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POLARIZED PARTIAL FREQUENCY REDISTRIBUTION IN SUBORDINATE LINES. I. RESONANCE SCATTERING WITH COLLISIONS

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ABSTRACT

Using a previously established theory, we derive a suitable form of the laboratory frame redistribution matrix for the resonance scattering in subordinate lines, allowing for the radiative as well as collisional broadening of both atomic levels involved. The lower level, though broadened, is assumed to be unpolarized. The elastic collisions both in the upper and lower levels are taken into account. We show that, in situations, when elastic collisions in the lower level can be neglected, the redistribution matrix for subordinate lines takes a form that is analogous to the corresponding case of resonance lines. Further, in the case of no-lower-level interactions (i.e., infinitely sharp lower level), we recover the redistribution matrix for resonance lines. We express the redistribution matrix for subordinate lines in terms of the irreducible spherical tensors for polarimetry. For practical applications in one-dimensional polarized radiative transfer problem, we derive the azimuth averaged subordinate line redistribution matrix.

Key words: atomic processes – line: formation – line: profiles – polarization – scattering

1. INTRODUCTION

While the problem of partial frequency redistribution (PRD) in resonance lines is well studied, the corresponding problem for subordinate lines has received little attention. This is largely due to the general belief that complete frequency redistribution is a good approximation to represent resonance scattering in subordinate lines. Even though this is somewhat justified for unpolarized scattering (see, e.g., Hubeny & Heinzel 1984; Mohan Rao et al. 1984), this is not the case when the polarization state of the radiation field is taken into account (see McKenna 1984; Nagendra 1994, 1995). A factorized form of the redistribution matrix for subordinate lines was used in Nagendra (1994, 1995). Unlike the case of resonance lines, a self-consistent expression of the laboratory frame redistribution matrix for subordinate lines taking into account the elastic collisions is still missing. The aim of the present paper is to derive such a redistribution matrix.

The problem of PRD in subordinate lines was originally addressed by Woolley & Stibbs (1953), who considered only radiatively broadened upper and lower levels. They derived the analytic form of the redistribution function for the subordinate lines in the atomic frame (AF) starting from the integral form presented by Woolley (1938). Later, using the technique of Fourier transform, Heinzel (1981) transformed the integral form of Woolley (1938) to the laboratory frame assuming a Maxwellian velocity distribution. This laboratory frame PRD function is denoted by $R_{\rm V}$. In the aforementioned papers only scattering of unpolarized radiation in subordinate lines was considered.

The polarized resonance scattering of radiation between two atomic levels broadened both radiatively and collisionally, was treated in an important paper by Omont et al. (1972). They derived AF collisional redistribution functions which are applicable to both resonance and subordinate lines. While for resonance lines their expression could be easily transformed to the laboratory frame, it was not the case for subordinate lines. Heinzel & Hubeny (1982) cleverly reformulated the quantum-mechanical AF collisional redistribution of Omont et al. (1972) for subordinate lines in the form of a linear combination of two redistribution functions $R_{\rm V}$ and $R_{\rm III}$.

In the case of resonance lines Domke & Hubeny (1988) derived the redistribution matrix for resonance scattering including collisions. Their work was based on the formalism of Omont et al. (1972). Bommier (1997a) derived a more elegant but equivalent expression for this PRD matrix with the master equation theory, which was later generalized by Bommier (1997b) to include arbitrary strength magnetic fields. This formulation is very general in the sense that it applies to polarized redistribution in resonance and also subordinate lines. For resonance lines (infinitely sharp lower level), explicit laboratory frame PRD matrices for arbitrary strength magnetic fields are derived in Bommier (1997b; see also Sampoorna et al. 2007a, 2007b; Sampoorna 2011). In the case of subordinate lines only the AF redistribution matrix for arbitrary strength fields including collisions is derived in Bommier (1997b, Equations (44)–(46)).

In the present paper we derive the laboratory frame redistribution matrix for resonance scattering in subordinate lines, taking into account collisions (both elastic and inelastic). We combine the collisional redistribution function of Heinzel & Hubeny (1982, who reformulated the expressions of Omont et al. 1972 to the case of subordinate lines), with the general redistribution matrix of Domke & Hubeny (1988, who particularized the results of Omont et al. 1972 to the case of polarization). Consequently, the redistribution matrix derived in this paper has the same physical limitations as the aforementioned studies. These limitations are the impact and isolated line approximations. Further, it is assumed that the lower level is unpolarized (i.e., all the magnetic substates of lower level are equally populated). Moreover, stimulated emission is neglected. See the monograph by Landi Degl'Innocenti & Landolfi (2004, and the references cited therein) for a sophisticated formalism where multi-level and multi-term atoms with polarization in all the levels are considered, but PRD effects are neglected. This formalism has been extended by Landi Degl'Innocenti et al. (1997) to include PRD effects in the absence of collisions, based on metalevel approach. More recently, Smitha et al. (2011) have derived the laboratory frame expression of the polarized PRD matrix for a two-term

atom with an infinitely sharp and unpolarized lower level and in the absence of collisions.

In Section 2, we recall the main equations of Domke & Hubeny (1988), but with slightly different notations. In Section 3, we briefly recall the collisional redistribution function for subordinate lines as presented by Heinzel & Hubeny (1982). In Section 4, we derive the redistribution matrix for subordinate lines using the results of Sections 2 and 3. In Section 5, we express the PRD matrix derived in Section 4 in terms of the irreducible spherical tensors for polarimetry introduced by Landi Degl'Innocenti (1984). This is essential to facilitate its use in polarized radiative transfer equation and to develop iterative techniques like polarized approximate lambda iteration (see, e.g., the reviews by Nagendra 2003; Nagendra & Sampoorna 2009). For applications in one-dimensional radiative transfer problem, we derive the azimuth averaged redistribution matrix in Section 6. Conclusions are presented in Section 7.

