MODIFIED TITIUS—BODE RELATION

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Abstract

The Titius-Bode empirical relation for planetary distances holds as far out as Uranus, but in the cases of Neptune and Pluto, the distances derived from this relation do not agree with the observed values. In the light of newly discovered Kowal's object 'Chiron' between Saturn and Uranus, the relation fails starting from 'Chiron.' There is therefore a need to modify the relation suitably in order to account for such developments in the solar system. Here an attempt is made to modify this relation so that it remains valid for all the planets. We also extend this relation 'inward.' The technique goes well in the cases of satellite-systems of Uranus, Saturn and Jupiter. The interesting implications of this relation are also discussed.

I. INTRODUCTION

Two centuries have now elapsed since Johann Danieal Titius von Wittenberg in 1766 first inserted his famous observation on planetary distances into a German translation of Charles Bonnet's *Contemplation de la Nature* (Nieto 1972). This observation can be stated in the following way : If the radius of the Earth's orbit is normalized to 10, then the radii of all planetary orbits can be given by

$$r_n = 4 + 3 \times 2^n, \tag{1}$$

where n is $= -\infty$ for Mercury and 0, 1, 2. . . . . . . for succeeding planets. Ever since, this 'Law' has been clouded by controversy for two reasons:

(1) the series of events that led to Johann Elert Bode being associated with the 'Law' instead of its author Titius, and (2) whether there is any physical significance to the 'Law.' During the period from Kepler to Kant the belief that regularities existed among the planetary orbits became ingrained among astronomers. This was important, for it set up the intellectual milieu from which the Titius Law was to come. Christian Freiherr von Wolf was the first person to state this law but his statement was not in the form of a geometric progression and it did not indicate a gap between Mars and Jupiter. Titius' historic insertion mentioned above, which contained what became known as the Titius-Bode Law, was also similar but showing one gap between Mars and Jupiter and was literally in the form of a geometric progression. However, Vikarius (Johann Friedrick) Wurm was the first man to put the Titius-Bode Law in the form of eq. (1). Later on, he also generalized it to get a better fit for the solar system and to apply it to other satellite-systems. The other research workers, who modified this law from time to time, or those who gave alternative formulations of this law so as to include all the systems of the then known objects into their discussion, were, in order: Gilbert (1802), Blagg (1913), Richardson (1945), Nieto (1972) and Dermott (1968).

In our discussion, we shall use this relation in the form given by Gilbert

$$D_n = d + ar^n, \tag{2}$$

$$n = -\infty, 0, 1, 2 \ldots \ldots$$

where D_n is the mean distance from the primary to a secondary labelled n. We try to obtain the spacing of all objects (planets or satellites) keeping the form of the relation unaltered but changing the values of the parameters d, a and r in such a way that the relation holds good for the next group of objects, thereby obtaining groups of objects characterised by sets of values of the parameters d, a and r. We also extend this relations 'inward', the formula for 'inward' extension being given by

$$\bar{D}_n = d - a\,r^n \tag{3}$$

$$n = -\infty, 0, 1, 2, \ldots \ldots \; p$$

with $D_{-\infty} = \bar{D}_{-\infty}$ and $d > a$.

It is clear that relation (2) holds good if the primary has three or less than three objects revolving around it. For 'inward' extension (3), the condition on d is, $d > a$. For $d > a$, $(d - a\,r^n)$ ($n = 0, 1, 2 \ldots \ldots p$) may be interpreted as the places for possible existence of secondaries between primary and the first observed secondary, provided the values of $(d - a\,r^n)$, ($n = 0, 1, 2, \ldots \ldots p$) do not fall within Roche's limit (Jeans 1960). If the values do fall well within Roche's limit, they may be thought to represent the rings around the primary. The Roche limit is the distance which equals 2.45 times the radius of the planet and within which a satellite gets disrupted into pieces by tidal forces; alternatively, that the matter within this distance would not get condensed into a satellite.

II. SATELLITE-SYSTEM OF URANUS

The planet Uranus has 5 known satellites : Applying relations (2) and (3), with $d = 131$, $a = 63$ and $r = 1.485$, we obtain the values shown in Table 1.

