial scale, and L is the scale of the equilibrium structure. An approximate scaling law between the ionization fraction and the scale of the structure is found. Further by expressing the not so well known ionization fraction in terms of the temperature of the system, assuming thermal equilibrium, relationships among the extensive parameters such as the scale, the neutral particle density, the flow velocity, the temperature, and the magnetic field of the equilibrium structure can be determined. There seems to be a good overlap between the Hall and the thermal equilibria. The validity of the neglect of the ion dynamics is discussed.
 [DOI: 10.1063/1.2345176]

I. INTRODUCTION

The study of plasmas with flows is as important for the laboratory plasmas[1–4] as it is for the space and astrophysical plasmas.[5–8] The flows are often associated with convective processes and or large scale atmospheric circulation. Plasmas in a gravitational field acquire the Keplerian rotation profile along with the additional possibility of the outflows in the form of jets. The interrelation of the flow and the magnetic field is of the paramount importance in the stability and the transport processes of these plasmas. The flow field in astrophysical plasmas such as the accretion disks is predominantly determined by the gravitational potential of the central object but an equivalent obvious source and form of the magnetic field does not exist, notwithstanding the fact that the magnetic field is absolutely necessary to account for the whole gamut of the observed phenomena such as the jets and the polarized radiation. There are some indicators for the permissible magnitude of the magnetic field such as the equipartition of energy, however, the choice of its configuration is often a matter of convenience. In the weakly ionized plasmas found around young stars and star forming regions, nonideal effects such as the Hall effect and the ambipolar diffusion become important contributors.[9,10] In this paper we investigate the coupled equilibria of the incompressible flow and the magnetic field of partially ionized differentially rotating plasmas with the Keplerian flows as the backdrop, including only the Hall effect as it operates in a density regime that is different from that in which the ambipolar diffusion dominates. The Hall effect introduces the characteristic ion-inertial length scale λ_i in an otherwise scale-free ideal magnetohydrodynamic (MHD) system. In Sec. II we derive the equilibrium equations describing the velocity and the magnetic field profiles along with the Bernoulli relation for the pressure profile. A few instructive examples of the Hall equilibrium are given in Sec. III. In Sec. IV we show that the equality of the two small parameters, (1) $\epsilon=\lambda_i/L$ and (2) the fractional ion mass density $\alpha=\rho_i/\rho_n$, produces a special case of the equilibrium with a unit magnetic Prandtl number.[11] It turns out to be a case of special astrophysical relevance. Since it is rather difficult to estimate fractional ionization, we replace its measurement with the temperature, assuming thermal equilibrium. Brief comments on the thermal equilibrium and its relation with the Hall equilibrium as well as the justification of the neglect of the ion dynamics in protoplanetary, protostellar, and dwarf novae disks are presented in Secs. V and VI, respectively. We end the paper with the conclusion.

II. HALL MHD OF PARTIALLY IONIZED PLASMA

We take the three component partially ionized plasma consisting of electrons (e), singly ionized ions (i) of uniform mass density ρ_i, and neutral particles (n) of uniform mass density ρ_n. The equation of motion of the electrons can be written as

[a]Present address: Indian Institute of Astrophysics, Bangalore, India. Electronic mail: vinod@iiap.res.in

$$m_e n_e\left[\frac{\partial \boldsymbol{V}_e}{\partial t}+(\boldsymbol{V}_e\cdot\nabla)\boldsymbol{V}_e\right]=-\nabla p_e-en_e\left[\boldsymbol{E}+\frac{\boldsymbol{V}_e\times\boldsymbol{B}}{c}\right]-\nu_{en}\rho_e(\boldsymbol{V}_e-\boldsymbol{V}_n),\qquad(1)$$

where ν_{en} is the electron-neutral collision frequency. For inertialess electrons ($m_e=0$), the electric field $\boldsymbol{E}$ is found to be

$$\boldsymbol{E}=-\frac{\boldsymbol{V}_e\times\boldsymbol{B}}{c}-\frac{\nabla p_e}{en_e}-\frac{\nu_{en}\rho_e(\boldsymbol{V}_e-\boldsymbol{V}_n)}{en_e}.\qquad(2)$$

