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Discrete Space Theory of Radiative Transfer: Applications

M. Srinivasa Rao

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Abstract The method of obtaining the solution of radiative transfer equation using discrete space theory (DST) is described with (1) interaction principle for different geometries (2) star product (3) calculation of radiation field at internal points. Some of the important steps to obtain the solution of radiative transfer equation in spherical symmetry is also mentioned. Applications of DST are discussed with their results in two cases (a) study of reflection effect in close binary systems and (b) to compute K1 emission line profiles from an N-type stars

Keywords radiative Transfer · discrete Space theory · close binaries · circumstellar envelope

1 Introduction

The study of transfer of radiation is pursued in several areas of scientific research such as astrophysics, reactor physics, meteorology and many other research fields. Astrophysics is one of the earliest areas of activity in whch serious studies took place. It was George Stokes (1852, 1862) first developed the concept of invariance in his glass plate problem. He showed how the reflectance remains invariant when an additional glass plate is added to a system of parallel plates of glass plates. Lord Rayleigh (1920) obtained transmission and reflection factors in more complicated situations such as dispersive medium.

The idea of addition of layer of arbitrary thickness to semi-infinite layer was proposed by Ambarzumian (1942) and noticed that the reflection characteristics remain invariant in such situations. This was taken up by Chandrasekhar (1960) and solved several problems through H-functions in a semi-infinite medium and X & Y functions in a finite medium. So layers with general properties can be added and their transmission and reflection properties can be calculated directly by utilizing the "Interaction Principle" Redheffer (1962), Preisendorfer (1965), Grant and Hunt (1969a, 1969b) is a generalized form of the invariance principle. Interaction principle is nothing but a manifestation of the conservation of radiant flux. It balances the emergent radiation

M. Srinivasa Rao: Indian Institute of Astrophysics, Koramangala, Bangalore 560034
Tel.: +91-80-25530672: Fax: +91-80-25534043: E-mail: msrao@iiap.res.in

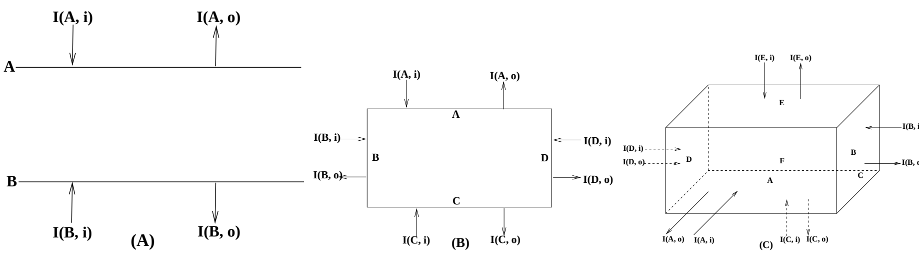

Fig. 1 Interaction principle (A) One dimensional slab (B) Two dimensional plane (C) Three dimensional cuboid

with that of reflected and transmitted input radiation together with the internally generated radiation.

2 Interaction principle

Let I represent the specific intensity and i and o represent the incident and output intensities as the boundary of A, B in one-dimensional geometry, A B C D in two-dimensional geometry and at surfaces A B C D E F in three-dimensional geometry. Let t and r represent the transmission and reflection operators and $S(A), S(B)$ represent the internal source term.

2.1 Interaction principle for (A) one (B) two and (C) three dimensional geometry

(A) The output intensities in terms of the input intensities for Fig. 1(A) as follows:

$$I(\mathrm{A}, o) = t(\mathrm{B}, \mathrm{A}) I(\mathrm{B}, i) + r(\mathrm{A}, \mathrm{A}) I(\mathrm{A}, i) + S(\mathrm{A}) \tag{1}$$

$$I(\mathrm{B}, o) = t(\mathrm{A}, \mathrm{B}) I(\mathrm{A}, i) + r(\mathrm{B}, \mathrm{B}) I(\mathrm{B}, i) + S(\mathrm{B}) \tag{2}$$

The above can be written in a matrix form

$$\begin{pmatrix} I(\mathrm{A}, o) \\ I(\mathrm{B}, o) \end{pmatrix} = \begin{pmatrix} r(\mathrm{A}, \mathrm{A}) & t(\mathrm{B}, \mathrm{A}) \\ t(\mathrm{A}, \mathrm{B}) & r(\mathrm{B}, \mathrm{B}) \end{pmatrix} \begin{pmatrix} I(\mathrm{A}, i) \\ I(\mathrm{B}, i) \end{pmatrix} + \begin{pmatrix} S(\mathrm{A}) \\ S(\mathrm{B}) \end{pmatrix}. \tag{3}$$