2. DOMKE–HUBENY REDISTRIBUTION FORMALISM

The quantum theory of resonance scattering of radiation by atoms undergoing collisions has been developed by Omont et al. (1972) in the density matrix formalism and under the assumption that the impact approximation is valid. Further they assumed an unpolarized lower level. For lines with m-degenerate levels they derived the probability density of scattering of an incident photon with frequency ω', propagation vector $\boldsymbol{n}'$, and polarization vector $\boldsymbol{\varepsilon}_1$ into a scattered photon represented by ω, $\boldsymbol{n}$, $\boldsymbol{\varepsilon}_2$. This quantity is denoted $F(\omega', \boldsymbol{n}', \boldsymbol{\varepsilon}_1 \rightarrow \omega, \boldsymbol{n}, \boldsymbol{\varepsilon}_2)$. In order to express F in the basis of Stokes parameters, Domke & Hubeny (1988) introduced the photon density matrix (which is a photon polarization matrix) and a suitable basis for the polarization vectors $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$. After an elaborate algebra Domke & Hubeny (1988) arrived at the following expression for the redistribution matrix in the AF :

$$\hat{\mathbf{F}}_{\rm AF}(\omega, \omega', \Theta) = \frac{2}{3}\{[F^{(0)}(\omega, \omega') - F^{(2)}(\omega, \omega')]\hat{\mathbf{P}}_{\rm is} + F^{(2)}(\omega, \omega')\hat{\mathbf{P}}_{\rm R}(\Theta) + F^{(1)}(\omega, \omega')\hat{\mathbf{P}}_{V}(\Theta)\}, \quad (1)$$

where Θ is the scattering angle between incident and scattered rays. $\hat{\mathbf{P}}_{\rm is}$ is the isotropic phase matrix, $\hat{\mathbf{P}}_{\rm R}(\Theta)$ is the Rayleigh phase matrix that describes scattering of the Stokes parameters I, Q, and U, and $\hat{\mathbf{P}}_{V}(\Theta)$ is the phase matrix for scattering of Stokes V parameter. The phase matrices $\hat{\mathbf{P}}_{\rm is}$, $\hat{\mathbf{P}}_{\rm R}(\Theta)$, and $\hat{\mathbf{P}}_{V}(\Theta)$ are given in the reference system defined by the scattering plane (see Equations (38)– (40) of Domke & Hubeny 1988). They can be transformed to a fixed polar reference system (also called atmospheric reference frame; see Chandrasekhar 1950). The functions $F^{(K)}(\omega, \omega')$ with $K = 0, 1, 2$ are the Kth multipole frequency redistribution functions. They are given by

$$F^{(K)}(\omega, \omega') = A_{fie}\, W(j_f\, j_i\, j_e\, K)\, f_{23}^{(K)}(\omega, \omega') + A_{fie}\, 3(2j_e+1) \sum_{K'=0}^{2} (-1)^{K+K'} (2K'+1) \times \begin{Bmatrix} 1 & 1 & K \\ 1 & 1 & K' \end{Bmatrix} \begin{Bmatrix} 1 & 1 & K' \\ j_f & j_i & j_e \end{Bmatrix}^2 f_1^{(K')}(\omega, \omega'), \quad (2)$$

where i, e, and f refer, respectively, to the initial, intermediate, and final atomic levels with total angular momentum quantum numbers j_i, j_e, and j_f. The functions $f_1^{(K)}$ and $f_{23}^{(K)}$ are called frequency profiles or line shape functions or elementary redistribution functions. They have the following form :

$$f_1^{(K)}(\omega, \omega') = \Re\left[\frac{\rm i}{\omega' - \omega - \omega_{if} - \Delta_{if}^{(K)} + {\rm i}\gamma_{if}^{(K)}} \times \frac{1}{\omega' - \omega_{ei} - \Delta_{ei}^{(1)} + {\rm i}\gamma_{ei}^{(1)}} \times \frac{1}{\omega - \omega_{ef} - \Delta_{ef}^{(1)} - {\rm i}\gamma_{ef}^{(1)}}\right], \quad (3)$$

$$f_{23}^{(K)}(\omega, \omega') = \frac{2}{\gamma_e^{(K)}} \frac{\gamma_{ei}^{(1)}}{\left(\omega' - \omega_{ei} - \Delta_{ei}^{(1)}\right)^2 + \left(\gamma_{ei}^{(1)}\right)^2} \times \frac{\gamma_{ef}^{(1)}}{\left(\omega - \omega_{ef} - \Delta_{ef}^{(1)}\right)^2 + \left(\gamma_{ef}^{(1)}\right)^2}, \quad (4)$$

where

$$\gamma_{ab}^{(K)} = \gamma_{ab}^{c(K)} + \frac{1}{2}\left(\Gamma_{\rm R}^{(a)} + \Gamma_{\rm R}^{(b)}\right), \quad (5)$$