This gives us Ohm's law. For $\alpha=(\rho_i/\rho_n)\ll 1$ the ion dynamics can be ignored. The ion force balance then becomes

$$0=-\nabla p_i+en_i\left[\boldsymbol{E}+\frac{\boldsymbol{V}_i\times\boldsymbol{B}}{c}\right]-\nu_{in}\rho_i(\boldsymbol{V}_i-\boldsymbol{V}_n),\qquad(3)$$

where ν_{in} is the ion-neutral collision frequency and the ion-electron collisions have been neglected for the low density ionized component. Substituting for $\boldsymbol{E}$ from Eq. (2), we find the relative velocity between the ions and the neutrals:

$$\boldsymbol{V}_n-\boldsymbol{V}_i=\frac{\nabla(p_i+p_e)}{\nu_{in}\rho_i}-\frac{\boldsymbol{J}\times\boldsymbol{B}}{c\nu_{in}\rho_i},\qquad(4)$$

$$\boldsymbol{J}=-en_e(\boldsymbol{V}_e-\boldsymbol{V}_i).\qquad(5)$$

The equation of motion of the neutral fluid is

$$\rho_n\left[\frac{\partial \boldsymbol{V}_n}{\partial t}+(\boldsymbol{V}_n\cdot\nabla)\boldsymbol{V}_n\right]=-\nabla p_n-\nu_{ni}\rho_n(\boldsymbol{V}_n-\boldsymbol{V}_i)-\rho_n\nabla\varphi_g+\mu\nabla^2\boldsymbol{V}_n,\qquad(6)$$

where φ_g is the gravitational potential and μ is the kinematic viscosity. Substituting from Eq. (4) and using $\nu_{in}\rho_i=\nu_{ni}\rho_n$, we find

$$\rho_n\left[\frac{\partial \boldsymbol{V}_n}{\partial t}+(\boldsymbol{V}_n\cdot\nabla)\boldsymbol{V}_n\right]=-\nabla p+\frac{\boldsymbol{J}\times\boldsymbol{B}}{c}-\rho_n\nabla\varphi_g+\mu\nabla^2\boldsymbol{V}_n,\qquad(7)$$

where $p=(p_n+p_i+p_e)$ is the total thermal pressure. Observe that the neutral fluid is subjected to the Lorentz force as a result of the strong ion-neutral coupling due to their collisions. The Faraday law of induction, on substituting for the electric field from Eq. (2), and the relative velocity of the ion and the neutral fluid from Eq. (4), becomes

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times\left[\left(\boldsymbol{V}_n-\frac{\boldsymbol{J}}{en_e}+\frac{\boldsymbol{J}\times\boldsymbol{B}}{c\nu_{in}\rho_i}-\frac{\nabla(p_e+p_i)}{\nu_{in}\rho_i}\right)\times\boldsymbol{B}\right]+\eta\nabla^2\boldsymbol{B},\qquad(8)$$

where $\eta=m_e\nu_{en}c^2/4\pi e^2n_e$ is the electrical resistivity predominantly due to electron-neutral collisions. One can easily identify the Hall term, $(\boldsymbol{J}/en_e)$, and the ambipolar diffusion term, $(\boldsymbol{J}\times\boldsymbol{B})$. The Hall term is much larger than the ambipolar term for large neutral particle densities or for $\nu_{in}\gg\omega_{ci}$, where ω_{ci} is the ion cyclotron frequency. We retain only the Hall effect and the resistivity as the nonideal effects in the induction equation. In this system the magnetic field is not frozen to any of the fluids and the ions and the neutrals move together. Equations (7) and (8) retaining only the Hall effect along with the mass conservation

$$\nabla\cdot\boldsymbol{V}=0\qquad(9)$$

form the basis of our investigation. We write the equations in a dimensionless form. The magnetic and the velocity fields are respectively normalized by a uniform field B_0 and the Alfvén speed $V_A=B_0/\sqrt{4\pi\rho_i}$. The time and the space variables are normalized, respectively, with the Alfvén travel time $t_A=L/V_A$, and a scale length L. The gravitational potential is normalized by V_A^2. The resistivity η and the kinematic viscosity μ are normalized by (LV_A). In these units, the following dimensionless equations,

