The Role of Space-Time Curvature in The Study of Plasma Processes near Neutron Stars and Black Holes

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Abstract

As neutron stars and black holes are very compact massive objects with possibly high magnetic fields, their surrounding space-time curvature should be treated important while considering plasma processes for explaining radiation emission nearby such objects. In this essay, we make a case for considering plasma processes on curved geometry (general relativistic framework), through considering detailed orbits of charged particles in electromagnetic fields on Schwarzschild and Kerr background.

Thanks to modern technology, during the last decade and a half, man's understanding of the universe has taken a big leap due to the birth of infrared, ultraviolet, X-ray and γ-ray astronomy getting access to the universe through the entire electromagnetic spectrum. Starting with sending small payloads through balloons, rockets and satellites, man has eventually reached a stage wherein an entire laboratory can be put into orbit to get the information from depths of space and time. With all the information that is coming in, astrophysics has now become a theorist's paradise wherein every new observation sees the birth of a new model. Of all the observations, the most popular amongst theoretical physicists as far as model building is concerned are the pulsars and compact X-ray sources. The central stars connected with these objects are the neutron stars and black holes. In either case, though the emission mechanism is very little understood, the consensus is in favour of emission due to plasma processes. The plasma processes considered so far have mainly been in the Newtonian framework, and very recently there have been some attempts to consider special relativistic effects. The popular model for a pulsar is that of a rotating neutron star with a co-rotating magnetosphere supporting the surrounding plasma, which gives out synchrotron radiation. In the case of compact X-ray sources, the emission is believed to be due to the thermalised plasma accreting onto the compact object (a neutron star or a black hole). When such accretion ensues because of the angular momentum of the system, the inflowing plasma acquires angular momentum and forms a disc around the compact objects. In this case too the discussions regarding the stability of discs have been confined to Newtonian formulation only.

In the case of neutron stars, even the most conservative estimate gives a mass of about 1 $M_{\odot}$ confined to a spheroid of radius $\sim$ 10 km. Thus the gravitational field nearby the star is very high and hence space-time curvature would certainly be prominent. We know that even in the case of the Sun and its surroundings the general relativistic effects are quite significant. Hence, it is desirable to reconsider some of these models in the framework of general relativity. In the case of black holes, the treatment should necessarily be general relativistic as the concept itself owes its origin to general relativity. Considering the plasma itself, the constituent particles are high energetic moving with relativistic speeds. With all these in the background, it is only natural to look for models wherein the plasma processes are considered on a curved background geometry.

We have started this study and as a basic step before considering plasma as a whole, we have looked into orbit theory, wherein we have made a detailed analysis of the orbits of charged particles in a magnetic field superimposed on curved background geometry, confining mainly to the equatorial plane of the central star which is responsible for the curvature of space time. We know that in the absence of all external fields even a charged particle just follows a geodesic in a curved space time as given by

$$U^{\mu}{}_{;\nu}\, U^{\nu} = 0,$$

wherein U^{μ} is the velocity-four-vector of the particle. If an electromagnetic field is also present, then the motion will no longer be along the geodesic, as it experiences Lorentz force and its motion will be described by the equation

$$U^{\mu}{}_{;\nu}\, U^{\nu} = (-e/M_0)\, F^{\mu}{}_{\nu}\, U^{\nu},$$

wherein e/M_0 is its charge to mass (proper) ratio and $F^{\mu}{}_{\nu}$ is the electromagnetic field tensor, derived from the vector potential A_{μ} as,

$$F_{\mu\nu} = (A_{\nu,\mu} - A_{\mu,\nu}).$$

In general, if one wants to consider the charged particle motion in general relativity, in principle one should consider the background space-time curvature produced by both the sources, gravitational as well as electromagnetic. This would involve solving the set of coupled Einstein-Maxwell equations in a self-consistent way. But this in general is a formidable task, even if a dipole magnetic field is to be taken as a source. However, there is a solution given by Bonnor (1966) for a dipole magnetic field, but this is not suitable if one wants to consider either a neutron star or a black hole for the source. In any case, for the purpose we have in mind, that is, studying the dynamics of a charged particle, we would not need a complete exact solution, as the contribution from the electromagnetic field to curvature is negligible compared to that of the gravitational field. Even the most intense magnetic field considered $\sim 10^{12}$ gauss, has negligiby small energy compared to the gravitational potential energy of $1 M_{\odot}$ on the surface of a neutron star. Thus it is sufficient if we start with a specified background like Schwarzschild or Kerr spacetime, then super-impose either a dipole magnetic field or an uniform magnetic field, and consider the effect of background curvature in the electromagnetic field by solving the covariant Maxwell's equations on the curved background. This procedure has been followed and some solutions are obtained by Ginzburg and Osernoi (1965), for dipole magnetic field on Schwarzschild background, by Chitre and Vishveshwara (1975), Petterson (1975), and King *et. al* (1975) for a multipole field on Kerr background and by Wald (1974) for an uniform magnetic field on Kerr background.