$$\gamma_{a}^{(K)} = \gamma_{a}^{c(K)} + \Gamma_{\rm R}^{(a)}, \quad (6)$$

with $a, b = i, e, f$. Here $\Gamma_{\rm R}^{(a)}$ is the radiative decay rate of the level a, $\gamma_{ab}^{c(K)}$ is the collisional relaxation rate of the K-multipole between levels a and b, $\gamma_a^{c(0)}$ is the rate of inelastic collisions from level a, $\gamma_a^{c(1)}$ and $\gamma_a^{c(2)}$ are, respectively, the rates of destruction of the orientation and alignment of level a, $\Delta_{ab}^{(K)}$ is the collisional frequency shift, and ω_{ab} is the frequency corresponding to the $a \rightarrow b$ transition. Other quantities appearing in Equation (2) are given by

$$A_{fie} = 2\,|\langle j_f||\boldsymbol{\mu}||j_e\rangle|^2\,|\langle j_i||\boldsymbol{\mu}||j_e\rangle|^2\,\frac{\rho(j_i)}{3(2j_e+1)}, \quad (7)$$

$$W(j_f\, j_i\, j_e\, K) = (-1)^{j_i - j_f}\, 3(2j_e+1) \times \begin{Bmatrix} 1 & 1 & K \\ j_e & j_e & j_f \end{Bmatrix} \begin{Bmatrix} 1 & 1 & K \\ j_e & j_e & j_i \end{Bmatrix}, \quad (8)$$

where $\langle j_a||\boldsymbol{\mu}||j_b\rangle$ is the reduced matrix element of the dipole operator and $\rho(j_i)$ is the population density of level j_i.

3. ELEMENTARY REDISTRIBUTION FUNCTIONS FOR SUBORDINATE LINES

The elementary redistribution function $f_1^{(K)}$ is not suitable for transformation to the laboratory frame. Therefore, Heinzel & Hubeny (1982) rewrote $f_1^{(K)}$ in a form that allows for a direct transformation to the laboratory frame. They considered both the non-degenerate and spatially degenerate cases. In the spatially degenerate case, they showed that Equations (3) and (4) can be rewritten in terms of the AF redistribution functions derived in Heinzel (1981) and Hummer (1962). According to Heinzel & Hubeny (1982), $f_1^{(K)}$ and $f_{23}^{(K)}$ can be written as

$$f_1^{(K)}(\omega, \omega') = \frac{2\pi^2}{2\gamma_{ei}^{(1)} - \gamma_i^{(K)}}\left[r_{\rm V}^{(K)} - r_{\rm III}^{\rm sl}\right], \quad (9)$$

$$f_{23}^{(K)}(\omega, \omega') = \frac{2\pi^2}{\gamma_e^{(K)}}\, r_{\rm III}^{\rm sl}, \quad (10)$$

where

$$r_{\rm V}^{(K)} \equiv r_{\rm V}\left(\omega,\ \omega',\ \omega_{ei}+\Delta_{ei}^{(1)};\ \gamma_i^{(K)},\ \gamma_{ei}^{(1)}\right), \tag{11}$$

$$r_{\rm III}^{\rm sl} \equiv r_{\rm III}\left(\omega,\ \omega',\ \omega_{ei}+\Delta_{ei}^{(1)};\ \gamma_{ei}^{(1)}\right). \tag{12}$$

In the above equations $r_{\rm V}$ and $r_{\rm III}$ are type-V and type-III AF redistribution functions derived in Heinzel (1981) and Hummer (1962), respectively. We recall that ω_{ei} is now replaced by $\omega_{ei}+\Delta_{ei}^{(1)}$ both in the case of type-V and type-III redistribution. For type-V redistribution the lower level width is now given by $\gamma_i^{(K)}$ and that of the upper level by $\gamma_{ei}^{(1)}$. For type-III redistribution the total damping width is given by $\gamma_{ei}^{(1)}$. For notational simplicity, we denote the type-III redistribution function for subordinate lines as $r_{\rm III}^{\rm sl}$, unlike Heinzel & Hubeny (1982) who denote it as $r_{\rm III}^{(1)}$. We remark that for resonance lines the elementary redistribution function f_1 is independent of K (see Equation (44) of Domke & Hubeny 1988). For subordinate lines the dependence of f_1 on K gives rise to several terms in the redistribution matrix (see Section 4 below).

4. COLLISIONAL REDISTRIBUTION MATRIX FOR SUBORDINATE LINES

Substituting Equations (9) and (10) into Equation (2), we obtain

$$F^{(K)}(\omega,\ \omega') = 2\pi^2 A_{fie}\left\{\sum_{K'=0}^{2}\frac{C_{KK'j_ij_e}}{2\gamma_{ei}^{(1)}-\gamma_i^{(K')}}\,r_{\rm V}^{(K')} + \left[\frac{W(j_i\ j_i\ j_e\ K)}{\gamma_e^{(K)}} - \sum_{K'=0}^{2}\frac{C_{KK'j_ij_e}}{2\gamma_{ei}^{(1)}-\gamma_i^{(K')}}\right] r_{\rm III}^{\rm sl}\right\}, \tag{13}$$

where we have used the fact that $j_f = j_i$ for a two-level atom. The coefficients $C_{KK'j_ij_e}$ are given by

$$C_{KK'j_ij_e} = (-1)^{K+K'}\,3(2j_e+1)\,(2K'+1) \times \begin{Bmatrix}1 & 1 & K\\ 1 & 1 & K'\end{Bmatrix}\begin{Bmatrix}1 & 1 & K'\\ j_i & j_i & j_e\end{Bmatrix}^2. \tag{14}$$