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times[(\boldsymbol{V}_n-\epsilon\nabla\times\boldsymbol{B})\times\boldsymbol{B}]+\eta\nabla^2\boldsymbol{B},\qquad(10)$$

$$\frac{\partial(\nabla\times\boldsymbol{V}_n)}{\partial t}=\nabla\times[\boldsymbol{V}_n\times(\nabla\times\boldsymbol{V}_n)-\alpha\boldsymbol{B}\times(\nabla\times\boldsymbol{B})-\mu\nabla\times(\nabla\times\boldsymbol{V}_n)],\qquad(11)$$

constitute the dissipative Hall MHD in the incompressible limit. Here $\epsilon=\lambda_i/L=c/\omega_{pi}L$, where $\omega_{pi}=(4\pi e^2n_i/m_i)^{1/2}$ is the ion plasma frequency and λ_i is the ion inertial length. Equation (11) has been obtained by taking the curl of the equation of motion of the neutral fluid. Adding Eqs. (10) and (11) begets

$$\frac{\partial(\boldsymbol{B}+\nabla\times\boldsymbol{V}_n)}{\partial t}=\nabla\times[\boldsymbol{V}_n\times(\boldsymbol{B}+\nabla\times\boldsymbol{V}_n)-(\alpha-\epsilon)\boldsymbol{B}\times(\nabla\times\boldsymbol{B})-\mu\nabla\times(\nabla\times\boldsymbol{V}_n)-\eta\nabla\times\boldsymbol{B}].\qquad(12)$$

Substitution of the equilibrium conditions [$\partial/\partial t=0$ in Eqs. (10) and (12)] in the stationary ($\partial/\partial t=0$) force balance of the neutral fluid [Eq. (7)] furnishes the generalized Bernoulli relation (in the dimensionless form)

$$\nabla(V_n^2/2+0.5\alpha\beta_0+\varphi_g-\psi_2+\psi_1)=0,\qquad(13)$$

where $\beta_0=8\pi p/B_0^2$ and ψ_j are the potentials corresponding to the energy density of the system. Thus the equilibrium flows and the fields ($\partial/\partial t=0$) must be determined from Eqs. (10)–(13).

III. FLOWS AND FIELDS IN THE HALL EQUILIBRIUM

The general Hall-equilibrium velocity and the magnetic fields are to be determined from

$$(\boldsymbol{V}_n-\epsilon\nabla\times\boldsymbol{B})\times\boldsymbol{B}-\eta\nabla\times\boldsymbol{B}=\nabla\psi_1,\qquad(14)$$

$$[\boldsymbol{V}_n\times(\boldsymbol{B}+\nabla\times\boldsymbol{V}_n)-(\alpha-\epsilon)\boldsymbol{B}\times(\nabla\times\boldsymbol{B})-\mu\nabla\times(\nabla\times\boldsymbol{V}_n)-\eta\nabla\times\boldsymbol{B}]=\nabla\psi_2,\qquad(15)$$

and

$$\nabla(V_n^2/2+0.5\alpha\beta_0+\varphi_g)=\nabla\psi_2-\nabla\psi_1=\boldsymbol{V}_n\times(\nabla\times\boldsymbol{V}_n)-\alpha\boldsymbol{B}\times(\nabla\times\boldsymbol{B})-\mu\nabla\times(\nabla\times\boldsymbol{V}_n).\qquad(16)$$

We delineate some illustrative and instructive solutions of these equations with the Keplerian motion as an essential component. We first begin with the one-dimensional, essentially radial equilibrium for ($\mu=0$).