(B) The output intensities in terms of the input intensities for Fig. 1(B) as follows:

$$I(\mathrm{A}, o) = r(\mathrm{A}, \mathrm{A}) I(\mathrm{A}, i) + t(\mathrm{B}, \mathrm{A})(\mathrm{B}, i) + t(\mathrm{C}, \mathrm{A}) I(\mathrm{C}, i) + t(\mathrm{D}, \mathrm{A}) I(\mathrm{D}, i) S(\mathrm{A}) \tag{4}$$

Similarly, for other sides I(B, o), I(C, o), I(D, o) and can also be written in matrix form Peraiah (1984)

(C) For three dimensional geometry the output intensities in terms of the input intensities for Fig. 1(C) are as follows:

$$\begin{aligned} I(\mathrm{A}, o) = {} & t(\mathrm{A}, \mathrm{B}) I(\mathrm{B}, i) + t(\mathrm{A}, \mathrm{C}) I(\mathrm{C}, i) + t(\mathrm{A}, \mathrm{D}) I(\mathrm{D}, i) + t(\mathrm{A}, \mathrm{E}) I(\mathrm{E}, i) \\ & + t(\mathrm{A}, \mathrm{F}) I(\mathrm{F}, i) + r(\mathrm{A}, \mathrm{A}) I(\mathrm{A}, i) + S(\mathrm{A}) \end{aligned} \tag{5}$$

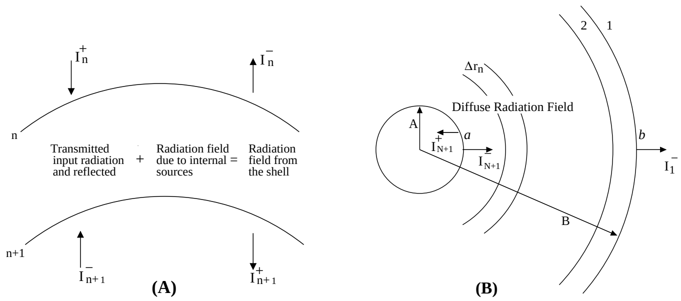

Fig. 2 Schematic diagram shows the (A) interaction principle for spherical case (B) diffuse radiation field

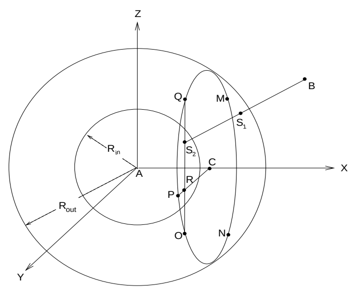

Fig. 3 Schematic diagram of the Cartesian co-ordinate system in X-Y-Z geometry; Srinivasa Rao & Varghese (2008)

Similarly, for other sides I(B, o),I(C, o),I(D, o), E(E, o), F(F, o). This can also be written in the matrix form Peraiah (1984).

For a spherically symmetric atmospheres output intensities can be written interms of input intensities. Let us consider a shell with boundaries (n, n+1) as shown in Fig. 2(A), the interaction principle can be written as follows:

$$\begin{pmatrix} \mathbf{I}^{+}_{n+1} \\ \mathbf{I}^{-}_{n} \end{pmatrix} = \mathbf{S}(n, n+1) \begin{pmatrix} \mathbf{I}^{+}_{n} \\ \mathbf{I}^{-}_{n+1} \end{pmatrix} + \sum(n, n+1) \tag{6}$$

refer Peraiah (2002, Chapter 6, Page 155) for matrix $\mathbf{S}$(n, n+1) and $\sum$ (n, n+1).

2.2 Star Product

If there is another shell with boundaries (n+1, n+2) adjacent to (n, n+1), interaction principle for this shell can be written as

$$\begin{pmatrix} \mathbf{I}^{+}_{n+2} \\ \mathbf{I}^{-}_{n+1} \end{pmatrix} = \mathbf{S}(n+1, n+2) \begin{pmatrix} \mathbf{I}^{+}_{n+1} \\ \mathbf{I}^{-}_{n+2} \end{pmatrix} + \sum(n+1, n+2) \tag{7}$$

where $\mathbf{S}$(n+1, n+2) is also similarly defined. If we combine the two shells (n, n+1) and (n+1, n+2), then the interaction principle for the combined shell is written as

$$\begin{pmatrix} \mathbf{I}^{+}_{n+2} \\ \mathbf{I}^{+}_{n+2} \\ \mathbf{I}^{-}_{n} \end{pmatrix} = \mathbf{S}(n, n+2) \begin{pmatrix} \mathbf{I}^{+}_{n} \\ \mathbf{I}^{-}_{n+2} \end{pmatrix} + \sum(n, n+2). \tag{8}$$

Redheffer (1962) calls $\mathbf{S}$(n, n+2) the star product of the two S-matrices $\mathbf{S}$(n, n+1) and $\mathbf{S}$(n+1, n+2) written as

$$\mathbf{S}(n, n+2) = \mathbf{S}(n, n+1) \star \mathbf{S}(n+1, n+2). \tag{9}$$

For other details refer Redheffer (1962).