The astrophysical redistribution matrix for a subordinate line is related to $\hat{\mathbf{F}}_{\rm AF}$ defined in Equation (1), by the following relation (see, e.g., Equation (49) of Domke & Hubeny 1988) :

$$\hat{\mathbf{R}}_{\rm AF}^{\rm sl}(\omega,\ \omega',\ \Theta) \equiv \frac{\hat{\mathbf{F}}_{\rm AF}(\omega,\ \omega',\ \Theta)}{\left[(4\pi^2/3)\,A_{fie}\,W(j_i\ j_i\ j_e\ 0)/\Gamma_{\rm R}^{(e)}\right]}, \tag{15}$$

where the superscript "sl" stands for subordinate line. Substituting Equations (1) and (13) into the above equation, we obtain

$$\hat{\mathbf{R}}_{\rm AF}^{\rm sl}(\omega,\ \omega',\ \Theta) = \hat{\mathbf{P}}_{\rm R}(\Theta)\left\{\sum_{K'=0}^{2}\alpha^{(K')}\bar{C}_{2K'j_ij_e}r_{\rm V}^{(K')} + \left[W_2\beta^{(2)} - \sum_{K'=0}^{2}\alpha^{(K')}\bar{C}_{2K'j_ij_e}\right] r_{\rm III}^{\rm sl}\right\} + \hat{\mathbf{P}}_{\rm is}\left\{\sum_{K'=0}^{2}\alpha^{(K')}\left(\bar{C}_{0K'j_ij_e}-\bar{C}_{2K'j_ij_e}\right) r_{\rm V}^{(K')} + \left[\beta^{(0)} - W_2\beta^{(2)} - \sum_{K'=0}^{2}\alpha^{(K')} \times \left(\bar{C}_{0K'j_ij_e}-\bar{C}_{2K'j_ij_e}\right)\right] r_{\rm III}^{\rm sl}\right\} + \hat{\mathbf{P}}_V(\Theta)\left\{\sum_{K'=0}^{2}\alpha^{(K')}\bar{C}_{1K'j_ij_e}r_{\rm V}^{(K')} + \left[W_1\beta^{(1)} - \sum_{K'=0}^{2}\alpha^{(K')}\bar{C}_{1K'j_ij_e}\right] r_{\rm III}^{\rm sl}\right\}, \tag{16}$$

where, following Domke & Hubeny (1988), Heinzel & Hubeny (1982), and Bommier (1997a, 1997b), we have defined

$$\alpha^{(K)} = \frac{\Gamma_{\rm R}^{(e)}}{2\gamma_{ei}^{(1)}-\gamma_i^{(K)}}, \tag{17}$$

$$\beta^{(K)} = \frac{\Gamma_{\rm R}^{(e)}}{\gamma_e^{(K)}}, \tag{18}$$

$$\bar{C}_{KK'j_ij_e} = \frac{C_{KK'j_ij_e}}{W(j_i\ j_i\ j_e\ 0)}, \tag{19}$$

and

$$W_K = \frac{W(j_i\ j_i\ j_e\ K)}{W(j_i\ j_i\ j_e\ 0)}. \tag{20}$$

Equation (16) has similar structure and the same physical interpretation as Equation (49) of Domke & Hubeny (1988) for resonance lines. However the branching ratio α, which is K-independent in the case of resonance line, now depends on K due to the presence of elastic collisions in the lower level (see below). Further the frequency redistribution terms are more complex in the case of subordinate lines than in the case of resonance lines.

We can deduce from Equations (41) and (44) of Ballagh & Cooper (1977) and Equation (2.17) of Heinzel & Hubeny (1982) that

$$2\gamma_{ab}^{c(1)} = \Gamma_{\rm E} + \Gamma_{\rm I}^{(a)} + \Gamma_{\rm I}^{(b)}, \tag{21}$$

where $a, b = i, e$. We recall that i denotes the initial level and e the intermediate level for a two-level atom. In Equation (21), $\Gamma_{\rm E}$ is the rate of elastic collisions and $\Gamma_{\rm I}^{(a)}$ is the inelastic collisional frequency for level a. Following Faurobert-Scholl (1992) and Nagendra (1994), we identify the notation $\gamma_a^{c(K)}$ with

$$\gamma_a^{c(K)} = \Gamma_{\rm I}^{(a)} + D_a^{(K)}, \tag{22}$$

where $a = i, e$, and $D_a^{(K)}$ are $2K$-multipole destruction rates for level a. Note that $D_a^{(0)} = 0$. Substituting Equations (5), (6), (21), and (22) into Equations (17) and (18), we obtain

$$\alpha^{(K)} = \frac{\Gamma_{\rm R}^{(e)}}{\Gamma_{\rm E}+\Gamma_{\rm I}^{(e)}+\Gamma_{\rm R}^{(e)}-D_i^{(K)}}, \tag{23}$$

$$\beta^{(K)} = \frac{\Gamma_{\rm R}^{(e)}}{\Gamma_{\rm R}^{(e)}+\Gamma_{\rm I}^{(e)}+D_e^{(K)}}. \tag{24}$$

Equation (23) shows that the K-dependence of α comes from the elastic collisions in the lower level. We note that our definition

of branching ratio $\beta^{(K)}$ for subordinate lines is identical to $\beta^{(K)}$ for resonance lines defined by Bommier (1997b).