We have used in our analysis the solutions of Ginzburg and Ozernoi for the Schwarzschild case and that of Petterson (dipole magnetic field) and Wald (uniform magnetic field) for the Kerr-case. In both the cases, our approach is similar and is as follows. The gravitational field as well as the electromagnetic field in either case is both axisymmetric and stationary (in fact in the Schwarzschild case, there is even a higher degree of symmetry than axisymmetry) and this fact leads to the existence of two Killing vectors, one time-like and one space-like. These in turn gives us two constants of motion, the time-like Killing vector corresponding to the energy of the particle and the space-like one corresponding to the canonical angular momentum of the particle. As the background geometry is assumed to be unaltered by the fields, the space-time metric provides one integral of motion, and thus using these two constants of motion in the metric, we can extract the third integral. By restricting the discussion to the motion in the equatorial plane ($\theta = \pi/2$, $d\theta/ds = 0$), all the first integrals can be explicitly written. Then by solving for $dr/ds = 0$, which gives the turning points of the orbits, we can get an expression for the effective potential for the r-motion of the particle. It is well-known that a study of effective potential curves (V_{eff}, r) gives one all the necessary information regarding the nature of the orbits. To obtain the actual orbits, we integrate the complete second order differential equation for r specifying the two constants of motion and the metric for the initial values. For fuller details we refer the reader to the papers concerned (Prasanna and Varma 1977 Prasanna and Vishveshwara 1978). The main conclusions regarding the orbits are as follows:

(i) **Dipole Magnetic Field on Schwarzschild Background:**

The presence of a magnetic field on the Schwarzschild geometry alters the character of the motion considerably. The orbits which are spiralling in the purely gravitational case are now turned around by the magnetic field (as in the non-realativistic case) and are thus stopped from spiralling in or out and are trapped in a region of the r-space determined by the radius of gyration. In this sense, the magnetic field stabilises the orbits. The effective potential curves display a minimum very near to the stellar surface, bound by two maxima (one due to the magnetic field and the other due to the centrifugal barrier) and in this potential well the particles are trapped wherein they execute Larmor motion. Depending on the energy, angular momentum and the magnetic field strength, the particles can be in stable gyrating orbits even very close to the event horizon. However, the weak field analysis is valid only outside the event horizon since at the event horizon there is a logarithmic singularity in the magnetic field components, and thus the discussions are limited to the case wherein the source of the magnetic field should be outside the event horizon. One could in fact have current loops just outside the event horizon but arbitrarily close to it as the source of such dipole magnetic fields as considered in this work.

(ii) **Dipole Field on the Kerr Background**

As almost all the celestrial bodies have a non-zero angular momentum, Kerr geometry is more suitable for discussions concerning neutron stars or black holes. Kerr space-time as given in Boyer-Lindquist co-ordinates read as

$$ds^2 = -\left(1 - \frac{2mr}{\Sigma}\right) c^2 dt^2 - (4amr/\Sigma)\ \mathrm{Sin}^2\theta\, dt\, d\varphi$$

$$+ (\Sigma/\Delta)\ dr^2 + \Sigma\, d\theta^2 + (B/\Sigma)\ \mathrm{Sin}^2\theta\, d\varphi^2$$

with

$$\Sigma = (r^2 + a^2 \mathrm{Cos}^2\theta), \quad \Delta = (r^2 - 2mr + a^2)$$

$$B = (r^2 + a^2)^2 - \Delta a^2 \mathrm{Sin}^2\theta, \ m = MG/C^2.$$

Petterson's solution for a stationary electromagnetic field on this background when restricted to dipole magnetic field and an induced quadrupole electric field is given by

$$A_t = \left(\frac{-3a\mu}{2\gamma^2\Sigma}\right)\Big\{[r(r-m) + (a^2 - mr)\mathrm{Cos}^2\theta]$$

$$\frac{1}{2\gamma}\ln\left(\frac{r-m+\gamma}{r-m-\gamma}\right) - (r - m\,\mathrm{Cos}^2\theta)\Big\}$$

$$A_\varphi = \left(\frac{-3\mu\,\mathrm{Sin}^2\theta}{4\gamma^2\Sigma}\right)\Big\{(r-m)a^2\mathrm{Cos}^2\theta + r$$

$$(r^2 + mr + 2a^2) - [r(r^3 - 2ma^2 + a^2 r) + \Delta$$

$$a^2\mathrm{Cos}^2\theta]\ \frac{1}{2\gamma}\ln\left(\frac{r-m+\gamma}{r-m-\gamma}\right)\Big\}$$

where μ is the magnetic dipole moment. The constants of motion obtained through the symmetry of the system are expressed as

