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m-STATE INTERFERENCE WITH PARTIAL FREQUENCY REDISTRIBUTION FOR POLARIZED LINE FORMATION IN ARBITRARY MAGNETIC FIELDS

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ABSTRACT

The present paper concerns the derivation of polarized partial frequency redistribution (PRD) matrices for scattering on a two-level atom in arbitrary magnetic fields. We generalize the classical theory of PRD that is applicable to a $J = 0 \rightarrow 1 \rightarrow 0$ scattering transition, to other types of atomic transitions with arbitrary quantum numbers. We take into account quantum interference between magnetic substates of a given upper J-state. The generalization proceeds in a phenomenological way, based on the direct analogy between the Kramers–Heisenberg scattering amplitude in quantum mechanics and the Jones scattering matrix in classical physics. The redistribution matrices derived from such a generalization of classical PRD theory are identical to those obtained from a summed perturbative quantum electrodynamic treatment of the atom–radiation interaction. Our semi-classical approach has the advantage that it is non-perturbative, more intuitive, and lends itself more easily to further generalization (like the inclusion of J-state interference in the PRD theory).

Key words: atomic processes – line: formation – line: profiles – magnetic fields – polarization

Online-only material: color figure

1. INTRODUCTION

The problem of atom–radiation interaction remains to this day an important problem of quantum physics. The problem of scattering of polarized radiation on atoms and molecules in arbitrary magnetic fields particularly is a front-line topic in solar physics, mainly due to the discovery of extremely rich structuring of the "second solar spectrum" (Stenflo & Keller 1996, 1997). The term "second solar spectrum" aptly refers to the linearly polarized spectrum of the Sun observed near the solar limb. It is formed due to anisotropic scattering of radiation on atoms and molecules. An atlas of this spectrum has been produced with high spectral resolution from the UV at 3160 Å to the red at 6995 Å (Gandorfer 2000, 2002, 2005).

The presence of a magnetic field modifies the second solar spectrum through the Hanle and Zeeman effects. The Hanle effect arises due to quantum interference between magnetic substates. It is most sensitive to the weak fields, when the Zeeman splitting is comparable to radiative width of the level under consideration. On the other hand the Zeeman effect is sensitive to strong fields, when the Zeeman splitting becomes comparable to the Doppler width of the line. The two effects nicely complement each other and thereby provide a diagnostic for the solar magnetic fields (see Stenflo 1994).

It is well known that the scattering polarization signatures of strong resonance lines can be modeled only when the so-called partial frequency redistribution (PRD) mechanism in scattering is taken into account. For example, recently a detailed modeling of Q/I spectra of the Ca I 4227 Å line has been carried out successfully by Anusha et al. (2010). The correlations in frequency, angle, and polarization between the incoming and outgoing photons in a scattering event are described by PRD. The theory of PRD was first developed for the scattering of unpolarized radiation (see Mihalas 1978; Hubeny 1985, for a review on the subject). This theory from a classical perspective was originally introduced by Zanstra (1941), who addressed the problem of non-magnetic collisional frequency redistribution in resonance lines.

The classical oscillator theory for frequency-coherent scattering of polarized radiation in the presence of magnetic fields was developed by Stenflo (1994, 1997, 1998). This theory was further extended by Bommier & Stenflo (1999, hereafter BS99) to handle PRD effects in the presence of arbitrary magnetic fields and collisions. They solved the time-dependent oscillator equation, in combination with a classical model for collisions (see Stenflo 1994, chapter 10), to derive polarized PRD matrices in the atomic rest frame. The corresponding laboratory frame redistribution matrices were derived in Sampoorna et al. (2007a, hereafter P1). We recall that the classical time-dependent oscillator theory of BS99 and P1 describes only the special case of a $J = 0 \rightarrow 1 \rightarrow 0$ scattering transition.

The quantum theory for the problem of redistribution of resonance radiation including the effects of collisions was developed by Omont et al. (1972). They used a quantum mechanical description of matter and radiation, and derived PRD functions in the atomic rest frame. A year later, these authors addressed the same problem but for the magnetized case (Omont et al. 1973). However, they did not present the explicit form of the polarized PRD matrices. Starting from the work of Omont et al. (1972), Domke & Hubeny (1988) derived expressions of the PRD matrices for resonance line polarization in a two-level atom with an unpolarized lower level. By applying a master equation theory, Bommier (1997a) derived a more elegant but equivalent expression for the non-magnetic PRD matrix. Moreover, Bommier (1997b, hereafter B97b) derived the PRD matrices in the presence of an arbitrary magnetic field for the case of a two-level model atom with an unpolarized lower level. An alternative theory based on the concept of metalevels or sublevels has been developed by Landi Degl'Innocenti et al. (1997), to handle polarized PRD scattering in the presence of magnetic fields, but in the absence of any collisions (elastic and inelastic). Sampoorna et al. (2007b, hereafter P2; see also BS99) showed that, for the particular case of a $J = 0 \rightarrow 1 \rightarrow 0$ scattering transition, the quantum electrodynamic (QED) theory of B97b and the classical

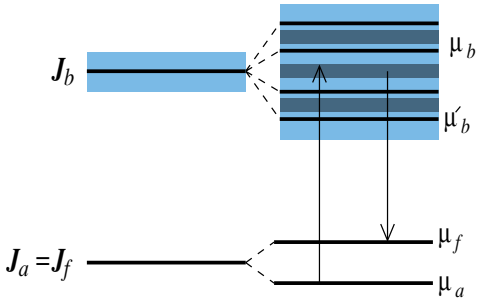

Figure 1. m-state interference phenomena in atomic transitions involving arbitrary J-states. The light shades represent the radiative widths of the levels and the dark shades refer to the interference between them. The lower level is assumed to be infinitely sharp.

(A color version of this figure is available in the online journal.)

oscillator theory give identical expression for the Hanle–Zeeman redistribution matrix. The term "Hanle–Zeeman" refers to the full field strength regime, from the zero field (resonance scattering), to the weak (Hanle effect), up to the strong field (Zeeman) regime.

In Section 5 of P1, the authors describe how the classical time-dependent oscillator theory for a $J = 0 \rightarrow 1 \rightarrow 0$ transition (normal Zeeman triplet) can be extended, to the more general case of transitions involving arbitrary quantum numbers. Such an extension proceeds in a phenomenological way, by drawing analogy between the Kramers–Heisenberg scattering amplitude for line scattering in quantum mechanics and the Jones matrix used in the classical theory of line scattering. In this paper, we present the mathematical basis for such a phenomenological extension and arrive at the Hanle–Zeeman redistribution matrix for the general case of a $J_a \rightarrow J_b \rightarrow J_a$ scattering transition (see Figure 1), where J_a and J_b are the total angular momentum quantum numbers of the lower and upper levels, respectively. It may be noted that the theory still uses the restriction of scattering on a two-level atom model with an infinitely sharp and unpolarized lower level.

The outline of the paper is as follows. In Section 2, we derive the ensemble-averaged coherency matrix in both the atomic and laboratory frames for a $J_a \rightarrow J_b \rightarrow J_a$ scattering transition. In the same section we also present the Mueller scattering matrix. In Section 3, we show the equivalence between the expressions for PRD matrices derived from our semi-classical approach and those derived from the QED theory of B97b. This equivalence is presented in the atomic rest frame. The expressions for the PRD matrices in the laboratory frame will be presented in Section 4. The Stokes profiles obtained from a single scattering experiment for a $J = 1/2 \rightarrow 3/2 \rightarrow 1/2$ scattering transition are shown and discussed in Section 5. Concluding remarks are drawn in Section 6.

2. HANLE–ZEEMAN REDISTRIBUTION MATRIX FOR A $J_a \rightarrow J_b \rightarrow J_a$ SCATTERING TRANSITION

2.1. Scattering Amplitude and Mueller Matrix for Frequency-coherent Scattering

Stenflo (1998; see also Stenflo 1994, 1997) has developed a theory of scattering that allows Mueller matrix for frequency-coherent scattering to be calculated for arbitrary magnetic fields, atomic multiplets, and scattering transitions (Rayleigh or Raman scattering). His theory is based on the Kramers–Heisenberg dispersion formula that gives differential cross-section for scattering of a photon by an atomic electron. It was originally derived by Kramers & Heisenberg (1925), before the advent of quantum mechanics, based on the correspondence principle applied to the classical dispersion formula for light. The actual quantum mechanical proof was given by Dirac (1927). The Kramers–Heisenberg dispersion formula is the basic expression of quantum mechanical scattering theory, and it comes as the second-order term in a time-dependent perturbation theory (see Loudon 1983; also Stenflo 1994). This formula is applicable to only frequency-coherent scattering. In Section 2.2, we show how it can be extended to include PRD in a phenomenological way (see also P1). Since we largely dwell upon the theoretical framework developed in Stenflo (1998), in this section we recall few important equations from that paper.

For allowed electric dipole transitions, the complex probability amplitude for scattering from a given initial magnetic substate characterized by quantum numbers J_a and μ_a into a final magnetic substate characterized by J_a and μ_f via all possible intermediate magnetic substates μ_b of the upper state J_b is given by the Kramers–Heisenberg formula (see Equation (3) in Stenflo 1998):

$$w_{\alpha\beta}(\mu_f \mu_a) \sim \sum_{\mu_b} (-1)^{q-q'} (-1)^{2r_{\rm ab}} f_{\rm ab} (2J_a + 1) \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix} \Phi_\gamma(\nu_{\mu_b\mu_f} - \xi) \varepsilon_q^{\alpha *} \varepsilon_{q'}^{\beta}, \quad (1)$$

where the quantities ε are the geometrical factors (see Equations (2) and (27) of Stenflo 1998) with α and β denoting the outgoing and incoming radiation, respectively. Owing to the property of 3-j symbols, q and q' in Equation (1) satisfy

$$q = \mu_f - \mu_b; \quad q' = \mu_a - \mu_b. \quad (2)$$

In Equation (1), $f_{\rm ab}$ gives the absorption oscillator strength between the lower (J_a) and upper (J_b) states, and the corresponding exponent $r_{\rm ab}$ determines the sign of the expression. They are defined in Stenflo (1994, pp. 192 and 199). Note that these factors depend only on the J, L, and S quantum numbers of the lower and upper states, and hence are constants for a given value of J_a and J_b. Therefore, they can be absorbed in the normalization constant along with the factor $(2J_a + 1)$, since there is no summation over J_a.