When the elastic collisions in the lower level are negligible, i.e., $D_i^{(K)} = 0$ for all K, $\alpha^{(K)}$ becomes independent of K and is identical to the α defined for resonance lines (see Equation (50) of Domke & Hubeny 1988, or Equation (88) of Bommier 1997b). Further $r_{\rm V}^{(K)}$ also becomes independent of K and is given by

$$r_{\rm V}^{(K)} = r_{\rm V}\left(\omega,\ \omega',\ \omega_{ei} + \Delta_{ei}^{(1)};\ \Gamma_{\rm R}^{(i)} + \Gamma_{\rm I}^{(i)},\ \gamma_{ei}^{(1)}\right). \tag{25}$$

Using Equation (43) of Domke & Hubeny (1988), one also has

$$\sum_{K'=0}^{2} \alpha^{(K')} \bar{C}_{KK'j_i j_e} = \alpha W_K. \tag{26}$$

The redistribution matrix (see Equation (16)) takes then the following simpler form :

$$\begin{aligned}\left[\hat{\mathbf{R}}_{\rm AF}^{\rm sl}(\omega,\ \omega',\ \Theta)\right]_{D_i^{(K)}=0} &= \hat{\mathbf{P}}_{\rm R}(\Theta)\left\{\alpha W_2 r_{\rm V} + W_2\left[\beta^{(2)} - \alpha\right] r_{\rm III}^{\rm sl}\right\} \\ &+ \hat{\mathbf{P}}_{\rm is}\left\{\alpha(1 - W_2) r_{\rm V} + [(\beta^{(0)} - \alpha) - W_2(\beta^{(2)} - \alpha)] r_{\rm III}^{\rm sl}\right\} \\ &+ \hat{\mathbf{P}}_V(\Theta)\left\{\alpha W_1 r_{\rm V} + W_1[\beta^{(1)} - \alpha] r_{\rm III}^{\rm sl}\right\}. \end{aligned} \tag{27}$$

The above equation is analogous to Equation (49) of Domke & Hubeny (1988), with the only difference that $r_{\rm II}$ is now replaced by $r_{\rm V}$ and $r_{\rm III}$ by $r_{\rm III}^{\rm sl}$. We recall that $r_{\rm III}^{\rm sl}$ has the same functional form as $r_{\rm III}$ for resonance lines, except that the damping width is now given by $\gamma_{ei}^{(1)}$ (see Equation (12)). Also note that our $\left(\beta^{(K)} - \alpha\right)$ is the same as $\beta^{(K)}$ of Domke & Hubeny (1988, see their Equation (51); see also Equation (101) of Bommier 1997b).

It is easy to verify that Equation (27) reduces to Equation (49) of Domke & Hubeny (1988) for no-lower-level interaction (i.e., infinitely sharp lower level). Because in this case $r_{\rm V} \to r_{\rm II}$ and $r_{\rm III}^{\rm sl} \to r_{\rm III}$.

Assuming that the velocity distribution of atoms in the lower level is Maxwellian, and the atomic velocity is unchanged during the scattering process, it is easy to transform the AF redistribution matrix to the laboratory frame. The resulting expression is similar to Equation (16), but with AF redistribution functions $r_{\rm V}^{(K)}$ and $r_{\rm III}^{\rm sl}$ replaced by corresponding laboratory frame functions $R_{\rm V}^{(K)}(x, \boldsymbol{n}, x', \boldsymbol{n}')$ and $R_{\rm III}^{\rm sl}(x, \boldsymbol{n}, x', \boldsymbol{n}')$, respectively. Here x' and x are, respectively, the incident and scattered frequencies in non-dimensional units, and $\boldsymbol{n}'$ and $\boldsymbol{n}$ denote the incident and scattered ray directions. We note that the laboratory frame functions $R_{\rm V}^{(K)}$ and $R_{\rm III}^{\rm sl}$ are derived in Heinzel & Hubeny (1982). They have the same functional form as their pure radiative counterpart derived in Heinzel (1981) and only the damping parameters are to be appropriately changed.

5. COLLISIONAL REDISTRIBUTION MATRIX IN TERMS OF IRREDUCIBLE SPHERICAL TENSORS

The redistribution matrix derived in Section 4 is written in terms of the Rayleigh and isotropic scattering phase matrices. It is advantageous to express them in terms of irreducible spherical tensors for polarimetry $\mathcal{T}_Q^K(i, \boldsymbol{n})$ introduced by Landi Degl'Innocenti (1984). Here $i = 0, 1, 2, 3$ refer to the Stokes parameters. The index K takes the values $K = 0, 1, 2$ and $-K \leqslant Q \leqslant +K$. In the following subsections we first recall the expression of the resonance scattering phase matrix in terms of $\mathcal{T}_Q^K(i, \boldsymbol{n})$, and then express the redistribution matrix derived by Domke & Hubeny (1988) for resonance lines in terms of spherical tensors. Finally, we express the redistribution matrix for subordinate lines in terms of $\mathcal{T}_Q^K(i, \boldsymbol{n})$.