1. For the case of the uniform axial magnetic field $\boldsymbol{B}=B_z e_z$, often assumed as an initial condition for various investigations, we find that Eq. (16) becomes

$$\nabla(V_n^2/2+0.5\alpha\beta_0+\varphi_g)=\boldsymbol{V}_n\times(\nabla\times\boldsymbol{V}_n), \tag{17}$$

which, for uniform pressure and the gravitational potential $\varphi_g=-GM/r$, gives a nonlinear Beltrami flow

$$\nabla\times\boldsymbol{V}_n=\lambda(r)\boldsymbol{V}_n, \tag{18}$$

$$\lambda(r)=\pm\frac{1}{2r}, \tag{19}$$

with

$$V_{nr}=0,$$

$$V_{n\theta}=(GM/r)^{1/2},$$

$$V_{nz}=\pm(GM/r)^{1/2},$$

where M is the mass generating the gravitational potential. Thus the equilibrium consists of an axial bipolar flow along with the Keplerian rotation for a uniform axial magnetic field and pressure.

2. It is easy to see that a force-free magnetic field configuration,

$$\nabla\times\boldsymbol{B}=\lambda_b\boldsymbol{B}, \tag{20}$$

$$B_r=0,$$

$$B_\theta=B_0J_1(\lambda_b r),$$

$$B_z=B_0J_0(\lambda_b r),$$

is also consistent with the flow field given in Eq. (18). Here J's are the Bessel functions.

3. For $\boldsymbol{V}_n=\sqrt{\alpha}\boldsymbol{B}$, an equilibrium with sub-Alfvénic flows obtains. The Bernoulli relation, Eq. (16), gives

$$V_{nr}=0,$$

$$V_{n\theta}=(GM/r)^{1/2}, \tag{21}$$

$$V_{nz}=V_{n\theta}\left(1-\frac{V_{nT}^2}{V_\theta^2}\right)^{1/2},$$

and the corresponding magnetic field is found to be

$$B_r=0,$$

$$B_\theta=(4\pi\rho_n GM/r)^{1/2}, \tag{22}$$

$$B_z=B_\theta\left(1-\frac{V_{nT}^2}{V_\theta^2}\right)^{1/2},$$

where $V_{nT}=2K_BT/m_n$ is the thermal speed of the neutral particle. Equation (21) shows that the axial flow diminishes at higher temperatures and so does the axial magnetic field [Eq. (22)]. We can define the plasma β,

$$\beta=\frac{8\pi n_n K_B T}{B_\theta^2+B_z^2}=\frac{T}{2T_c-T}, \tag{23}$$

where $T_c(r)=GMm_n/2K_Br$. The plasma β of the equilibrium is, therefore, much less than 1 for $T\ll T_c$, and increases as T approaches T_c, becoming 1 at $T=T_c$. The axial velocity and the axial magnetic field vanish at $T=T_c$. The Keplerian velocity $V_{n\theta}=\sqrt{2}c_s$, where c_s is the isothermal sound speed. The equilibrium ceases to exist for $T>T_c$ as the axial components become imaginary quantities. The incompressibility condition also approaches its limit of validity. Thus T_c represents a critical temperature beyond which this equilibrium does not exist. Since the Bernoulli relation Eq. (16) for this case is symmetric in the axial and the azimuthal components of the velocity, one can also envisage an equilibrium with $V_{n\theta}$ and B_θ replaced by V_{nz} and B_z, respectively. In this case the hotter plasmas have lower angular momenta and the transition at $T=T_c$ is to a pure axial flow.