The area-normalized profile function is given by

$$\Phi_\gamma(\nu_{\mu_b\mu_f} - \xi) = \frac{1/(\pi {\rm i})}{\nu_{\mu_b\mu_f} - \xi - {\rm i}\gamma/(4\pi)}, \quad (3)$$

where ξ is the frequency of the outgoing photon in the atomic rest frame and

$$\nu_{\mu_b\mu_f} = \nu_0 + (g_b\mu_b - g_a\mu_f)\nu_L. \tag{4}$$

Here, $h\nu_0$ is the energy difference between the upper state J_b and lower state J_a in the absence of magnetic fields, g_b and g_a are the Landé factors of the upper and the lower states, ν_L is the Larmor frequency, and γ is the damping constant that accounts for the broadening of the excited state. The lower state in this formulation is assumed to be infinitely sharp (see Stenflo 1998).

In the classical theory of scattering, the transformation from the incident to the scattered Stokes vector is described by the Mueller scattering matrix (see Stenflo 1994). It is given by

$$\mathbf{M} = \mathbf{T}\mathbf{W}\mathbf{T}^{-1}, \tag{5}$$

where (see Equation (2) of Stenflo 1997)

$$\mathbf{W} = \sum_{\mu_a,\mu_f} \boldsymbol{w}(\mu_f\mu_a) \otimes \boldsymbol{w}^*(\mu_f\mu_a). \tag{6}$$

The symbol "$\otimes$" stands for the tensor product and "$*$" for the complex conjugation. In this paper, we assume that there is no atomic polarization in the initial state a when summing over all the initial and final magnetic substates represented by μ_a and μ_f, respectively. The matrices $\mathbf{T}$ and $\mathbf{T}^{-1}$ in Equation (5) are purely mathematical transformation matrices and their explicit form can be found in Stenflo (1998). The form of the tensor product $\boldsymbol{w}(\mu_f\mu_a)\otimes\boldsymbol{w}^*(\mu_f\mu_a)$ is also given in Equation (10) of Stenflo (1998). The normalized Mueller matrix is nothing but the Hanle–Zeeman scattering matrix in the particular case of frequency-coherent scattering and is termed as the Hanle–Zeeman redistribution matrix in the general case of PRD.

2.2. Phenomenological Extension to Include PRD

The phenomenological extension of Equation (1) to the case of PRD is achieved by treating each radiative emission transition between magnetic substates μ_b and μ_f by a damped oscillation that is truncated by collisions (see P1). In other words, in Equation (1) we make the following replacement for the profile function:

$$\Phi_\gamma(\nu_{\mu_b\mu_f} - \xi) \longrightarrow \tilde{r}_{\mu_f\mu_b\mu_a}, \tag{7}$$

where $\tilde{r}_{\mu_f\mu_b\mu_a}$ denotes the Fourier-transformed solution of the time-dependent oscillator equation and is given by

$$\tilde{r}_{\mu_f\mu_b\mu_a} = \int_{t_0}^{t_0+t_c} r_{\mu_f\mu_b\mu_a}(t,\xi')\mathrm{e}^{2\pi\mathrm{i}\xi t}dt. \tag{8}$$

The limits of the Fourier integral in Equation (8) are taken as finite, to accommodate the effects of elastic collisions (see BS99). The collision interval is taken to be t_c. Within this interval, the oscillator remains undisturbed. The elastic collision causes phase scrambling generally leading to depolarization. The Fourier integral has non-zero contributions only during the time interval $[t_0, t_0 + t_c]$, where t_0 and $t_0 + t_c$ are the time points at which collision events occur. In Equation (8), ξ' refers to the frequency of the incoming photon in the atomic rest frame. In the case of a $J = 0 \rightarrow 1 \rightarrow 0$ scattering transition, $r_{\mu_f\mu_b\mu_a}$ represents solution of the classical oscillator equation and is given by Equations (16)–(18) of BS99. As suggested in P1, we may generalize Equations (16)–(18) of BS99 to a $\mu_a \rightarrow \mu_b \rightarrow \mu_f$ scattering transition as follows:

$$r_{\mu_f\mu_b\mu_a}(t,\xi') = r^{\mathrm{stat}}_{\mu_f\mu_b\mu_a}(t,\xi') + Cr^{\mathrm{trans}}_{\mu_f\mu_b\mu_a}(t,\xi')\mathrm{e}^{\mathrm{i}\delta}, \tag{9}$$

where C and δ are the amplitude and phase of the oscillator. The stationary solution of the oscillator equation is given by

$$r^{\mathrm{stat}}_{\mu_f\mu_b\mu_a}(t,\xi') = \frac{\mathrm{e}^{-2\pi\mathrm{i}\xi' t}}{2\pi\xi' - (2\pi\nu_{\mu_b\mu_a} - \mathrm{i}\gamma/2)}, \tag{10}$$

and the transitory solution is given by

$$r^{\mathrm{trans}}_{\mu_f\mu_b\mu_a}(t,\xi') = \frac{\mathrm{e}^{-2\pi\mathrm{i}(\nu_{\mu_b\mu_f} - \mathrm{i}\gamma/2)t}}{2\pi\xi' - (2\pi\nu_{\mu_b\mu_a} - \mathrm{i}\gamma/2)}. \tag{11}$$

Clearly, we have associated $\nu_{\mu_b\mu_a}$ to the absorption profile part of the solution and $\nu_{\mu_b\mu_f}$ to the emission profile part of the solution (which we obtain after taking a Fourier transform as described by Equation (8)). Such a generalization of classical oscillator solution to a $\mu_a \rightarrow \mu_b \rightarrow \mu_f$ transition is also consistent with the energy conservation described by Equations (9.10) and (9.11) of Stenflo (1994). Now taking the Fourier transform of Equations (10) and (11), we obtain

$$\tilde{r}^{\mathrm{stat}}_{\mu_f\mu_b\mu_a} = \Phi'_\gamma(\nu_{\mu_b\mu_a} - \xi')\delta(\xi - \xi' - \nu_{af}) \tag{12}$$

and

$$\tilde{r}^{\mathrm{trans}}_{\mu_f\mu_b\mu_a} = \Phi'_\gamma(\nu_{\mu_b\mu_a} - \xi')\Phi_\gamma(\nu_{\mu_b\mu_f} - \xi)[1 - \mathrm{e}^{-\mathrm{i}(2\pi\nu_{\mu_b\mu_f} - \mathrm{i}\gamma/2 - 2\pi\xi)t_c}]. \tag{13}$$

$\Phi'_\gamma(\nu_{\mu_b\mu_a} - \xi')$ is given by Equation (3) but with ξ and $\nu_{\mu_b\mu_f}$ replaced respectively by ξ' and $\nu_{\mu_b\mu_a}$. Note that to be consistent with the energy conservation, namely, Equation (9.10) of Stenflo (1994), we have introduced ν_{af}—the energy difference between the magnetic sub-states μ_a and μ_f in the delta function appearing in Equation (12). It is given by

$$\nu_{af} = g_a(\mu_a - \mu_f)\nu_L. \tag{14}$$

2.3. Atomic Frame Coherency Matrix

From Equation (6), it is clear that the **W** matrix depends on bilinear products of the form (ignoring the unimportant proportionality factors)

$$w_{\alpha\beta}(\mu_f\mu_a)w^*_{\alpha'\beta'}(\mu_f\mu_a) \sim \sum_{\mu_b\mu'_b}(-1)^{q-q'}(-1)^{q''-q'''}\varepsilon^{\alpha*}_{q}\varepsilon^{\alpha'}_{q''}\varepsilon^{\beta}_{q'}\varepsilon^{\beta'*}_{q'''}\langle\tilde{r}_{\mu_f\mu_b\mu_a}\tilde{r}^*_{\mu_f\mu'_b\mu_a}\rangle\begin{pmatrix}J_b & J_a & 1\\ -\mu_b & \mu_a & -q'\end{pmatrix}$$
$$\times\begin{pmatrix}J_b & J_a & 1\\ -\mu'_b & \mu_a & -q'''\end{pmatrix}\begin{pmatrix}J_b & J_a & 1\\ -\mu_b & \mu_f & -q\end{pmatrix}\begin{pmatrix}J_b & J_a & 1\\ -\mu'_b & \mu_f & -q''\end{pmatrix}, \tag{15}$$

where the ensemble-averaged coherency matrix elements $\langle\tilde{r}_{\mu_f\mu_b\mu_a}\tilde{r}^*_{\mu_f\mu'_b\mu_a}\rangle$ can be derived using Equations (12) and (13), following exactly the same procedure which is described in detail in BS99. Here, we present only the final expressions. Thus, the ensemble-averaged coherency matrix elements in the atomic frame are given by

$$\langle\tilde{r}_{\mu_f\mu_b\mu_a}\tilde{r}^*_{\mu_f\mu'_b\mu_a}\rangle = A\cos\beta_{\mu'_b-\mu_b}e^{i\beta_{\mu'_b-\mu_b}}\Phi^{\gamma+\gamma_c}_{\mu_b\mu'_b\mu_a}(\xi')\delta(\xi-\xi'-v_{af}) + B\cos\beta_{\mu'_b-\mu_b}\cos\alpha_{\mu'_b-\mu_b}e^{i(\beta_{\mu'_b-\mu_b}+\alpha_{\mu'_b-\mu_b})}\Phi^{\gamma+\gamma_c}_{\mu_b\mu'_b\mu_a}(\xi')\Phi^{\gamma+\gamma_c}_{\mu_b\mu'_b\mu_f}(\xi), \tag{16}$$

where γ_c is the collisional damping constant. The Hanle angles $\beta_{\mu'_b-\mu_b}$ and $\alpha_{\mu'_b-\mu_b}$ are defined respectively by

$$\tan\beta_{\mu'_b-\mu_b} = \frac{g_b(\mu'_b-\mu_b)2\pi\nu_L}{\gamma+\gamma_c}, \tag{17}$$

$$\tan\alpha_{\mu'_b-\mu_b} = \frac{g_b(\mu'_b-\mu_b)2\pi\nu_L}{\gamma+\gamma_c/2}. \tag{18}$$