5.1. Multipolar Components of the Resonance Scattering Phase Matrix

The resonance scattering phase matrix can be written as a linear combination of Rayleigh phase matrix multipolar components $\hat{\mathbf{P}}_{\rm R}^{(K)}(\boldsymbol{n}, \boldsymbol{n}')$ and is given by (see Landi Degl'Innocenti 1984; Bommier 1997b)

$$\hat{\mathbf{P}}(\boldsymbol{n}, \boldsymbol{n}') = \sum_{K=0}^{2} W_K(j_i, j_e) \hat{\mathbf{P}}_{\rm R}^{(K)}(\boldsymbol{n}, \boldsymbol{n}'), \tag{28}$$

where $W_K(j_i, j_e)$ is the same as W_K defined in Equation (20), and

$$\left[\hat{\mathbf{P}}_{\rm R}^{(K)}(\boldsymbol{n}, \boldsymbol{n}')\right]_{ij} = \sum_{Q=-K}^{+K} (-1)^Q \mathcal{T}_Q^K(i, \boldsymbol{n}) \mathcal{T}_{-Q}^K(j, \boldsymbol{n}'), \tag{29}$$

where $i, j = 0, 1, 2, 3$. In terms of the phase matrices $\hat{\mathbf{P}}_{\rm is}$, $\hat{\mathbf{P}}_{\rm R}$, and $\hat{\mathbf{P}}_V$ introduced in Section 4, the resonance scattering phase matrix can be written as (see Frisch 1996)

$$\hat{\mathbf{P}}(\boldsymbol{n}, \boldsymbol{n}') = W_2 \hat{\mathbf{P}}_{\rm R} + (1 - W_2)\hat{\mathbf{P}}_{\rm is} + W_1 \hat{\mathbf{P}}_V. \tag{30}$$

Comparing Equations (28) and (30), it is easy to show that

$$\hat{\mathbf{P}}_{\rm is} = \hat{\mathbf{P}}_{\rm R}^{(0)}, \tag{31}$$

$$\hat{\mathbf{P}}_{\rm R} = \hat{\mathbf{P}}_{\rm R}^{(0)} + \hat{\mathbf{P}}_{\rm R}^{(2)}, \tag{32}$$

and

$$\hat{\mathbf{P}}_V = \hat{\mathbf{P}}_{\rm R}^{(1)}. \tag{33}$$

5.2. The Case of Lines with an Infinitely Sharp Lower Level (Resonance Lines)

For resonance lines, Equation (27) takes the form

$$\begin{aligned}\hat{\mathbf{R}}_{\rm LF}^{\rm rl}(x, \boldsymbol{n}, x', \boldsymbol{n}') &= \hat{\mathbf{P}}_{\rm R}\left\{\alpha W_2 R_{\rm II} + W_2\left[\beta^{(2)} - \alpha\right] R_{\rm III}\right\} \\ &+ \hat{\mathbf{P}}_{\rm is}\left\{\alpha(1 - W_2) R_{\rm II} + [(\beta^{(0)} - \alpha) - W_2(\beta^{(2)} - \alpha)] R_{\rm III}\right\} \\ &+ \hat{\mathbf{P}}_V\left\{\alpha W_1 R_{\rm II} + W_1\left[\beta^{(1)} - \alpha\right] R_{\rm III}\right\}, \end{aligned} \tag{34}$$

where the superscript "rl" stands for resonance line and the subscript "LF" stands for laboratory frame. In Equation (34), $R_{\rm II}$ and $R_{\rm III}$ denote the type-II and type-III laboratory frame redistribution functions of Hummer (1962). Equation (34) is the same as Equation (49) of Domke & Hubeny (1988), except that our $\left(\beta^{(K)} - \alpha\right)$ is equal to their $\beta^{(K)}$ and the redistribution functions are now written in laboratory frame. Substituting Equations (31)–(33) into the above equation, we obtain

$$\hat{\mathbf{R}}_{\rm LF}^{\rm rl}(x, \boldsymbol{n}, x', \boldsymbol{n}') = \sum_{K=0}^{2} W_K[\alpha R_{\rm II} + (\beta^{(K)} - \alpha) R_{\rm III}] \hat{\mathbf{P}}_{\rm R}^{(K)}(\boldsymbol{n}, \boldsymbol{n}'). \tag{35}$$

The above redistribution matrix for resonance lines is the same as that derived by Bommier (1997a) applying a master equation theory.

5.3. The Case of Lines with Finite Width of Lower Level (Subordinate Lines)

To simplify the algebra we introduce the notations

$$\delta^{(K)} = \sum_{K'=0}^{2} \alpha^{(K')} \bar{C}_{KK'j_i j_e}, \tag{36}$$

and

$$\tilde{R}_{\rm V}^{(K)} = \sum_{K'=0}^{2} \alpha^{(K')} \bar{C}_{KK'j_i j_e} R_{\rm V}^{(K')}. \tag{37}$$

Using Equations (36) and (37), the redistribution matrix for subordinate lines (see Equation (16)) can be rewritten in the laboratory frame as

$$\begin{aligned}\hat{\mathbf{R}}^{\rm sl}_{\rm LF}(x,\boldsymbol{n},x',\boldsymbol{n}') &= \hat{\mathbf{P}}_{\rm R}\left\{\tilde{R}_{\rm V}^{(2)} + \left[W_2\beta^{(2)} - \delta^{(2)}\right]R^{\rm sl}_{\rm III}\right\} \\ &+ \hat{\mathbf{P}}_{\rm is}\left\{\left[\tilde{R}_{\rm V}^{(0)} - \tilde{R}_{\rm V}^{(2)}\right] + [(\beta^{(0)} - W_2\beta^{(2)}) - (\delta^{(0)} - \delta^{(2)})]R^{\rm sl}_{\rm III}\right\} \\ &+ \hat{\mathbf{P}}_{V}\left\{\tilde{R}_{\rm V}^{(1)} + [W_1\beta^{(1)} - \delta^{(1)}]R^{\rm sl}_{\rm III}\right\}. \end{aligned} \tag{38}$$