4. Let us consider a case of viscous equilibrium again with $\boldsymbol{V}_n=\sqrt{\alpha}\boldsymbol{B}$ and $\mu\neq0$. The Bernoulli relation Eq. (16) becomes

$$\nabla\left(\frac{V_n^2}{2}+0.5\alpha\beta_0+\varphi_g\right)=\mu\nabla^2\boldsymbol{V}_n, \tag{24}$$

with a possible solution

$$V_{nr}=-\frac{\mu}{r},$$

$$V_{n\theta}=\text{const}, \tag{25}$$

$$V_{nz}=\text{const},$$

$$\frac{d}{dr}(0.5\alpha\beta_0+\varphi_g)=0. \tag{26}$$

The radial inflow in the presence of viscosity emerges as an essential component of the flow. This equilibrium exists for a plasma with angular rotation speed inversely proportional to the radial coordinate r. This case is reminiscent of the flow exhibited by the flat rotation curves of the galaxies.

5. Consider the nonviscous equilibrium for $(\boldsymbol{B}+\nabla\times\boldsymbol{V}_n)=0$ for which $\nabla\psi_2=0$ and the Bernoulli relation becomes

$$\nabla\left(\frac{V_n^2}{2}+0.5\alpha\beta_0+\varphi_g\right)=\boldsymbol{V}_n\times(\nabla\times\boldsymbol{V}_n)+\alpha(\nabla\times(\nabla\times\boldsymbol{V}_n))\times(\nabla\times\boldsymbol{V}_n). \tag{27}$$

For the nonlinear Beltrami flow $\nabla\times\boldsymbol{V}_n=\lambda(r)\boldsymbol{V}_n$, Eq. (18), the corresponding magnetic field and the pressure profile are found to be

$$\boldsymbol{B}=-\lambda(r)\boldsymbol{V}_n, \tag{28}$$

$$\frac{d}{dr}[0.5\alpha\beta_0] = -\alpha\varphi_g\frac{d\lambda^2}{dr}. \tag{29}$$

6. Another interesting case is obtained for $\boldsymbol{V}_n=\epsilon\nabla\times\boldsymbol{B}$, $\nabla\psi_1=0=\nabla\psi_2$, and $\mu=\eta=0$. From Eqs. (10) and (11) we get

$$\epsilon^2\nabla^2\boldsymbol{B} - \alpha\boldsymbol{B} = a\nabla\times\boldsymbol{B}, \tag{30}$$

where a is a constant. The solution is a superposition of fields at two different spatial scales $(\lambda_+^{-1},\lambda_-^{-1})$ with

$$\boldsymbol{B} = c_+\boldsymbol{B}_+ + c_-\boldsymbol{B}_-, \tag{31}$$

$$\nabla\times\boldsymbol{B}_+ = \lambda_+\boldsymbol{B}_+, \tag{32}$$

$$\nabla\times\boldsymbol{B}_- = \lambda_-\boldsymbol{B}_-, \tag{33}$$

where c's are constants and

$$\lambda_\pm = -\frac{a}{2\epsilon^2} \pm \frac{1}{2\epsilon^2}\sqrt{(a^2-4\alpha\epsilon^2)}. \tag{34}$$

This is also known as the double Beltrami solution in which the Hall effect removes the degeneracy of the force-free or the so-called Taylor state. We have given a few examples of the possible equilibrium structures that may be obtained in some of the astrophysical and laboratory plasmas.

IV. A SPECIAL CASE

Let us consider a special case for which $\alpha=\epsilon$ and $\mu=\eta$, Eqs. (10) and (12) have an identical form of the type

$$\frac{\partial\boldsymbol{\Omega}_j}{\partial t} = \nabla\times[\boldsymbol{U}_j\times\boldsymbol{\Omega}_j - \chi_j\nabla\times\boldsymbol{\Omega}_j], \tag{35}$$

where $j=1,2$ and

$$\boldsymbol{\Omega}_1 = \boldsymbol{B}, \quad \boldsymbol{U}_1 = \boldsymbol{V}_n - \epsilon\nabla\times\boldsymbol{B}, \quad \chi_1 = \eta, \tag{36}$$

$$\boldsymbol{\Omega}_2 = \boldsymbol{B} + \nabla\times\boldsymbol{V}_n, \quad \boldsymbol{U}_2 = \boldsymbol{V}_n, \quad \chi_2 = \mu. \tag{37}$$