A and B are branching ratios given in Equations (40) and (41) of BS99. They give the fraction of scattering process that are coherent (A) and incoherent (B) in nature. The classical generalized profile function is defined as

$$\Phi^{\gamma}_{\mu_b\mu'_b\mu_f}(\xi) = \frac{1}{2}[\Phi_\gamma(\nu_{\mu_b\mu_f}-\xi)+\Phi^*_\gamma(\nu_{\mu'_b\mu_f}-\xi)]. \tag{19}$$

When deriving Equation (16), we have made use of the following relation:

$$\Phi_\gamma(\nu_{\mu_b\mu_f}-\xi)\Phi^*_\gamma(\nu_{\mu'_b\mu_f}-\xi) = \frac{4}{\gamma-ig_b(\mu'_b-\mu_b)2\pi\nu_L}\Phi^{\gamma}_{\mu_b\mu'_b\mu_f}(\xi). \tag{20}$$

2.4. Laboratory Frame Coherency Matrix

Equation (16) can be transformed to the laboratory frame following exactly the same procedure as described in Section 2.2 of P1 (see also Section 3.3 of B97b). Thus, the ensemble-averaged coherency matrix elements in the laboratory frame are given by

$$\langle\tilde{r}_{\mu_f\mu_b\mu_a}\tilde{r}^*_{\mu_f\mu'_b\mu_a}\rangle = A\cos\beta_{\mu'_b-\mu_b}e^{i\beta_{\mu'_b-\mu_b}}\left[h^{\rm II}_{\mu_b\mu'_b}(\mu_f\mu_a)+if^{\rm II}_{\mu_b\mu'_b}(\mu_f\mu_a)\right] + B\cos\beta_{\mu'_b-\mu_b}\cos\alpha_{\mu'_b-\mu_b}e^{i(\beta_{\mu'_b-\mu_b}+\alpha_{\mu'_b-\mu_b})}$$
$$\times\left\{\Re\left[h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\right]-\Im\left[f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\right]+i\left(\Im\left[h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\right]+\Re\left[f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\right]\right)\right\}. \tag{21}$$

The various auxiliary quantities appearing in the above equation for the case of Hummer's type II redistribution are given by

$$h^{\rm II}_{\mu_b\mu'_b}(\mu_f\mu_a) = \frac{1}{2}\left[R^{\rm II,\,H}_{\mu_f\mu_b\mu_a}+R^{\rm II,\,H}_{\mu_f\mu'_b\mu_a}\right], \tag{22}$$

$$f^{\rm II}_{\mu_b\mu'_b}(\mu_f\mu_a) = \frac{1}{2}\left[R^{\rm II,\,F}_{\mu_f\mu'_b\mu_a}-R^{\rm II,\,F}_{\mu_f\mu_b\mu_a}\right], \tag{23}$$

where the magnetic redistribution functions of type II are given by

$$R^{\rm II,\,H}_{\mu_f\mu_b\mu_a}(x,x',\Theta) = \frac{1}{\pi\sin\Theta}\exp\left\{-\left[\frac{x-x'+x_{af}}{2\sin(\Theta/2)}\right]^2\right\}H\left(\frac{a}{\cos(\Theta/2)},\frac{v_{\mu_b\mu_a}+v'_{\mu_b\mu_a}+x_{af}}{2\cos(\Theta/2)}\right), \tag{24}$$

$$R^{\rm II,\,F}_{\mu_f\mu_b\mu_a}(x,x',\Theta) = \frac{1}{\pi\sin\Theta}\exp\left\{-\left[\frac{x-x'+x_{af}}{2\sin(\Theta/2)}\right]^2\right\}F\left(\frac{a}{\cos(\Theta/2)},\frac{v_{\mu_b\mu_a}+v'_{\mu_b\mu_a}+x_{af}}{2\cos(\Theta/2)}\right). \tag{25}$$

In the above equations $H(a, x)$ and $F(a, x)$ are the Voigt and Faraday–Voigt functions, Θ is the scattering angle (the angle between incident and scattered ray; see Figure 2). The dimensionless quantities appearing in Equations (24) and (25) are given by

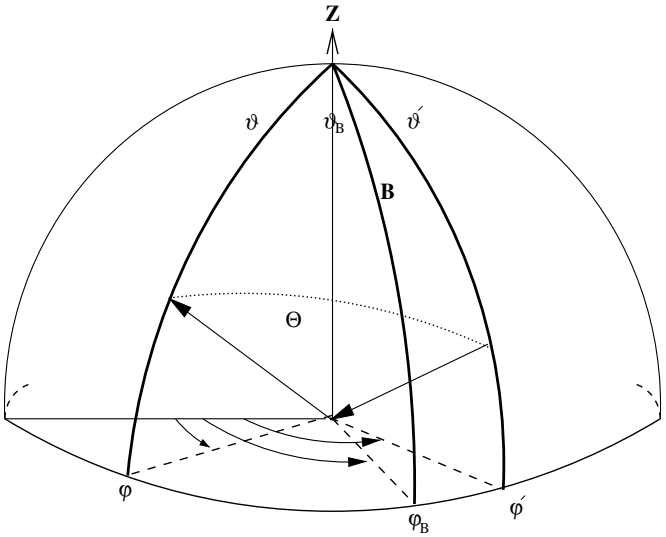

Figure 2. Geometry showing the scattering process in a coordinate system where the magnetic field makes an angle ϑ_B with respect to the polar Z-axis and has an azimuth of φ_B. (ϑ', φ') refer to the incident ray and (ϑ, φ) to the scattered ray defined with respect to the polar Z-axis. Θ is the scattering angle.

$$x = \frac{\nu_0 - \nu}{\Delta\nu_D}; \quad v_{\mu_b\mu_a} = x + (g_b\mu_b - g_a\mu_a)\frac{\nu_L}{\Delta\nu_D}, \quad a = \frac{\gamma + \gamma_c}{4\pi\Delta\nu_D}, \tag{26}$$

which are, respectively, the emission frequency, magnetic shift, and damping parameter. $\Delta\nu_D$ is the Doppler width and $x_{af} = \nu_{af}/\Delta\nu_D$.

Now the various auxiliary quantities appearing in Equation (21) for the case of type III redistribution are given by

$$h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a) = \Re\big[h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big] + {\rm i}\Im\big[h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big], \tag{27}$$

where the real ($\Re$) and imaginary ($\Im$) parts are defined through

$$\Re\big[h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big] = \frac{1}{4}\big[R^{\rm III,\,HH}_{\mu'_b\mu_a,\mu'_b\mu_f} + R^{\rm III,\,HH}_{\mu'_b\mu_a,\mu_b\mu_f} + R^{\rm III,\,HH}_{\mu_b\mu_a,\mu'_b\mu_f} + R^{\rm III,\,HH}_{\mu_b\mu_a,\mu_b\mu_f}\big], \tag{28}$$

$$\Im\big[h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big] = \frac{1}{4}\big[R^{\rm III,\,FH}_{\mu'_b\mu_a,\mu'_b\mu_f} + R^{\rm III,\,FH}_{\mu'_b\mu_a,\mu_b\mu_f} - R^{\rm III,\,FH}_{\mu_b\mu_a,\mu'_b\mu_f} - R^{\rm III,\,FH}_{\mu_b\mu_a,\mu_b\mu_f}\big]. \tag{29}$$

Similarly, we have

$$f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a) = \Re\big[f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big] + {\rm i}\Im\big[f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big], \tag{30}$$

where the real ($\Re$) and imaginary ($\Im$) parts are defined through

$$\Re\big[f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big] = \frac{1}{4}\big[R^{\rm III,\,HF}_{\mu'_b\mu_a,\mu'_b\mu_f} - R^{\rm III,\,HF}_{\mu'_b\mu_a,\mu_b\mu_f} + R^{\rm III,\,HF}_{\mu_b\mu_a,\mu'_b\mu_f} - R^{\rm III,\,HF}_{\mu_b\mu_a,\mu_b\mu_f}\big], \tag{31}$$

$$\Im\big[f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)\big] = \frac{1}{4}\big[R^{\rm III,\,FF}_{\mu'_b\mu_a,\mu'_b\mu_f} - R^{\rm III,\,FF}_{\mu'_b\mu_a,\mu_b\mu_f} - R^{\rm III,\,FF}_{\mu_b\mu_a,\mu'_b\mu_f} + R^{\rm III,\,FF}_{\mu_b\mu_a,\mu_b\mu_f}\big]. \tag{32}$$

The magnetic redistribution functions of type III appearing in Equations (28)–(32) are given by

$$R^{\rm III,\,HH}_{\mu_b\mu_a,\mu'_b\mu_f}(x,\,x',\,\Theta) = \frac{1}{\pi^2\sin\Theta}\int_{-\infty}^{+\infty} du\,{\rm e}^{-u^2}\left[\frac{a}{a^2 + (v'_{\mu_b\mu_a} - u)^2}\right] H\left(\frac{a}{\sin\Theta},\,\frac{v_{\mu'_b\mu_f}}{\sin\Theta} - u\cot\Theta\right), \tag{33}$$

$$R^{\rm III,\,HF}_{\mu_b\mu_a,\mu'_b\mu_f}(x,\,x',\,\Theta) = \frac{1}{\pi^2\sin\Theta}\int_{-\infty}^{+\infty} du\,{\rm e}^{-u^2}\left[\frac{a}{a^2 + (v'_{\mu_b\mu_a} - u)^2}\right] F\left(\frac{a}{\sin\Theta},\,\frac{v_{\mu'_b\mu_f}}{\sin\Theta} - u\cot\Theta\right), \tag{34}$$

$$R^{\rm III,\,FH}_{\mu_b\mu_a,\mu'_b\mu_f}(x,\,x',\,\Theta) = \frac{1}{\pi^2\sin\Theta}\int_{-\infty}^{+\infty} du\,{\rm e}^{-u^2}\left[\frac{(v'_{\mu_b\mu_a} - u)}{a^2 + (v'_{\mu_b\mu_a} - u)^2}\right] H\left(\frac{a}{\sin\Theta},\,\frac{v_{\mu'_b\mu_f}}{\sin\Theta} - u\cot\Theta\right), \tag{35}$$

and

$$R^{\rm III,\,FF}_{\mu_b\mu_a,\mu'_b\mu_f}(x,\,x',\,\Theta) = \frac{1}{\pi^2\sin\Theta}\int_{-\infty}^{+\infty} du\,{\rm e}^{-u^2}\left[\frac{(v'_{\mu_b\mu_a} - u)}{a^2 + (v'_{\mu_b\mu_a} - u)^2}\right] F\left(\frac{a}{\sin\Theta},\,\frac{v_{\mu'_b\mu_f}}{\sin\Theta} - u\cot\Theta\right). \tag{36}$$