Substituting Equations (31)–(33) into the above equation, we obtain

$$\hat{\mathbf{R}}^{\rm sl}_{\rm LF}(x,\boldsymbol{n},x',\boldsymbol{n}') = \sum_{K=0}^{2}\left\{\tilde{R}_{\rm V}^{(K)} + [W_K\beta^{(K)} - \delta^{(K)}]R^{\rm sl}_{\rm III}\right\}\hat{\mathbf{P}}_{\rm R}^{(K)}(\boldsymbol{n},\boldsymbol{n}'). \tag{39}$$

In the case of no elastic collisions in the lower level, Equations (36) and (37) reduce to (see Equation (26))

$$[\delta^{(K)}]_{D_i^{(K)}=0} = W_K\alpha, \tag{40}$$

$$\left[\tilde{R}_{\rm V}^{(K)}\right]_{D_i^{(K)}=0} = W_K\alpha R_{\rm V}. \tag{41}$$

Using Equations (40) and (41) in Equation (39), we obtain

$$\begin{aligned}\left[\hat{\mathbf{R}}^{\rm sl}_{\rm LF}(x,\boldsymbol{n},x',\boldsymbol{n}')\right]_{D_i^{(K)}=0} &= \sum_{K=0}^{2} W_K \\ &\times \left[\alpha R_{\rm V} + (\beta^{(K)} - \alpha)R^{\rm sl}_{\rm III}\right]\hat{\mathbf{P}}_{\rm R}^{(K)}(\boldsymbol{n},\boldsymbol{n}'). \end{aligned} \tag{42}$$

Clearly the above equation is analogous to Equation (35).

6. AZIMUTH AVERAGED COLLISIONAL REDISTRIBUTION MATRIX

The assumption of a planar axisymmetric geometry is commonly used in the modeling of stellar atmospheres, in particular when there are uncertainties about the shape of the emitting region. In this section, we derive an azimuth averaged redistribution matrix for subordinate lines that can be used with axisymmetric radiative transfer equation.

For resonance lines, azimuth averaged redistribution functions were used by Milkey et al. (1975, and references cited therein). For polarized scattering in resonance lines, an azimuth averaged redistribution matrix was first used by Dumont et al. (1977) and Faurobert (1987) for the type-I and type-II redistribution, respectively (see also Wallace & Yelle 1989, for methods of computing azimuth averaged type-II redistribution function). Domke & Hubeny (1988) discuss the same problem with a better treatment of collisions ($R_{\rm II}$ and $R_{\rm III}$). More recently Frisch (2010) has proposed a method of performing azimuth averaging of angle-dependent PRD matrices using irreducible spherical tensors. This technique is based on Fourier azimuthal expansion of the redistribution functions. Also it uses spherical tensor expansion of the angular phase matrices (in contrast to the Fourier decomposition method of Chandrasekhar 1950; Faurobert-Scholl 1991; Nagendra et al. 1998). In the following, we apply the method of Frisch (2010) to the problem of deriving azimuth averaged PRD matrix for subordinate lines.

In an axisymmetric medium, the Stokes $(I,\ Q)$ are sufficient to represent the polarization state of the radiation field (see, e.g., Chandrasekhar 1950). Therefore, in Equation (29) the Stokes parameters indices i and j take values 0 and 1. Following Frisch (2010, Section 2), we can rewrite Equation (29) as

$$\left[\hat{\mathbf{P}}_{\rm R}^{(K)}(\boldsymbol{n},\boldsymbol{n}')\right]_{ij} = \sum_{Q\geqslant 0}^{K} c_Q\, \tilde{\mathcal{T}}_Q^K(i,\theta)\tilde{\mathcal{T}}_Q^K(j,\theta')\cos Q(\chi-\chi'), \tag{43}$$

where $i,\ j = 0,\ 1$ and $c_Q = 2-\delta_{0Q}$. $(\theta',\ \chi')$ and $(\theta,\ \chi)$ represent the incident and scattered ray directions with respect to the polar Z-axis. The relation between $\tilde{\mathcal{T}}_Q^K(i,\theta)$ and $\mathcal{T}_Q^K(i,\boldsymbol{n})$ is given in Equation (5) of Frisch (2010). Following Domke & Hubeny (1988) the angle-dependent PRD function $R(x,x',\Theta)$ can be expanded in an azimuthal Fourier series as

$$R(x,x',\Theta) = \sum_{k\geqslant 0}^{\infty}\mathcal{R}^{(k)}(x,\theta,x',\theta')\,\cos k\Delta, \tag{44}$$

where $\Delta = \chi - \chi'$. The Fourier coefficients are given by

$$\mathcal{R}^{(k)}(x,\theta,x',\theta') = \frac{c_Q}{2\pi}\int_0^{2\pi} R(x,\theta,x',\theta',\Delta)\cos k\Delta\, d\Delta. \tag{45}$$

Applying the above azimuthal expansion to $\tilde{R}_{\rm V}^{(K)}$ and $R^{\rm sl}_{\rm III}$, we obtain

$$\tilde{R}_{\rm V}^{(K)} = \sum_{k\geqslant 0}^{\infty}\widetilde{\mathcal{R}}_{\rm V}^{(k,K)}\,\cos k\Delta, \tag{46}$$

$$R^{\rm sl}_{\rm III} = \sum_{k\geqslant 0}^{\infty}\mathcal{R}^{(k),{\rm sl}}_{\rm III}\,\cos k\Delta, \tag{47}$$

where

$$\widetilde{\mathcal{R}}_{\rm V}^{(k,K)} = \sum_{K'=0}^{2}\alpha^{(K')}\bar{C}_{KK'j_ij_e}\mathcal{R}_{\rm V}^{(k,K')}(x,\theta,x',\theta'). \tag{48}$$

The Fourier coefficients $\mathcal{R}^{(k),{\rm sl}}_{\rm III}$ and $\mathcal{R}_{\rm V}^{(k,K')}$ have an expression similar to Equation (45), but with $R^{\rm sl}_{\rm III}$ and $R_{\rm V}^{(K')}$ in place of R.