$(U_\varphi + e A_\varphi) = l \qquad (U_t + e A_t) = -E$

which may be written with U^φ and U^t as

$$U^\varphi = \frac{d\varphi}{ds} = \Delta^{-1}\left\{\left(1-\frac{2mr}{\Sigma}\right)\frac{(l - eA_\varphi)}{\sin^2\theta} + \frac{2mar}{\Sigma}(E + eA_t)\right\}$$

$$U^t = \frac{dt}{ds} = (\Sigma\,\Delta)^{-1}\left\{[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta](E + eA_t) - 2mra\,(l - eA_\varphi)\right\}$$

Restricting oursevles to the study of motion in the equatorial plane, we get, on using these in the metric

$$(U^\rho)^2 \equiv \left(\frac{d\rho}{d\sigma}\right)^2 = \frac{1}{\rho^3}\left\{[\rho(\rho^2+\alpha^2)+2\alpha^2](E + A_\tau)^2 - \rho\Delta - 4\alpha(E+A_\tau)(L-\bar{A}_\varphi) - (\rho-2)(L-\bar{A}_\varphi)^2\right\}$$

wherein

$$\rho = r/m,\ \sigma = s/m,\ \alpha = a/m,\ L = l/m,\ \bar{A}_\varphi = \frac{eA_\varphi}{m}$$

Solving for $U^\rho = 0$, we get the expression for the effective potential

$$V_{eff} = -A_\tau + K/R$$

$$K = [2\alpha(L-\bar{A}_\varphi) + \Delta^{\frac{1}{2}}\{\rho^2(L-\bar{A}_\varphi)^2 + \rho R\}^{\frac{1}{2}}]$$

$$R = (\rho^3 + \alpha^2\rho + 2\alpha^2)$$

Plotting V_{eff} against ρ we can get the structure of the potential well wherein the particles are trapped in bound orbits. Unlike in the Schwarzschild case, here we have apart from the event horizon at $\rho = 1 + \sqrt{1-\alpha^2}$, the ergosurface which in the equatorial plane is at $\rho = 2$. Thus, depending on the various physical parameters like a, l, E and μ, the potential well lies completely outside or inside or partly inside and partly outside $\rho = 2$. Actual numerical integration of the orbit equations gave us a very interesting result that among the bound orbits not all exhibit gyration of the particle. Detailed analysis showed that the particles gyrate only if they are completely outside the ergo-surface. This result is just the effect of inertial frame dragging of the Kerr-geometry. When a particle gyrates, then during every Larmor circle the particle's angular velocity will be prograde for one half and retrograde for the other half with respect to the angular velocity of the central star. It is well-known that in the Kerr space-time the ergo-surface is a limiting surface for the retrograde motion on and behind which no

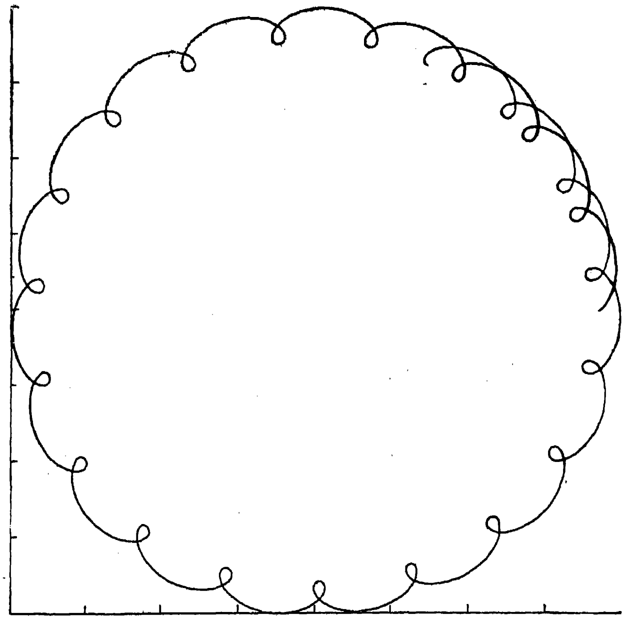

Fig. 1 : Charged particle orbits in a dipole magnetic field superposed on the Kerr background geometry as seen in the equatorial plane of the Kerr object. The figure shows the gyrations outside the ergosurface for $\alpha = 0.99$, $\lambda = 1000$, L=500, E=30, $\rho_0 = 3.51763$ $\rho_1 = 3.39123$ and $\rho_2 = 3.81977$.