TABLE 1

The distance of the satellite from the planet Uranus in 1000 km

		Calculated Value	Observed Value (Frederick and Baker 1976)
......		$\bar{D}_1 = 37$	......
......		$\bar{D}_0 = 68$	......
Miranda		$D_{-\infty} = 131$	131
Ariel		$D_0 = 194$	192
......		$D_1 = 225$	......
Umbriel		$D_2 = 270$	267
......		$D_3 = 337$	......
Titania		$D_4 = 437$	438
Oberon		$D_5 = 586$	586

According to 'inward' extension relation (3), between Uranus and the first observed satellite Miranda, there could be two satellites of Uranus at distances of 68,000 km and 37,000 km. Recently, Elliot *et al.* (1977) and Bhattacharyya and Kuppuswamy (1977) have reported the observations of rings around Uranus with outer edge extending upto about 54,000 km from the planet.

The radius of Uranus is 27,000 km (Marsden 1977) and the Roche's limit for the planet is about 66,000 km. The calculated distance from the relation (3), for the unknown satellite corresponding to $n = 0$, between Uranus and Miranda falls outside the Roche's limit of the planet, and therefore the object at a distance of 68,000 km would exist there. The second unknown satellite corresponding to $n = 1$ is at a distance of 37,000 km which is well within the Roche's limit and therefore it may be thought to represent the ring structure around the planet.

According to relation (2), there could exist two more unknown satellites of Uranus, one between Ariel and Umbriel at a distance of 225,000 km and the other between Umbriel and Titania at a distance of 337,000 km from the planet (Table 1). It is interesting to note that a minor planet at a mean distance of 2.8A U was predicted by Titius-Bode relation between Mars and Jupiter and in fact Ceres at a mean distance of 2.8 AU was discovered later by Giuseppe Piazzi in 1801. Extending this argument for satellite system of Uranus, there could exist two more unknown satellites of Uranus at places mentioned above. Note that so far all the known satellites of Uranus belong to a single group.

III SATELLITE-SYSTEM OF SATURN

The planet Saturn has 10 known satellites. The first group consists of six known satellites. Applying relations (2) and (3), with $d = 186$, $a = 27$, $r = 1.435$, we obtain Table 2.

These values of the parameters have been chosen to match the distance of Mimas, the second known satellite of Saturn. The distance for the nearest satellite to Saturn Janus is obtained with the help of 'inward' extension putting $n = 0$ for Janus. The second group contains three known satellites Titan, Hyperion and Iapetus. The last satellite Phoebe can be looked upon as forming the third group of a single object at a distance of 12,952,000 km. In Table 2 are given the calculated values of distances of the first group of satellites from the planet and their observed distances in thousands of kilometers.

TABLE 2

The distance of the Satellite from the Planet Saturn in 1000 km

	Calculated Value	observed value (Fredrick and Baker 1976)
First Group		
......	$D_5 = 22$	......
......	$\bar{D}_4 = 72$	......
......	$\bar{D}_3 = 106$	......
......	$\bar{D}_2 = 130$	......
......	$\bar{D} = 147$	......
Janus	$\bar{D}_0 = 159$	158
Mimas	$D_{-\infty} = 186$	186
......	$D_0 = 213$	......
......	$D_1 = 225$	......
Enceladus	$D_2 = 242$	238
......	$D_3 = 266$	......
Tethys	$D_4 = 301$	295
......	$D_5 = 350$	......
Dione	......	378
......	$D_6 = 422$	......
Rhea	$D_7 = 524$	527

According to 'inward' extension relation (3) between Saturn and first observed satellite Janus, there could be five satellites of Saturn at distances of about 147,000, 130,000, 106,000, 72,000 and 22,000 km. Roche's limit for Saturn is about 1,48,000 km. The calculated distance from the relation (3), for the unknown satellite corresponding to $n = 1$ between Saturn and Janus falls just at the edge of Roche's limit of the planet, and therefore the object at the distance 147,000 km might have been under the process of disruption into small fragments contributing something to the ring structure round the planet.

Fountain and Larson (1977) recently re-examined photographs of Saturn taken in 1966 with Lunar and Planetary Laboratory's 154 cm Catalina reflector and found evidence on the photographs of at least one other satellite. Combining the LPL observations with those made by Texerau, Walker and Dollfus, they confirmed the new satellite's existence and calculated its distance to be 151,300 km from the planet. Further observations will be possible in 1979 and 1980 when the rings will be again seen edge-on.