Note that the two "vorticities," $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$, differ by the vorticity $(\nabla\times\boldsymbol{V}_n)$ of the neutral fluid and the two "velocities" $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ differ by the Hall velocity $\boldsymbol{V}_H=-\boldsymbol{J}/en_e$. Equation (37) exhibits that the vorticity $\nabla\times\boldsymbol{V}_n$ and the magnetic field $\boldsymbol{B}$ share the same status and the "field" $\boldsymbol{\Omega}_j$ is frozen to the "flow" $\boldsymbol{U}_j$ in the absence of dissipation. The equilibria of the system contained in Eq. (35) can be described as

$$\boldsymbol{U}_j\times\boldsymbol{\Omega}_j - \chi_j\nabla\times\boldsymbol{\Omega}_j = \nabla\psi_j \tag{38}$$

or

$$(\boldsymbol{V}_n - \epsilon\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \mu\nabla\times\boldsymbol{B} = \nabla\psi_1 \tag{39}$$

and

$$\boldsymbol{V}_n\times(\boldsymbol{B}+\nabla\times\boldsymbol{V}_n) - \mu\nabla\times(\boldsymbol{B}+\nabla\times\boldsymbol{V}_n) = \nabla\psi_2. \tag{40}$$

Equations (39) and (40) describe the dissipative equilibrium. The condition $\alpha=\epsilon$ determines the scale of the equilibrium structure for a given neutral fluid density and the ionization fraction in the form of a scaling law given as

$$L \sim n_{14}^{-1/2}\left(\frac{n_i}{n_n}\right)^{-3/2}\ \text{cm}, \tag{41}$$

where an average ion mass $m_i=30m_p$ and the neutral particle mass $m_n=2.33m_p$ have been used with m_p as the mass of a proton and $n_n=n_{14}\times10^{14}\ \text{cm}^{-3}$, representative values for the partially ionized plasmas in protoplanetary disks. Thus for a typical value of the ionization fraction, 10^{-10}, the scale L of the equilibrium structure turns out to be of the order of 100 a.u., a typical size indeed. For the dwarf novae disks[12] with neutral density $n_{14}=4\times10^3$ and the ionization fraction $(n_i/n_n)=1.3\times10^{-8}$, the scale L turns out to be 10^{10} cm, which is again of the expected magnitude.

In order to examine the second condition of the equality of the kinematic viscosity μ and the electrical resistivity η or the unit magnetic Prandtl number, we assume that the electrical resistivity is due to the electron-neutral collisions and is given by

$$\eta = 234(n_n/n_e)T^{1/2}\ \text{cm}^2\ \text{s}^{-1} = 10^{14}T_3\ \text{cm}^2\ \text{s}^{-1}, \tag{42}$$

where $T=T_3\times10^3$ K. The kinematic viscosity is given by

$$\mu = 2.2\times10^{-16}\frac{T^{5/2}}{\rho_n}\ \text{cm}^2\ \text{s}^{-1} = 18T_3^{5/2}n_{14}^{-1}\ \text{cm}^2\ \text{s}^{-1}. \tag{43}$$

Thus it seems that the viscosity is far too small to be equal to the resistivity. We take this as a pointer to the anomalous viscosity. For the standard model[13] of the turbulent viscosity, $\mu_t\sim\alpha_d c_s H$, where c_s is the sound speed, H is the scale height, and α_d is an undetermined multiplier representing the angular momentum transport efficiency. For $c_s=2\times10^5T_3\ \text{cm s}^{-1}$ and $H=0.1$ a.u., we find that $\mu=\eta$ if $\alpha_d\sim5\times10^{-4}$ for protoplanetary disks. For the dwarf novae disks with $T=3000$ K, $n_e=8.5\times10^{11}\ \text{cm}^{-3}$, $n_n=3.7\times10^{17}\ \text{cm}^{-3}$, $c_s=3.5\times10^5\ \text{cm s}^{-1}$, $H=7.7\times10^7$ cm, we find $\alpha_d=2\times10^{-4}$. The dissipative Hall equilibrium, therefore, appears to be an aftermath of a turbulent state.