We note that $f^{\rm II}_{\mu_b\mu'_b}(\mu_f\mu_a)$, $\Im[h^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)]$, and $f^{\rm III}_{\mu_b\mu'_b}(\mu_f\mu_a)$ are non-zero only when $\mu_b \neq \mu'_b$. Furthermore, these auxiliary quantities defined above satisfy the following symmetry relations:

$$h^{\rm II}_{\mu'_b\mu_b}(\mu_f\mu_a) = h^{\rm II}_{\mu_b\mu'_b}(\mu_f\mu_a), \quad f^{\rm II}_{\mu'_b\mu_b}(\mu_f\mu_a) = -f^{\rm II}_{\mu_b\mu'_b}(\mu_f\mu_a), \quad h^{\rm III}_{\mu'_b\mu_b}(\mu_f\mu_a) = h^{\rm III\,*}_{\mu_b\mu'_b}(\mu_f\mu_a), \quad f^{\rm III}_{\mu'_b\mu_b}(\mu_f\mu_a) = -f^{\rm III\,*}_{\mu_b\mu'_b}(\mu_f\mu_a). \tag{37}$$

Using Equations (15) and (21) in Equations (5) and (6), we obtain the Hanle–Zeeman redistribution matrix for the general $J_a \to J_b \to J_a$ scattering transition in the laboratory frame.

3. EQUIVALENCE OF THE REDISTRIBUTION MATRICES DERIVED FROM QED AND THE SEMI-CLASSICAL THEORIES

Our aim in this section is to show that the redistribution matrix derived in B97b for a general $J_a \to J_b \to J_a$ transition (see her Equations (49) and (51) for the infinitely sharp lower level, arbitrary line case) is equivalent to the one derived in Section 2 using a semi-classical approach. Establishing this equivalence is very crucial to prove the correctness of the proposed PRD theory for a general $J_a \to J_b \to J_a$ transition. To this end, we need to expand the redistribution matrix derived above as a sum of its multipolar components. Furthermore, such an expansion becomes essential for the type III redistribution, as we can then assign the proper multipole index K (where $K = 0, 1, 2$) to the branching ratio B, and the Hanle angle $\alpha_{\mu'_b-\mu_b}$, both of which depend on the depolarizing collisions $D^{(K)}$ (see, e.g., Equation (41) of BS99).

In P2, it was shown that the multipolar expansion of the redistribution matrix can be achieved by introducing the irreducible spherical tensors $\mathcal{T}^K_Q(i, \boldsymbol{n})$ of Landi Degl'Innocenti (1984), where i refers to the Stokes parameters ($i = 0, 1, 2, 3$), $\boldsymbol{n}$ ($= \vartheta, \varphi$) to the ray direction with respect to the polar Z-axis (see Figure 2), and $K = 0, 1, 2$ with $-K \leqslant Q \leqslant +K$. Here, we follow the same procedure described in detail in Appendix C of P2. In Appendices A and B of the present paper, we describe in detail how to apply the procedure given in P2, to the problem at hand, but for zero magnetic field case, with and without frequency redistribution. Here, we consider the case of non-zero magnetic field. In the following sub-sections, we first recall the redistribution matrix elements presented by B97b in her notations, and then using the procedure described in Appendices A and B, we cast our equations given in Section 2 for non-zero magnetic field, into a form similar to that of B97b, to establish the equivalence.

3.1. Atomic Frame Redistribution Matrices Derived from QED Theory

For a $J \to J' \to J$ scattering transition with infinitely sharp lower level J, the QED redistribution matrix elements in the atomic frame are given by

$$\mathbf{R}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) = \mathbf{R}^{\rm II}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) + \mathbf{R}^{\rm III}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}). \tag{38}$$

The elements of the type II redistribution matrix are given by Equation (51) of B97b. Using her Equation (12), Equation (51) of B97b can be re-written as

$$\begin{aligned}\mathbf{R}^{\rm II}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) = \sum_{K'K''QMM'NN''pp'p''p'''} & 3(2J'+1)\sqrt{(2K'+1)(2K''+1)}\frac{\Gamma_R}{\Gamma_R+\Gamma_I+\Gamma_E+{\rm i}\omega_L g_{J'}Q}(-1)^{J'-N-1+Q}(-1)^{J'-N''-1+Q} \\ & \times \begin{pmatrix} J & 1 & J' \\ -N & -p & M \end{pmatrix}\begin{pmatrix} J & 1 & J' \\ -N & -p' & M' \end{pmatrix}\begin{pmatrix} J & 1 & J' \\ -N'' & -p'' & M \end{pmatrix}\begin{pmatrix} J & 1 & J' \\ -N'' & -p''' & M' \end{pmatrix}\begin{pmatrix} 1 & 1 & K' \\ -p & p' & Q \end{pmatrix} \\ & \times \begin{pmatrix} 1 & 1 & K'' \\ -p'' & p''' & Q \end{pmatrix}\delta(\xi-\xi'-\nu_{NN''})\frac{1}{2}[\phi(\nu_{J'M',JN}-\xi')+\phi^*(\nu_{J'M,JN}-\xi')](-1)^Q\mathcal{T}^{K''}_Q(i,\boldsymbol{n})\mathcal{T}^{K'}_{-Q}(j,\boldsymbol{n}'),\end{aligned} \tag{39}$$

where M, M' and N, N'' denote the magnetic sub-states of the upper level J', and lower level J, respectively, and $\omega_L = 2\pi\nu_L$. The profile function is given by (see Equation (2) of B97b, where we neglect the Lamb shift term Δ_{ba})

$$\phi(\nu_{J'M,JN}-\xi) = \frac{2}{(\Gamma_R+\Gamma_I+\Gamma_E)/2 - 2\pi{\rm i}(\nu_{J'M,JN}-\xi)}, \tag{40}$$

with $\nu_{J'M,JN} = \nu_0 + (g_{J'}M - g_J N)\nu_L$ and $\nu_{NN''} = g_J(N-N'')\nu_L$.

The elements of the redistribution matrix of type III are given by (see Equation (49) of B97b)

$$\begin{aligned}\mathbf{R}^{\rm III}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) = \sum_{KK'K''Q} & \frac{\Gamma_R}{\Gamma_R+\Gamma_I+D^{(K)}+{\rm i}\omega_L g_{J'}Q}\frac{[\Gamma_E - D^{(K)}]}{\Gamma_R+\Gamma_I+\Gamma_E+{\rm i}\omega_L g_{J'}Q} \\ & \times (-1)^Q\mathcal{T}^{K''}_Q(i,\boldsymbol{n})\mathcal{T}^{K'}_{-Q}(j,\boldsymbol{n}')\Phi^{K,K''}_Q(J,J';\xi)\Phi^{K,K'}_Q(J,J';\xi'),\end{aligned} \tag{41}$$

where the quantum generalized profile functions $\Phi_Q^{K,K''}(J, J'; \xi)$ and $\Phi_Q^{K,K'}(J, J'; \xi')$ are related to the profile function defined in Equation (40), by the following expression (see Equations (22) and (12) of B97b):

$$\sum_K \Phi_Q^{K,K''}(J, J'; \xi)\Phi_Q^{K,K'}(J, J'; \xi') = \sum_{MM'NN''pp'p''p'''} 3(2J'+1)\sqrt{(2K'+1)(2K''+1)}(-1)^{J'-N-1+Q}(-1)^{J'-N''-1+Q}$$
$$\times \begin{pmatrix} J & 1 & J' \\ -N & -p & M \end{pmatrix}\begin{pmatrix} J & 1 & J' \\ -N & -p' & M' \end{pmatrix}\begin{pmatrix} J & 1 & J' \\ -N'' & -p'' & M \end{pmatrix}$$
$$\times \begin{pmatrix} J & 1 & J' \\ -N'' & -p''' & M' \end{pmatrix}\begin{pmatrix} 1 & 1 & K' \\ -p & p' & Q \end{pmatrix}\begin{pmatrix} 1 & 1 & K'' \\ -p'' & p''' & Q \end{pmatrix}$$
$$\times \frac{1}{2}[\phi(\nu_{J'M',JN''} - \xi) + \phi^*(\nu_{J'M,JN''} - \xi)]\frac{1}{2}[\phi(\nu_{J'M',JN} - \xi') + \phi^*(\nu_{J'M,JN} - \xi')]. \quad (42)$$

In the above equations, Γ_R is the radiative de-excitation rate of the upper level, Γ_I is the inelastic de-excitation rate, and Γ_E is the elastic collisional rate. The depolarizing collisional rate is denoted by $D^{(K)}$.

3.2. Atomic Frame Redistribution Matrices Derived from the Semi-classical Theory

Following the same procedure as discussed in Appendices A and B for the case of zero magnetic field, we now write the redistribution matrix for the non-zero magnetic field case, in terms of the irreducible spherical tensors.