It seems that the satellite at the mean distance 147,000 km might not have been completely disrupted, but may be still under the process of disruption and may come out of Roche's limit at any moment, because it is just at the edge of Roche's limit. The distance of 11th satellite of Saturn proposed by Fountain and Larson is 1,51,300 km from the planet but still observation is not final. Therefore, we feel that the hypothetical satellite at the mean distance 1,47,000 km may be Fountain and Larson's object.

The distances of other four unknown satellites between Saturn and Janus fall well within the Roche's limit of the planet and therefore we may interpret them to represent the four rings around the planet. According to relation (2), there could exist five more unknown satellites of Saturn at places shown in the Table 2.

Note that the relation omits the satellite Dione. This omission may imply that Dione is not an original satellite of Saturn, but is a captured one and that this relation reflects the conditions of formation rather than the results of evolution. It is likely that another asteriod belt may exist between Saturn and Uranus to which recently discovered Kowal's object ' Chiron ' also belongs and that the originally, Dione was a member of this belt (see also sec. IV).

IV. SOLAR SYSTEM

The Table 3 shows that the newly discovered tiny planet—Kowal's object ' Chiron ' follows the relation at a mean distance of 14.1 AU. It is likely that a belt of asteroides may be found between Saturn and Uranus The distance of Neptune agrees with the relation. One remarkable point about this relation is that it omits Pluto which is at a mean distance of 39.4 AU. This may be attributed to the feeling held by many astronomers (Lyttleton 1961; Hoyle 1975) that Pluto is not the original planet of the Sun but is caputred one (originally, Pluto was a satellite of Neptune) and that this relation reflects the condition of formation rather than the results of evolution. If there exist Xth and XIth planet beyond Pluto this relation places them to be at the distances of about 52 AU and 99 AU respectively.

' Inward ' extension of the relation to the solar system for locating possible planets between the Sun and Mercury shows that there is a possibility of a planet at 0.1 AU. The radius of the Sun is 6,95,000 km and therefore the Roche's limit for the sun is 17,02,750 km. This shows that the above mentioned unknown planet is beyond the Roche's limit for the Sun and therefore a planet may exist there according to this relation. This hypothetical planet will have a revolution period of about $11^d\ 13^h$ around the Sun and revolution speed 94.5 km/sec. The minimum elongation of the planet would be about 5.8°. Bruman (1968) has considered a mass distributed uniformly around the Sun, extending at least to the mean distance of Saturn and having a density such that a sphere

TABLE 3

Modified Titius—Bode relation in the Solar System.

Distance of the Planet from the Sun in AU

First Group ($d = 0.4, a = 0.3, r = 2$)

Planet	Calculated value	Observed value
......	$\bar{D}_0 = 0.1$	......
Mercury	$D_{-\infty} = 0.4$	0.4
Venus	$D_0 = 0.7$	0.7
Earth	$D_1 = 1.0$	1.0
Mars	$D_2 = 1.6$	1.5
Minor Planets	$D_3 = 2.8$	2.8
Jupiter	$D_4 = 5.2$	5.2

Second Group ($d = 9.5, a = 4.6, r = 2.1$)

Planet	Calculated Value	Observed Value
Saturn	$D_{-\infty} = 9.5$	9.5
Chiron	$D_0 = 14.1$	13.7
Uranus	$D_1 = 19.2$	19.2
Neptune	$D_2 = 29.8$	30.1
Pluto	...	39.4
X Planet (if its exists)	$D_3 = 52.1$	......
XI Planet (if it exists)	$D_4 = 99$	......

of radius 1 AU has the same mass as the Earth. Comparing the mean distance R of such planet from the Sun, against the radius R^1 of the uniform spherical distribution having the same mass *i.e.*

$$R^1 = (1\ \text{AU})(\Sigma\ \text{Mass})^{1/3}$$

he finds a good agreement between R and R^1. Thus the simple postulate of spherical mass appears related to the present positions of the first six planets (93 per cent of the solar system aside from the Sun itself) with an accuracy of ± 30 percent or so. As R and R^1 agree very closely, using the above mentioned formula of Bruman and writing $R^1 = 0.1$ AU, we find the mass of the hypothetical planet to be approximately equal to 0.001 $M_{\oplus}$.