The azimuth averaged redistribution matrix is defined by (see Equation (65) of Domke & Hubeny 1988)

$$\left\langle\hat{\mathbf{R}}^{\rm sl}_{\rm LF}\right\rangle_{\rm Az.Av.} = \frac{1}{2\pi}\int_0^{2\pi}\hat{\mathbf{R}}^{\rm sl}_{\rm LF}(x,\theta,x',\theta',\Delta)\,d\Delta. \tag{49}$$

Substituting Equations (43), (46), and (47) into Equation (39) and the resulting equation in Equation (49), we obtain after some algebra

$$\begin{aligned}\left[\left\langle\hat{\mathbf{R}}^{\rm sl}_{\rm LF}\right\rangle_{\rm Az.Av.}\right]_{ij} &= \sum_{K=0}^{2}\sum_{Q\geqslant 0}^{K}\left\{\widetilde{\mathcal{R}}_{\rm V}^{(Q,K)} + \left[W_K\beta^{(K)} - \delta^{(K)}\right]\mathcal{R}^{(Q),{\rm sl}}_{\rm III}\right\} \\ &\times \tilde{\mathcal{T}}_Q^K(i,\theta)\tilde{\mathcal{T}}_Q^K(j,\theta'). \end{aligned} \tag{50}$$

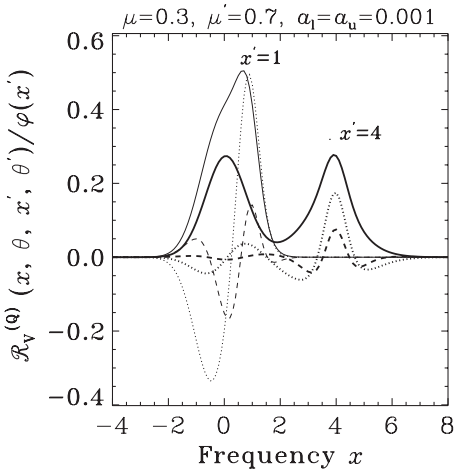

Figure 1. Frequency dependence of azimuthal Fourier coefficient of type-V redistribution function for $\mu = 0.3$, $\mu' = 0.7$, and for different values of Q. The damping parameters $a_l = a_u = 10^{-3}$. Thin lines correspond to $x' = 1$ and the thick lines to $x' = 4$. The solid line represents $Q = 0$, the dotted line $Q = 1$, and the dashed line $Q = 2$.

The azimuth averaged redistribution matrix for the special cases of lower level with no elastic collisions (i.e., $D_i^{(K)} = 0$) and infinitely sharp lower level (resonance line) can be recovered from Equation (50) (see Sections 4 and 5).

From Equation (50), it is clear that only Fourier coefficients of order $k = Q = 0, 1, 2$ are sufficient to fully represent the azimuth averaged redistribution matrix. For the sake of illustration, we have calculated azimuthal Fourier coefficients of type-V redistribution function. The upper and lower level radiative widths are parameterized as $a_l = a_u = 10^{-3}$. Figure 1 shows $\mathcal{R}_V^{(Q)}(x, \theta, x', \theta')/\varphi(x')$ for $\mu = \cos\theta = 0.3$ and $\mu' = \cos\theta' = 0.7$ and for $x' = 1$ (thin lines) and $x' = 4$ (thick lines). $\varphi(x')$ denotes the Voigt absorption profile function. As in the case of type-II and type-III redistribution functions (see, e.g., Domke & Hubeny 1988; Sampoorna et al. 2011; Nagendra & Sampoorna 2011), the azimuthal Fourier coefficients of the type-V redistribution function decrease with the increasing azimuthal order Q. For $x' = 4$ the $\mathcal{R}_V^{(0)}$ exhibits a double maxima, one at $x = x'$ and other at $x = 0$, which is typical of the type-V redistribution function (see, e.g., Frisch 1980; Heinzel 1981; Heinzel & Hubeny 1983). See Frisch (1980) for a detailed physical interpretation of this double maxima exhibited by the type-V function.

7. CONCLUSIONS

Here we have derived laboratory frame expressions for the polarized redistribution matrix for subordinate lines including collisions. Our approach is based on the earlier works by Omont et al. (1972), Domke & Hubeny (1988), and Heinzel & Hubeny (1982). As in the aforementioned papers, the lower level is assumed to be unpolarized. The scalar collisional redistribution function derived by Heinzel & Hubeny (1982) for subordinate lines is used to derive the polarized PRD matrix. An alternative approach is that of Bommier (1997a, 1997b), where AF polarized PRD matrices are given.

The laboratory frame redistribution matrix is written in a form suitable for application in polarized line formation theories. This purpose is better served by formulating the redistribution matrix in terms of the irreducible spherical tensors for polarimetry. Azimuth averaged redistribution matrix is also derived keeping in view the astrophysical applications where the radiation field is axisymmetric. The collisional frequency redistribution is considered in sufficient detail keeping in view the use of these matrices in modeling the linearly polarized spectrum of the Sun.

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