In the presence of a non-zero magnetic field, Equation (B2) takes the following form in the atomic frame:

$$T^{\rm S}_{\mu\nu,\rho\sigma}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) = (2J_a+1)^2 \sum_{\mu_a\mu_f\mu_b\mu_b'} (-1)^{q-q'}(-1)^{q''-q'''}\mathcal{E}^{\rm S}_{qq''}(\mu, \nu, \boldsymbol{n})\mathcal{E}^{\rm S}_{q'''q'}(\sigma, \rho, \boldsymbol{n}')\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix}$$
$$\times \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_a & -q''' \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_f & -q'' \end{pmatrix}\langle \tilde{r}_{\mu_f\mu_b\mu_a}\tilde{r}^*_{\mu_f\mu_b'\mu_a}\rangle, \quad (43)$$

where $\langle \tilde{r}_{\mu_f\mu_b\mu_a}\tilde{r}^*_{\mu_f\mu_b'\mu_a}\rangle$ is given by Equation (16). Now transforming to the Stokes formalism (see Appendix A), we find, after some algebra, that the redistribution matrix in the atomic frame is given by Equation (38). For the type II redistribution the expression is

$$\mathbf{R}^{\rm II}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) = \frac{2}{3}(2J_a+1)^2 \sum_{K'K''Q\mu_a\mu_f\mu_b\mu_b'} \sqrt{(2K'+1)(2K''+1)}(-1)^{q''-q'} A\cos\beta_Q e^{i\beta_Q}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix}$$
$$\times \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_a & -q''' \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_f & -q'' \end{pmatrix}\begin{pmatrix} 1 & 1 & K'' \\ q & -q'' & -Q \end{pmatrix}\begin{pmatrix} 1 & 1 & K' \\ q''' & -q' & Q \end{pmatrix}$$
$$\times \delta(\xi - \xi' - \nu_{af})\Phi^{\gamma+\gamma_c}_{\mu_b\mu_b'\mu_a}(\xi')\left[\mathcal{T}^{K''}_Q(i, \boldsymbol{n})\right]^{\rm S}\left[\mathcal{T}^{K'}_{-Q}(j, \boldsymbol{n}')\right]^{\rm S} \quad (44)$$

and for the type III redistribution the corresponding expression is

$$\mathbf{R}^{\rm III}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) = \frac{2}{3}(2J_a+1)^2 \sum_{K'K''Q\mu_a\mu_f\mu_b\mu_b'} \sqrt{(2K'+1)(2K''+1)}(-1)^{q''-q'} B\cos\beta_Q\cos\alpha_Q e^{i(\beta_Q+\alpha_Q)}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix}$$
$$\times \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_a & -q''' \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_f & -q'' \end{pmatrix}\begin{pmatrix} 1 & 1 & K'' \\ q & -q'' & -Q \end{pmatrix}\begin{pmatrix} 1 & 1 & K' \\ q''' & -q' & Q \end{pmatrix}$$
$$\times \Phi^{\gamma+\gamma_c}_{\mu_b\mu_b'\mu_a}(\xi')\Phi^{\gamma+\gamma_c}_{\mu_b\mu_b'\mu_f}(\xi)\left[\mathcal{T}^{K''}_Q(i, \boldsymbol{n})\right]^{\rm S}\left[\mathcal{T}^{K'}_{-Q}(j, \boldsymbol{n}')\right]^{\rm S}. \quad (45)$$

In the above equations $Q = \mu_b' - \mu_b$. Note that $\Phi^{\gamma+\gamma_c}_{\mu_b\mu_b'\mu_f}(\xi)$ appearing in the above equations is defined in Equation (19). Comparing the classical profile function defined in Equation (3) with the quantum profile function given in Equation (40), it is easy to verify that

$$\Phi_{\gamma+\gamma_c}(\nu_{\mu_b\mu_f} - \xi) = \phi^*(\nu_{J_b\mu_b,J_a\mu_f} - \xi). \quad (46)$$

Also, we identify that $\gamma = \Gamma_R + \Gamma_I$ and $\gamma_c = \Gamma_E$. Using Equation (46) and Equation (C22) of P2 in Equation (44) and changing $Q \to -Q$, we find

$$\mathbf{R}^{\rm II}_{ij}(\xi, \boldsymbol{n}, \xi', \boldsymbol{n}', \boldsymbol{B}) = \frac{2}{3}(2J_a+1)^2 \sum_{K'K''Q\mu_a\mu_f\mu_b\mu_b'qq'q''q'''} \sqrt{(2K'+1)(2K''+1)}(-1)^{q''-q'}(-1)^Q \frac{\Gamma_R}{\Gamma_R+\Gamma_I+\Gamma_E+iQg_b\omega_L}$$
$$\times \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_a & -q''' \end{pmatrix}\begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_f & -q'' \end{pmatrix}\begin{pmatrix} 1 & 1 & K'' \\ q & -q'' & Q \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 1 & K' \\ q' & -q''' & Q \end{pmatrix}\delta(\xi - \xi' - \nu_{af})\frac{1}{2}\left[\phi(\nu_{J_b\mu_b',J_a\mu_a} - \xi') + \phi^*(\nu_{J_b\mu_b,J_a\mu_a} - \xi')\right](-1)^Q\mathcal{T}^{K''}_Q(i, \boldsymbol{n})\mathcal{T}^{K'}_{-Q}(j, \boldsymbol{n}'). \quad (47)$$

Note that in the above equation we have introduced the summations over q, q', q'', and q'''. These summations are redundant, as their values are fixed by the property of the 3-j symbols and determined by the magnetic quantum numbers of the upper and lower levels. Nevertheless, it is not incorrect to either retain or drop the summation over these indices. In the above equation, we have used the relation

$$A\cos\beta_Q \mathrm{e}^{-\mathrm{i}\beta_Q} = \frac{\Gamma_R}{\Gamma_R+\Gamma_I+\Gamma_E+\mathrm{i}Qg_b\omega_L}. \tag{48}$$

Now we need to compare our Equation (47) derived from a semi-classical approach with the corresponding QED expression given in Equation (39). The two equations are identical, if we set in our Equation (47) $J_b = J'$, $J_a = J$, $\mu_b = M$, $\mu_b' = M'$, $\mu_a = N$, $\mu_f = N''$, $q = -p''$, $q'' = -p'''$, $q' = -p$, $q''' = -p'$, and make use of the properties of the 3-j symbols, along with the factor $2(2J+1)^2/[9(2J'+1)]$ being absorbed into the normalization constant. Thus, in the case of type II redistribution, our generalization of the Kramers–Heisenberg formula to handle PRD gives exactly the same expression as the QED theory.

To prove the equivalence of the QED expression and the corresponding semi-classical expression for $\mathbf{R}^{\mathrm{III}}$, we have to follow all the steps used to prove a similar equivalence as in the case of $\mathbf{R}^{\mathrm{II}}$. An application of these steps to Equation (45) gives

$$\begin{aligned}\mathbf{R}^{\mathrm{III}}_{ij}(\xi,\boldsymbol{n},\xi',\boldsymbol{n}',\boldsymbol{B}) = &\sum_{K'K''QMM'NN''pp'p''p'''} 3(2J'+1)\sqrt{(2K'+1)(2K''+1)}B\cos\beta_Q\cos\alpha_Q \mathrm{e}^{-\mathrm{i}(\beta_Q+\alpha_Q)}(-1)^{J'-N-1+Q}(-1)^{J'-N''-1+Q}\\
&\times\begin{pmatrix}J & 1 & J'\\ -N & -p & M\end{pmatrix}\begin{pmatrix}J & 1 & J'\\ -N & -p' & M'\end{pmatrix}\begin{pmatrix}J & 1 & J'\\ -N'' & -p'' & M\end{pmatrix}\begin{pmatrix}J & 1 & J'\\ -N'' & -p''' & M'\end{pmatrix}\begin{pmatrix}1 & 1 & K'\\ -p & p' & Q\end{pmatrix}\\
&\times\begin{pmatrix}1 & 1 & K''\\ -p'' & p''' & Q\end{pmatrix}\frac{1}{2}[\phi(\nu_{J'M',JN''}-\xi)+\phi^*(\nu_{J'M,JN''}-\xi)]\frac{1}{2}[\phi(\nu_{J'M',JN}-\xi')+\phi^*(\nu_{J'M,JN}-\xi')]\\
&\times(-1)^Q\mathcal{T}^{K''}_{Q}(i,\boldsymbol{n})\mathcal{T}^{K'}_{-Q}(j,\boldsymbol{n}').\end{aligned} \tag{49}$$

We can now apply Equation (42) to the above equation and re-write Equation (49) as

$$\mathbf{R}^{\mathrm{III}}_{ij}(\xi,\boldsymbol{n},\xi',\boldsymbol{n}',\boldsymbol{B}) = \sum_{KK'K''Q} B^{(K)}\cos\beta_Q\cos\alpha^{(K)}_Q \mathrm{e}^{-\mathrm{i}[\beta_Q+\alpha^{(K)}_Q]}(-1)^Q\mathcal{T}^{K''}_{Q}(i,\boldsymbol{n})\mathcal{T}^{K'}_{-Q}(j,\boldsymbol{n}')\Phi^{K,K''}_{Q}(J,J';\xi)\Phi^{K,K'}_{Q}(J,J';\xi'), \tag{50}$$

after identifying $\gamma_c/2 = D^{(K)}$. It is easy to verify that

$$B^{(K)}\cos\beta_Q\cos\alpha^{(K)}_Q\mathrm{e}^{-\mathrm{i}[\beta_Q+\alpha^{(K)}_Q]} = \frac{\Gamma_R}{\Gamma_R+\Gamma_I+D^{(K)}+\mathrm{i}\omega_L g_{J'}Q}\frac{[\Gamma_E-D^{(K)}]}{\Gamma_R+\Gamma_I+\Gamma_E+\mathrm{i}\omega_L g_{J'}Q}. \tag{51}$$

Substituting Equation (51) into Equation (50), we recover the QED expression of B97b given in Equation (41). The Hanle–Zeeman PRD matrices derived from our semi-classical approach are not new. Indeed they were previously derived by Bommier (1997b) using a rigorous QED theory. However the present work provides an alternative approach to this problem, and also the semi-classical theory is now generalized to treat transitions involving arbitrary J-quantum numbers. This semi-classical PRD theory can also be extended to treat J-state interference, as will be demonstrated in a forthcoming paper.