V. SATELLITE-SYSTEM OF JUPITER

The planet Jupiter has 14 known satellites of which, depending upon their distances from the planet, the first 13 satellites form three distinct spatial groups. The distance of satellite XIV is not known. If the last satellite XIV belongs to the third group, then its distance can be calculated, using the relation (2) for the third group, otherwise, it can be looked upon as forming the fourth group of a single object away from other three groups.

TABLE 4

Distance of the Satellite from the planet Jupiter in 1000 km

Satellite	Calculated value	Observed value (Fredrick and Baker 1976)
First Group : (d = 181, a = 240, r = 1.922)		
V	$D_{-\infty}$ = 181	181
I (I_0)	D_0 = 421	422
......	D_1 = 642	......
II (Europa)	......	671
III (Ganymede)	D_2 = 1067	1070
IV (Callisto)	D_3 = 1884	1880
Second Group : (d = 10170, a = 1300, r = 1.099)		
XIII	$D_{-\infty}$ = 10170	10170
VI	D_0 = 11470	11470
......	D_1 = 11599	......
X	......	11710
VII	D_2 = 11741	11740
Third Group : (d = 20700, a = 1375, r = 1.17)		
XII	$D_{-\infty}$ = 20700	20700
......	D^0 = 22075	......
VI	D_1 = 22309	22350
......	D_2 = 22580	......
......	D_3 = 22903	......
VIII	D_4 = 23277	23300
IX	D_5 = 23715	23700

The relation (2) for the first group omits Europa and that for the second group omits the satellite X which may be interpreted as captured ones. According to this relation, there could exist one more unknown satellite in each of the first two groups and three other unknown satellites in the third group at places shown in the Table 4.

For the first group of satellite system of Jupiter, d < a, and hence 'inward' extension for the first group of this system is not possible. This situation may be interpreted as that there could not exist any satellite or rings around Jupiter on inner side of the first observed satellite V lable n = — ∞. On similar lines of argument we may say that there could not exist any satellite or rings around Neptune or Mars on inner sides of their first observed satellites.

VI. THE χ^2-TEST

Tables 1, 2, 3 and 4 clearly indicate a good agreement between the observed and the calculated values of the distances, of course with certain number of possible

TABLE 5*

x^2 Test

Name of the system	Degrees of freedom	Table value of x^2	Calculated value of x^2	% level of significance
1. Uranus	5	0.412	0.0567	95.5
2. Saturn : (First Group)	11	2.603	0.2125	99.5
3. Sun : (First Group)	3	0.0717	0.0066	99.5
4. Sun : (Second Group)	3	0.0717	0.0145	99.5
5. Jupiter : (First Group)	2	0.0201	0.0192	99
6. Jupiter: (Second Group)	1	0.000157	0.0000851	99
7. Jupiter : (Third Group)	4	0.207	0.1073	99.5

* The table shows the compatibility of theoretical and observed values for all systems at high level of significance.

gaps, of secondaries around their respective primaries. Applying the χ^2–test to test the goodness of fit to groups of all four systems, we get the results as given in Table 5.

The table shows the compatibility of theoretical and observed values for all systems at high level of significance.

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References :

Bhattacharyya, J. C. and Kuppuswamy, K., 1977, *Nature* **267**, 330.

Burman, J. R. 1968, *Icarus*, **8**, 508.

Dermott, S. F., 1968, *Mon. Not. R. astr. Soc.*, **141**, 363.

Elliot, J. L. Dunham, E. and Mink, D. 1977, *Nature*, **267**, 328.

Fountain, J. W. and Larson, S. M. 1977, *Science*, **197**, 915.

Fredrick, L. W. and Baker, R. H. 1976, *Astronomy*, 10th edition, Nostrand Company, p. 532.

Hoyle, F., 1975, *Astronomy and Cosmology*, W. H. Freeman and Co. San Franscisco, pp. 441—451.

Jeans, J. 1960, *Universe Around Us*. 4th edition, Cambridge University Press.

Lyttleton, R. A. 1961, *Man's view of the Universe*, Micheal Joseph, London.

Marsden B. G., 1977, *IAU circular Nos.* 3048 (March, 14). 3051 (March, 21), 3040 (Feb., 23) 3151 (Dec., 13), 3156 (Dec., 30).

Nieto, M. M. 1972, *The Titius—Bode Law of Planetary Distance : It's History and Theory*, Pergamon Press, Oxford