V. THERMAL EQUILIBRIUM

We see that the fractional ionization factor (n_i/n_n) plays an important role in the Hall equilibrium. Since it is rather difficult to fix it quantitatively as a host of ionization processes may contribute to it, it may be better to replace its measure by an effective equivalent temperature. Identifying the net ionization with thermal ionization, assuming thermal equilibrium, we can use Saha's ionization equation to relate the ionization fraction to the temperature as

$$\frac{n_i}{n_n} = 2.4\times10^{15}T^{3/2}\frac{n_{Na}}{n_i n_n}\exp(-U_{Na}/K_BT), \tag{44}$$

where n_{Na} is the number density of neutral atoms (sodium atoms in astrophysical conditions) and U_{Na} (=5.2 eV for sodium) is its ionization energy. Using the equilibrium condition $\alpha=\epsilon$, the scale L of the structure (with sodium ions) can be related to the temperature as

$$L = 1.2\times10^{-3}n_{14}^{1/4}(n[\text{Na}]_{-2})^{-3/4}T_3^{-9/8}\exp(45.15/T_3)\ \text{cm}. \tag{45}$$

This structure has an ionization fraction of

$$\frac{n_i}{n_n} = 87 n_{14}^{-1/2} (n[\mathrm{Na}]_{-2})^{1/2} T_3^{3/4} \exp(-30.1/T_3), \quad (46)$$

where we have assumed $n[\mathrm{Na}] \equiv n_{\mathrm{Na}}/n_n = 10^{-2}$ and other symbols have their usual meaning. One notices the extreme sensitivity of the size L for the temperature T. Thus at $T_3=1$, $n_{14}=1$, $n[\mathrm{Na}]_{-2}=1$, the scale L turns out to be 3.4×10^3 a.u. and the ionization fraction $\sim 7 \times 10^{-12}$, characteristics of the weakly ionized star forming regions. We find $L=51$ a.u. with an ionization fraction of $\sim 10^{-12}$ at $T_3=1.1$. For the dwarf novae disks with $n_n=3.73 \times 10^{17}$ cm^{-3}, $T_3=1.6$, the scale L turns out to be 10^{10} cm and the ionization fraction $\sim 10^{-8}$, again very reasonable values.

VI. INCLUSION OF THE ION DYNAMICS

The ion dynamics becomes important as the fractional ion mass density α increases. The condition for neglecting the ion dynamics in a partially ionized plasma is that the electron-neutral collision frequency remains larger than the electron-ion collision frequency. The electron-neutral collision frequency ν_{en} and the electron-ion collision frequency ν_{ei} are given as[14]

$$\nu_{en} = 8.28 \times 10^{-10} n_n T^{1/2}\ \mathrm{s}^{-1}, \quad (47)$$

$$\nu_{ei} = 16 n_i T^{-3/2}\ \mathrm{s}^{-1}. \quad (48)$$

The condition $\nu_{en} > \nu_{ei}$ gives

$$\frac{n_i}{n_n} < 5 \times 10^{-11} T^2 = 5 \times 10^{-5} T_3^2. \quad (49)$$

Thus α must be smaller than 6.4×10^{-4} at $T_3=1$, the condition amply satisfied in the cases considered here. We realize that as the temperature and therefore the ionization fraction increases, the scale of the equilibrium structure decreases and the structure will eventually become unobservable. The ion-dynamics dominated Hall equilibrium would furnish structures on the scale of the ion inertial scale, which decreases as the ion density increases. Thus the Hall effect may not be on an observable scale.