4. LABORATORY FRAME REDISTRIBUTION MATRICES IN TERMS OF THE IRREDUCIBLE SPHERICAL TENSORS

In Section 3, we have shown that the Hanle–Zeeman redistribution matrices derived from a semi-classical theory proposed in this paper are equivalent to those derived from the QED theory of B97b in the atomic rest frame. In the present section, we transform the redistribution matrices given in Equations (38), (39), and (41) to the laboratory frame following exactly the same steps as described in Sections 2.1 and 2.2 of P2 (see also Section 3.3 of B97b). After these transformations, the final expression for the Hanle–Zeeman redistribution matrix in the laboratory frame can be written as

$$\mathbf{R}_{ij}(x,\boldsymbol{n},x',\boldsymbol{n}',\boldsymbol{B}) = \mathbf{R}^{\mathrm{II}}_{ij}(x,\boldsymbol{n},x',\boldsymbol{n}',\boldsymbol{B}) + \mathbf{R}^{\mathrm{III}}_{ij}(x,\boldsymbol{n},x',\boldsymbol{n}',\boldsymbol{B}). \tag{52}$$

The function $\delta(\xi-\xi'-\nu_{NN''})\phi(\nu_{J'M',JN}-\xi')$ appearing in Equation (39) in the atomic rest frame transforms to $R^{\mathrm{II,H}}_{N''M'N}+\mathrm{i}R^{\mathrm{II,F}}_{N''M'N}$ in the laboratory frame, where the latter are the magnetic redistribution functions of type II defined in Equations (24) and (25). Thus, the elements of the type II redistribution matrix, in terms of the auxiliary functions defined in Equations (22) and (23), can be written as

$$\begin{aligned}\mathbf{R}^{\mathrm{II}}_{ij}(x,\boldsymbol{n},x',\boldsymbol{n}',\boldsymbol{B}) = &\sum_{K'K''QMM'NN''pp'p''p'''} 3(2J'+1)\sqrt{(2K'+1)(2K''+1)}\frac{\Gamma_R}{\Gamma_R+\Gamma_I+\Gamma_E+\mathrm{i}\omega_L g_{J'}Q}(-1)^{J'-N-1+Q}(-1)^{J'-N''-1+Q}\\
&\times\begin{pmatrix}J & 1 & J'\\ -N & -p & M\end{pmatrix}\begin{pmatrix}J & 1 & J'\\ -N & -p' & M'\end{pmatrix}\begin{pmatrix}J & 1 & J'\\ -N'' & -p'' & M\end{pmatrix}\begin{pmatrix}J & 1 & J'\\ -N'' & -p''' & M'\end{pmatrix}\begin{pmatrix}1 & 1 & K'\\ -p & p' & Q\end{pmatrix}\\
&\times\begin{pmatrix}1 & 1 & K''\\ -p'' & p''' & Q\end{pmatrix}\left[h^{\mathrm{II}}_{MM'}(N''N)+\mathrm{i}f^{\mathrm{II}}_{MM'}(N''N)\right](-1)^Q\mathcal{T}^{K''}_{Q}(i,\boldsymbol{n})\mathcal{T}^{K'}_{-Q}(j,\boldsymbol{n}').\end{aligned} \tag{53}$$

From Equation (41), we see that the type III redistribution matrix involves product of two generalized profile functions. The generalized profile function of a line at frequency ξ' connecting the lower level J to the upper level J' is given by (see Equation (36) of Landi Degl'Innocenti et al. 1991)

$$\Phi_Q^{K,K'}(J,J';\xi') = \sqrt{3(2J'+1)(2K+1)(2K'+1)} \sum_{MM'Npp'} (-1)^{J'-N-1+Q} \begin{pmatrix} J' & J & 1 \\ -M & N & p \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M' & N & p' \end{pmatrix} \times \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix} \begin{pmatrix} 1 & 1 & K' \\ -p & p' & Q \end{pmatrix} \frac{1}{2}[\phi(\nu_{J'M',JN} - \xi') + \phi^*(\nu_{J'M,JN} - \xi')]. \quad (54)$$

We denote the product of two generalized profile functions $\Phi_Q^{K,K''}(J,J';\xi)\Phi_Q^{K,K'}(J,J';\xi')$ after transforming them to the laboratory frame, as $\mathcal{R}_{Q,\mathrm{III}}^{K'',K,K'}(x,x',\Theta,B)$. This function can be expressed in terms of the magnetic redistribution functions of type III introduced in Equations (33)–(36). For example, $\phi(\nu_{J'M',JN''} - \xi)\phi(\nu_{J'M,JN} - \xi')$ in the atomic rest frame transforms to

$$R^{\mathrm{III,HH}}_{MN,M'N''} - R^{\mathrm{III,FF}}_{MN,M'N''} + \mathrm{i}\left(R^{\mathrm{III,HF}}_{MN,M'N''} + R^{\mathrm{III,FH}}_{MN,M'N''}\right),$$

in the laboratory frame. Thus, the type III redistribution matrix in the laboratory frame may be written as

$$\mathbf{R}^{\mathrm{III}}_{ij}(x,\boldsymbol{n},x',\boldsymbol{n}',\boldsymbol{B}) = \sum_{KK'K''Q} \frac{\Gamma_R}{\Gamma_R+\Gamma_I+D^{(K)}+\mathrm{i}\omega_L g_{J'}Q} \frac{[\Gamma_E - D^{(K)}]}{\Gamma_R+\Gamma_I+\Gamma_E+\mathrm{i}\omega_L g_{J'}Q} (-1)^Q \mathcal{T}_Q^{K''}(i,\boldsymbol{n}) \mathcal{T}_{-Q}^{K'}(j,\boldsymbol{n}') \mathcal{R}_{Q,\mathrm{III}}^{K'',K,K'}(x,x',\Theta,B), \quad (55)$$

where the laboratory frame redistribution function $\mathcal{R}_{Q,\mathrm{III}}^{K'',K,K'}(x,x',\Theta,B)$ is given by

$$\begin{aligned}
\mathcal{R}_{Q,\mathrm{III}}^{K'',K,K'}(x,x',\Theta,B) &= 3(2J'+1)(2K+1)\sqrt{(2K'+1)(2K''+1)} \sum_{MM'M''M'''NN''pp'p''p'''} (-1)^{J'-N-1+Q}(-1)^{J'-N''-1+Q} \\
&\times \begin{pmatrix} J & 1 & J' \\ -N & -p & M \end{pmatrix} \begin{pmatrix} J & 1 & J' \\ -N & -p' & M' \end{pmatrix} \begin{pmatrix} J & 1 & J' \\ -N'' & -p'' & M'' \end{pmatrix} \begin{pmatrix} J & 1 & J' \\ -N'' & -p''' & M''' \end{pmatrix} \begin{pmatrix} 1 & 1 & K' \\ -p & p' & Q \end{pmatrix} \begin{pmatrix} 1 & 1 & K'' \\ -p'' & p''' & Q \end{pmatrix} \\
&\times \begin{pmatrix} J' & J' & K \\ M' & -M & Q \end{pmatrix} \begin{pmatrix} J' & J' & K \\ M''' & -M'' & Q \end{pmatrix} \frac{1}{4}\big[R^{\mathrm{III,HH}}_{M'N,M'''N''} + R^{\mathrm{III,HH}}_{M'N,M''N''} + R^{\mathrm{III,HH}}_{MN,M'''N''} + R^{\mathrm{III,HH}}_{MN,M''N''} \\
&+ \mathrm{i}\left(R^{\mathrm{III,FH}}_{M'N,M'''N''} + R^{\mathrm{III,FH}}_{M'N,M''N''} - R^{\mathrm{III,FH}}_{MN,M'''N''} - R^{\mathrm{III,FH}}_{MN,M''N''}\right) + \mathrm{i}\left(R^{\mathrm{III,HF}}_{M'N,M'''N''} - R^{\mathrm{III,HF}}_{M'N,M''N''} + R^{\mathrm{III,HF}}_{MN,M'''N''} - R^{\mathrm{III,HF}}_{MN,M''N''}\right) \\
&- R^{\mathrm{III,FF}}_{M'N,M'''N''} + R^{\mathrm{III,FF}}_{M'N,M''N''} + R^{\mathrm{III,FF}}_{MN,M'''N''} - R^{\mathrm{III,FF}}_{MN,M''N''}\big]. \quad (56)
\end{aligned}$$

The results presented in Section 5 are computed using Equations (53) and (55).

5. SINGLE SCATTERING EXPERIMENT

Here, we present the Stokes profiles resulting from a single scattering experiment for a $J = 1/2 \rightarrow 3/2 \rightarrow 1/2$ scattering transition. As in P2 (see their Section 5), we consider an atom illuminated along the polar-Z axis (see Figure 2) by an unpolarized radiation field $(I_{\mathrm{inc}}, 0, 0, 0)^{\mathrm{T}}$ that is frequency independent across the spectral line. We choose the magnitude of this incident intensity I_{inc} to be unity. In this case, the scattered Stokes vector $(I, Q, U, V)^{\mathrm{T}}$ is simply given by the first column of the redistribution matrix. These elements can then be easily integrated over x' to give single scattered Stokes parameters.