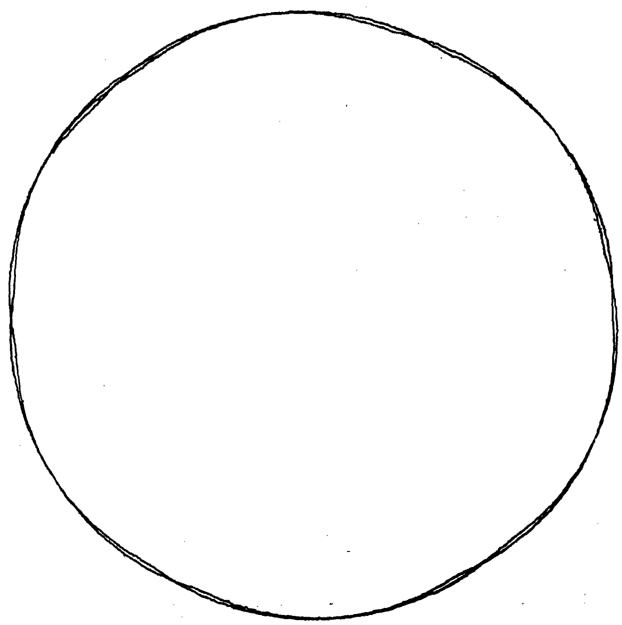

Fig. 2 : Similar to Fig. 1. But here no gyration is seen as the particle is confined to within the ergosurface. Here $\alpha = 0.99$, $\lambda = 50$, L = 500, E = 152, $\rho_0 = 1.36570$, $\rho_1 = 1.35715$ and $\rho_2 = 1.39003$.

particle can have retrograde motion. Thus the particle cannot gyrate if it enters the ergosurface. Figures (1) and (2) show the actual orbits for the two distinct cases of gyrating and non-gyrating bound orbit. In fact, we can see analytically this result very clearly as follows: when a particle gyrates, its $(d\varphi / d\sigma)$ has to go through zero for some $\rho = \rho_g$ for which $(d\rho / d\sigma)$ should be real. Thus, if we take $(d\varphi / d\sigma)_{\rho = \rho_g} = 0$, and use the ensuing relation in $(d\rho / d\sigma)^2$, we get

$$(d\rho / d\sigma)^2 = (\triangle / \rho_g^2) \left\{ (1 - 2/\rho_g)^{-1} [E + (A_\tau)_{\rho = \rho_g}]^2 - 1 \right\}$$

$(d\rho / d\sigma)$ will be real if and only if $(1 - 2/\rho_g) > 0$ which can be true only for $\rho_g > 2$.

The orbits for the case of the uniform magnetic field also have similar features. The difference between the two different fields appears mainly in the effective potential which reflects in the structure of the potentia well. In the dipole case, the effective potential goes to 1 as $\rho \to \infty$, whereas in the case of the uniform field, it goes as ρ^2 for large ρ. As had been found in the case of the Schwarzschild background, the essential role that the magnetic field plays is to stabilise the orbits. The case of the uniform fields could correspond to the galactic magnetic field surrounding compact objects. The dipole field could correspond to the intrinsic field of a compact object if it is a neutron star, whereas in the case of black holes it has to result from current rings exterior to the event horizon but can be very close to it.

Having analysed the motion for the stationary case, the immediate generalisation that needs looking into is the case when the particle is radiating, as the energy and angular momentum in that case are no longer constants of motion. When the plasma as a whole is considered, one normally resorts to Alfven's guiding centre approximation which uses certain adiabatic invariants. Now in the curved space-time formulation, the motion of adiabatic invariants has to be carefully looked into. The guiding centre motion on curved spacetime needs to be analysed before we can indulge in using the guiding centre approximation for studying the dynamical behaviour of plasma. In fact, a very important question arises in the case of Kerr black holes. As we have seen, if there is no gyration possible for the particle on and inside the ergosphere, then there is nothing like a guiding centre for such particles. Since there is no analogue of such a situation in non-relativistic formulation, the treatment for dynamical stability of a plasma disc very close to a Kerr black-hole (some part of which is extending into the ergosphere) would have to be perhaps entirely different than the one used hitherto. In any case, it is clear that in analysing the dynamics of the stability of plasma discs around neutron stars and black holes the final formulation should necessarily be general relativistic (at least parametrised post Newtonian) wherein the contribution from the magnetic field as well as the space-time curvature are taken with equal imporance.

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