VII. CONCLUSION

The Hall effect in weakly ionized rotating plasmas plays a decisive role in determining the equilibrium flows, magnetic fields, and pressure profiles. A variety of velocity and magnetic field profiles including the so-called double Beltrami emerge in the equilibrium. A few examples, although by no means exhaustive, have been given. The inclusion of the gravitational potential necessarily leads to an axial flow along with the Keplerian rotation. This indicates the likely generation of jet structures. Inward radial flow emerges as an essential feature of the viscous equilibrium. It is possible, sometimes, to arrive at the stability characteristics from the very nature of the equilibrium solution. For example, a linear force-free magnetic field along with the linear Beltrami flow represents the minimum energy state and is therefore stable whereas the nonlinear force-free magnetic field with linear or nonlinear Beltrami flow does not represent the minimum energy state and thus liable to instability. One may observe that the Keplerian flow along with the axial flow discussed in cases 1 and 2 is describable by a nonlinear Beltrami flow and therefore could become unstable. The full implications of the Bernoulli relation (16) along with the equilibrium conditions (14) and (15) can be grasped only with numerical evaluation, which we plan to take up immediately.

A special case of equilibrium exists for a structure size determined from the equality of the Hall parameter ϵ and the ionization fraction α for a unit value of the magnetic Prandtl number demanding an anomalous viscosity. The scales, the ionization fractions, and the temperatures determined for this special case furnishes an expected range of values for some of the astrophysical plasmas. In addition, the scaling law between the spatial scale and the ionization fraction could provide a good handle on the average characteristics of weakly ionized plasmas in protoplanetary disks over a range of physical conditions. The critical state with unit plasma β represents the end state of the magnetically supported structures. The thermal equilibrium is found to have a good overlap with the Hall equilibrium over a range of temperatures in the neighborhood of 1000 K.

The neglect of the ion dynamics has been shown to be valid for protoplanetary, protostellar, and the dwarf novae disks. Once the ion dynamics takes over, the Hall effect operates on rather microscopic unobservable spatial scales. Having obtained these equilibria, it goes without saying that this is where one should begin to explore the much sought after instabilities. Since a special case of the Hall equilibrium already demands an anomalous viscosity, the instabilities and hence the turbulence responsible for producing an anomalous viscosity may be different from those to which the Hall equilibrium itself may be subjected. We believe this first attempt at investigating the new equilibria of the partially ionized rotating plasmas will prove to be instructive and insightful.

ACKNOWLEDGMENTS

The authors thank the referee for very many useful suggestions. The authors express their deep gratitude to Dr. Baba Varghese for his help in the preparation of the manuscript.

[1] E. Hameiri and R. Torasso, Phys. Plasmas **11**, 4934 (2004).
[2] L. Guazzotto, R. Betti, J. Manickam, and S. Kaye, Phys. Plasmas **11**, 604 (2004).
[3] S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).
[4] Z. Yoshida and S. M. Mahajan, Phys. Rev. Lett. **88**, 095001 (2002).
[5] K. Staciewicz, P. K. Shukla, G. Gustaffson, S. Buchert, B. Lavraud, B. Thide, and Z. Klos, Phys. Rev. Lett. **90**, 085002 (2003).
[6] K. Staciewicz, Phys. Rev. Lett. **93**, 125004 (2004).
[7] V. Krishan and S. M. Mahajan, Sol. Phys. **220**, 29 (2004).
[8] V. Krishan and S. M. Mahajan, J. Geophys. Res. **109**, A11105 (2004).
[9] S. A. Balbus and C. Terquem, Astrophys. J. **552**, 235 (2001).
[10] R. Numata, Z. Yoshida, and T. Hayashi, Comput. Phys. Commun. **164**, 291 (2004).
[11] G. K. Batchelor, Proc. R. Soc. London, Ser. A **201**, 405 (1950).
[12] C. F. Gammie and K. Menou, Astrophys. J. **492**, L78 (1998).
[13] N. I. Shakura and R. A. Sunyaev, Astron. Astrophys. **24**, 337 (1973).
[14] H. Alfvén and C.-G. Fälthammar, 1962, *Cosmic Electrodynamics* (Clarendon, Oxford, 1962), p. 180.