The radiative width of the upper level is parameterized as $a_{\mathrm{R}} = \Gamma_R/(4\pi\Delta\nu_{\mathrm{D}})$, which is related to the total damping parameter a through

$$a = a_{\mathrm{R}}\left[1 + \left(\frac{\Gamma_I + \Gamma_E}{\Gamma_R}\right)\right]. \quad (57)$$

We assume the inelastic collision rate Γ_I to be zero (i.e., a pure scattering medium). The depolarizing collisional rates $D^{(2)} = 0.5\Gamma_E$ and $D^{(1)} = 0$. We present the scattered Stokes profiles for several values of the field strength parameter υ_{H} $(= \nu_L/\Delta\nu_{\mathrm{D}})$. In Figure 3, we show the x'-integrated Stokes I and fractional polarizations $(Q/I,\ U/I,\ V/I)$. The model parameters used are $(a_{\mathrm{R}},\ \vartheta_{\mathrm{B}},\ \varphi_{\mathrm{B}},\ \Theta,\ \Gamma_E/\Gamma_R) = (0.004,\ 90^\circ,\ 45^\circ,\ 60^\circ,\ 0.01)$. Our choice of Γ_E/Γ_R corresponds to an uneven mix of Hummer's type II and type III redistribution with a dominant contribution from type II scattering. Field strength parameter υ_{H} is varied such that we cover the entire field strength regime of very weak to strong fields.

Figure 3(a) corresponds to a pure line case without any continuum. Thus, the linear polarization (Q/I and U/I) go to Rayleigh scattering in the far wings (i.e., $Q/I \approx 0.3$ and $U/I = 0$). At the line center and the near wings, there is a gradual transition from the Hanle effect in the weak fields to the Zeeman effect in the strong fields. Following Stenflo (1998, see his Figure 4) in Figure 3(b) we show the Stokes profiles obtained in the presence of a background continuum emission that is weakly polarized by non-magnetic Rayleigh scattering. Let P denote the fractional polarization ($-Q/I$, $-U/I$, or V/I) in the absence of continuum. Then the polarization P' in the presence of a background continuum can be written as (see Equation (58) of Stenflo 1998)

$$P' = \frac{I}{I+c}P + \frac{c}{I+c}b, \quad (58)$$

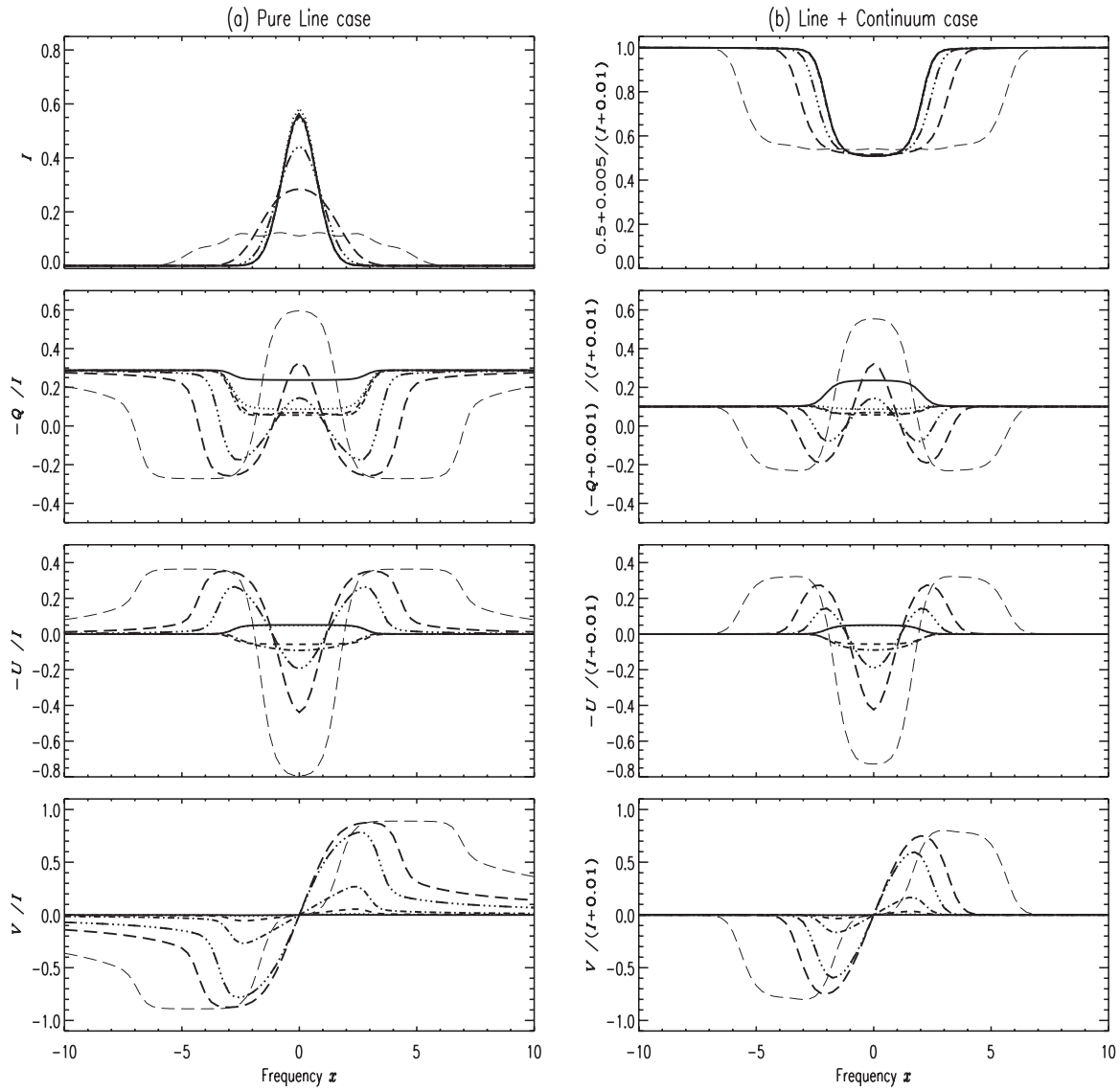

Figure 3. Stokes profile I and the fractional polarization profiles Q/I, U/I, and V/I for a $J = 1/2 \rightarrow 3/2 \rightarrow 1/2$ scattering transition, after integration over the incoming frequencies (x'). The model parameters are (a_R, ϑ_B, φ_B, Θ, Γ_E/Γ_R) = (0.004, 90°, 45°, 60°, 0.01). Different line types correspond to v_H = 0.0008 (solid), 0.004 (dotted), 0.02 (dashed), 0.1 (dot-dashed), 0.5 (dash-triple-dotted), 1.0 (long-dashed), 2.5 (thin long-dashed). Panel (a) corresponds to the pure line case and panel (b) to the case of a non-zero continuum. In the latter case the line polarization approaches the level of continuum polarization in the far wings.

where c is a constant that represents the background continuum and b represents the degree of non-magnetic continuum polarization. The Stokes I in the presence of continuum is modeled by assuming LTE and using a Milne–Eddington model (see Stenflo 1998). In this case one can show that (see Equation (61) of Stenflo 1998)

$$I'/I'_c = 1 - \beta + \frac{c}{I + c}\beta, \tag{59}$$

where I' denotes the total intensity, I'_c is the continuum intensity, and β is a limb-darkening parameter. The polarized profiles plotted in Figure 3(b) are computed by substituting in Equation (58) the I, Q, U, and V data shown in Figure 3(a), with $c = 0.01$ and $b = 0.1$ (for Stokes Q) or $b = 0$ (for Stokes U and V). The intensity profiles have been obtained using Equation (59) with $\beta = 0.5$, which is a Milne–Eddington model for Stokes I. Clearly, Q'/I' approaches the continuum polarization in the far wings. Also, the profile shapes in the presence of a background continuum seem more realistic than those in the pure line case shown in Figure 3(a).

6. CONCLUSIONS

In the present paper, we have extended the classical PRD theory of Bommier & Stenflo (1999; see also Sampoorna et al. 2007a), which is applicable for a $J = 0 \rightarrow 1 \rightarrow 0$ scattering transition, to the case of atomic transitions of the type $J_a \rightarrow J_b \rightarrow J_a$, with arbitrary values of J_a and J_b quantum numbers. The path to such a generalization was already indicated in Sampoorna et al. (2007a). The generalization actually proceeds in a phenomenological way and uses the direct analogy between the Kramers–Heisenberg dispersion formula in quantum mechanics and the classical Jones matrix. Here, we demonstrate that this semi-classical approach

indeed gives the Hanle–Zeeman PRD matrices that are in full agreement with those derived from a QED theory (Bommier 1997b). We also present the explicit form of the PRD matrices in the laboratory frame. Moreover, we illustrate the nature of the Hanle–Zeeman redistribution matrix for a single-scattering event, and for a $J = 1/2 \rightarrow 3/2 \rightarrow 1/2$ scattering transition, with and without a background continuum. In a forthcoming paper, we show that the PRD theory presented here can also be generalized to include the quantum interference between the J-states.

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APPENDIX A

NON-MAGNETIC ANGULAR PHASE MATRIX FOR A $J_a \rightarrow J_b \rightarrow J_a$ TRANSITION

Starting from the semi-classical approach of Stenflo (1998), we derive in this appendix the Rayleigh phase matrix for a $J_a \rightarrow J_b \rightarrow J_a$ transition in terms of the irreducible tensors for polarimetry $\mathcal{T}^K_Q(i, \boldsymbol{n})$ introduced by Landi Degl'Innocenti (1984), where i refers to the Stokes parameters ($i = 0, 1, 2, 3$) and $K = 0, 1, 2$ with $-K \leqslant Q \leqslant +K$. We show that we indeed recover the quantum mechanical expression for the Rayleigh phase matrix derived by Hamilton (1947). The procedure described below is used in Section 3 to express the more general expressions for the Hanle–Zeeman redistribution matrices in terms of $\mathcal{T}^K_Q(i, \boldsymbol{n})$.

When the magnetic field is zero, the profile function $\Phi_\gamma(\nu_{\mu_b\mu_f} - \xi)$ becomes independent of the magnetic sub-states μ_b and μ_f as $\nu_L = 0$ (see Equations (3) and (4)). Therefore, we can disregard the profile function in Equation (1), as we are interested only in the angular phase matrix. Thus, when the magnetic field is zero Equation (1) takes the form

$$w_{\alpha\beta}(\mu_f\mu_a) \sim \sum_{\mu_b} (-1)^{q-q'}(2J_a+1) \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix} \varepsilon^{\alpha *}_q \varepsilon^{\beta}_{q'}. \tag{A1}$$

A procedure to express the Mueller matrix for the Hanle effect in the $J = 0 \rightarrow 1 \rightarrow 0$ scattering transition in terms of the irreducible spherical tensors was presented in Appendix C of P2. We now apply the same procedure to the present case.

Using Equation (3.84) of Stenflo (1994) and Equations (C1) and (C2) of P2, we can write the electric field E_μ of the scattered ray as

$$E_\mu \sim \sum_{\rho\mu_b} (-1)^{q-q'}(2J_a+1) \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix} [e_\mu(\boldsymbol{n})]^*_q [e_\rho(\boldsymbol{n}')]_{q'} E'_\rho, \tag{A2}$$

where E'_ρ is the electric field of the incident radiation, $[e_\mu(\boldsymbol{n})]_q$ is defined in Equation (C1) of P2, and the indices μ and ρ take values ± 1. The above equation is written in the basis defined in Equation (C3) of P2. Now, the elements of the coherency matrix may be written as

$$I^{\rm S}_{\mu\nu} = \sum_{\mu_a\mu_f} E_\mu E^*_\nu. \tag{A3}$$

The superscript S in the above equation and elsewhere in the paper mean that the concerned quantities refer to the semi-classical approach. These quantities are always the complex conjugate of the respective quantities expressed in the notation of Landi Degl'Innocenti & Landolfi (2004). Substituting Equation (A2) in the above equation and using Equation (C6) of P2, we obtain

$$I^{\rm S}_{\mu\nu} \sim \sum_{\rho\sigma} T^{\rm S}_{\mu\nu,\rho\sigma}(\boldsymbol{n}, \boldsymbol{n}') I'^{\rm S}_{\rho\sigma}, \tag{A4}$$

where

$$\begin{aligned} T^{\rm S}_{\mu\nu,\rho\sigma}(\boldsymbol{n}, \boldsymbol{n}') = (2J_a+1)^2 \sum_{\mu_a\mu_f\mu_b\mu'_b} & (-1)^{q-q'}(-1)^{q''-q'''} \mathcal{E}^{\rm S}_{qq''}(\mu, \nu, \boldsymbol{n}) \mathcal{E}^{\rm S}_{q'''q'}(\sigma, \rho, \boldsymbol{n}') \\ & \times \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu'_b & \mu_a & -q''' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu'_b & \mu_f & -q'' \end{pmatrix}. \end{aligned} \tag{A5}$$

In the above equation $\mathcal{E}^{\rm S}_{qq''}(\mu, \nu, \boldsymbol{n})$ is a reducible spherical tensor. As described in P2, we now make a transformation from the coherency matrix formalism to the Stokes vector formalism. The scattered Stokes vector is then given by (see Equation (C13) of P2)

$$S_i = \sum_{j=0}^{3} \mathbf{P}_{ij}(\boldsymbol{n}, \boldsymbol{n}') S'_j, \tag{A6}$$

where

$$\mathbf{P}_{ij}(\boldsymbol{n}, \boldsymbol{n}') = \frac{1}{2} \sum_{\mu\nu\rho\sigma} (\boldsymbol{\sigma}_i)_{\nu\mu} (\boldsymbol{\sigma}_j)_{\rho\sigma} T^{\rm S}_{\mu\nu,\rho\sigma}(\boldsymbol{n}, \boldsymbol{n}'). \tag{A7}$$

The (2×2) matrices $\boldsymbol{\sigma}_i$ are defined in Equation (C12) of P2. Substituting Equation (A5) into the above equation and then using Equations (C15) and (C18) of P2, we obtain

$$\mathbf{P}_{ij}(\boldsymbol{n},\boldsymbol{n}') = \frac{2}{3}(2J_a+1)^2 \sum_{K K' Q \mu_a \mu_f \mu_b \mu_b'} (-1)^{q''-q'} \sqrt{(2K+1)(2K'+1)} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_a & -q''' \end{pmatrix} \times \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_f & -q'' \end{pmatrix} \begin{pmatrix} 1 & 1 & K \\ q & -q'' & -Q \end{pmatrix} \begin{pmatrix} 1 & 1 & K' \\ q''' & -q' & Q \end{pmatrix} \left[\mathcal{T}^K_Q(i,\boldsymbol{n})\right]^{\mathrm{S}} \left[\mathcal{T}^{K'}_{-Q}(j,\boldsymbol{n}')\right]^{\mathrm{S}}, \tag{A8}$$

where $[\mathcal{T}^K_Q(i,\boldsymbol{n})]^{\mathrm{S}}$ is related to $\mathcal{T}^K_Q(i,\boldsymbol{n})$ of Landi Degl'Innocenti (1984) by a complex conjugation (see Equation (C22) of P2). The six 3-j symbols appearing in the above equation can be contracted into two 6-j symbols by first applying Equation (2.42) and then Equation (2.34) of Landi Degl'Innocenti & Landolfi (2004). After some algebra we obtain

$$\mathbf{P}_{ij}(\boldsymbol{n},\boldsymbol{n}') = \frac{2}{3}(2J_a+1)^2 \sum_{K Q} (-1)^Q \begin{Bmatrix} 1 & 1 & K \\ J_b & J_b & J_a \end{Bmatrix}^2 \left[\mathcal{T}^K_Q(i,\boldsymbol{n})\right]^{\mathrm{S}} \left[\mathcal{T}^K_{-Q}(j,\boldsymbol{n}')\right]^{\mathrm{S}}. \tag{A9}$$

Following Equation (10.11) of Landi Degl'Innocenti & Landolfi (2004), we define

$$w^{(K)}_{J_b J_a} = (-1)^{1+J_a+J_b} \sqrt{3(2J_b+1)} \begin{Bmatrix} 1 & 1 & K \\ J_b & J_b & J_a \end{Bmatrix}. \tag{A10}$$

Note that the atomic depolarizability factor is simply given by $W_K(J_a, J_b) = [w^{(K)}_{J_b J_a}]^2$ (see Equation (10.17) of Landi Degl'Innocenti & Landolfi 2004). Using Equation (A10) and Equation (C22) of P2, we obtain, after some algebra,

$$\mathbf{P}_{ij}(\boldsymbol{n},\boldsymbol{n}') = \sum_K W_K(J_a, J_b) \left[\mathbf{P}^{(K)}_{\mathrm{R}}(\boldsymbol{n},\boldsymbol{n}')\right]_{ij}, \tag{A11}$$

where we have absorbed the factor $2(2J_a+1)^2/[9(2J_b+1)]$ in the normalization constant. The multipolar components of the Rayleigh phase matrix are then given by

$$\left[\mathbf{P}^{(K)}_{\mathrm{R}}(\boldsymbol{n},\boldsymbol{n}')\right]_{ij} = \sum_Q (-1)^Q \mathcal{T}^K_Q(i,\boldsymbol{n}) \mathcal{T}^K_{-Q}(j,\boldsymbol{n}'). \tag{A12}$$

Equations (A11) and (A12) are the same as Equations (25) and (31) of B97b.

APPENDIX B

NON-MAGNETIC REDISTRIBUTION MATRIX FOR A $J_a \rightarrow J_b \rightarrow J_a$ TRANSITION

We now set the magnetic field to zero in the expressions of Section 2, derived using a semi-classical approach. Following the same approach as in Appendix A, we derive the non-magnetic PRD matrix in terms of the irreducible spherical tensors.

Since the magnetic field is set to zero, the ensemble-averaged coherency matrix elements $\langle \tilde{r}_{\mu_f \mu_b \mu_a} \tilde{r}^*_{\mu_f \mu_b' \mu_a} \rangle$ are independent of the magnetic sub-states and in the laboratory frame are given by (see Equation (21), where now the Hanle angles are zero)

$$\left\langle \tilde{r}_{\mu_f \mu_b \mu_a} \tilde{r}^*_{\mu_f \mu_b' \mu_a} \right\rangle = A R_{\mathrm{II}}(x, x', \Theta) + B R_{\mathrm{III}}(x, x', \Theta), \tag{B1}$$

where R_{II} and R_{III} are the type II and type III redistribution functions of Hummer (1962). Following the same steps as in Appendix A, we find that

$$T^{\mathrm{S}}_{\mu\nu,\rho\sigma}(x,\boldsymbol{n},x',\boldsymbol{n}') = (2J_a+1)^2 \sum_{\mu_a \mu_f \mu_b \mu_b'} (-1)^{q-q'} (-1)^{q''-q'''} \mathcal{E}^{\mathrm{S}}_{qq''}(\mu,\nu,\boldsymbol{n}) \mathcal{E}^{\mathrm{S}}_{q'''q'}(\sigma,\rho,\boldsymbol{n}') \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_a & -q' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b & \mu_f & -q \end{pmatrix} \times \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_a & -q''' \end{pmatrix} \begin{pmatrix} J_b & J_a & 1 \\ -\mu_b' & \mu_f & -q'' \end{pmatrix} [A R_{\mathrm{II}}(x, x', \Theta) + B R_{\mathrm{III}}(x, x', \Theta)]. \tag{B2}$$

Note that the introduction of scalar frequency redistribution functions do not change in any way the steps leading to the final equations obtained in Appendix A. The only difference is that we now have two terms, one corresponding to type II and the other corresponding to type III. Furthermore, for type III redistribution the branching ratio B needs to be kept inside the summation, and a proper index K finally needs to be assigned, as done in Section 3.1 of P2. Hence following the same steps as described in Appendix A, we finally obtain (after transforming to the Stokes formalism) the non-magnetic PRD matrix as

$$\mathbf{R}_{ij}(x,\boldsymbol{n},x',\boldsymbol{n}') = \sum_K [A R_{\mathrm{II}}(x, x', \Theta) + B^{(K)} R_{\mathrm{III}}(x, x', \Theta)] W_K(J_a, J_b) \left[\mathbf{P}^{(K)}_{\mathrm{R}}(\boldsymbol{n},\boldsymbol{n}')\right]_{ij}. \tag{B3}$$

The above equation is the same as Equation (109) of Bommier (1997a), which is actually in the atomic frame. Transformation of her Equation (109) to the laboratory frame, however, gives the same expression as given above.

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