3 Method of obtaining the solution of transfer equation

We shall consider equation of transfer with monochromatic radiation field scattering isotropically and follow the steps to obtain the solution of transfer equation
(i) We divide the medium as shown in Fig. 2(B) into number of "cells" whose thickness is less than or equal to the critical (τ_{crit}). The critical thickness is determined on the basis of physical characteristics of the medium. τ_{crit} ensures the stability and uniqueness of the solution.
(ii) Now the integration of the transfer equation is performed on the "cell" which is two-dimensional radius - angle grid bounded by $[r_n, r_{n+1}] \times [\mu_{j-\frac{1}{2}}, \mu_{j+\frac{1}{2}}]$ where $\mu_{j+\frac{1}{2}} = \sum_{k=1}^{j} C_k, j = 1, 2 \ldots, J$, where C_k are the weights of Gauss Legendre formula.
(iii) By using the interaction principle described in Peraiah and Grant (1973), we obtain the reflection and transmission operators over the "cell".
(iv) Finally we combine all the cells by the star algorithm described in Peraiah and Grant (1973) and obtain the radiation field.
(v) Using discrete space theory the solution is developed for a non-LTE two level approximation case. We represent the integration by summation, derivatives by the differences for frequency (x), angle (μ) and radius (r).
The above mentioned steps (i)-(iv) has to be followed to obtain the solution of transfer equation in spherical symmetry and for details refer Peraiah (2002, Chapter 6, Page 179).

4 Applications of radiative transfer equation in spherical symmetry:

The DST method is a finite difference, discrete ordinance method which is a very accurate and easy to generalize to different physical problems. It is applied for 1) reflection effect in close binaries for calculating the self radiation of the primary component in a binary system, 2) K1 emission lines from circumstellar shells of N-Type star R Scl and 3) Stellar winds in O and B type stars: Hydrogen atmosphere 4) To study the effects of aberration and advection in a plane parallel medium in motion and 5) Non-magnetic resonance scattering polarization in spherically symmetric media, polarized line formation in moving media, CRD, PRD and many more other applications. Presently first two cases are presented in the next section.

5 Results and discussion

- **Calculation of irradiation in 3D geometry, Fig. 3:**
  Peraiah (1983), Peraiah & Srinivasa Rao(1983, 1998, 2002), Srinivasa Rao & Peraiah (2000) studied effect of reflection in a close binary system in two dimensional geometry when secondary component is (1) a point source (2) an extended source and

(3) effect on spectral lines. They concluded that the reflection effect phenomenon has to be considered while modeling the close binary system. Since we all know three dimensional approach is more realistic than the two dimensional. recently, Srinivasa Rao & Varghese (2008) calculated the reflected radiation from an extended surface of the primary component of a close binary system in a 3-dimensional Cartesian coordinate geometry. They assumed that the secondary component is point source and moves in circular orbits with respect to the primary component. The specific intensity of the radiation field on the primary component is estimated along the line of sight observer at infinity Fig. 3.

The coordinates of the secondary component was set as (x_2, y_2, z_2) where the radiation is incident on the primary component. The following cases were considered to calculate the direction cosines of the lines which are parallel to the Z-axis and also parallel to the line of sight. In their calculations secondary component positioned at different places like; Case 1: $x_2 = \mathrm{R}, \quad y_2 = 0, \quad z_2 = 0;$ Case 2: $x_2 = \mathrm{R}\sin\frac{\pi}{4}, \quad y_2 = 0, \quad z_2 = \mathrm{R}\cos\frac{\pi}{4};$ and Case 3: $x_2 = 0, \quad y_2 = 0, \quad z_2 = \mathrm{R}$. It is noticed that the radiation field changes depending upon the position of the secondary component and shown in the surface contours, Srinivasa Rao & Varghese (2008)

- **Calculation of K1 emission line profiles from circumstellar envelope:** It is commonly observed that many super giants of O and B type stars and red giants have circumstellar shells and it formed due to various physical phenomena. We rarely obtain high resolution observations of the emission lines formed in these shells. Gustaffson et. al (1997, hereafter GEKOO) studied three N-type stars R Scl, VAql and X TrA of the K1 769.9 nm resonance line. The emission line profiles of K1 presented in the figure 2 of GEKOO are observed at two different position angles and these show significantly different characteristics.

  DST radiative transfer is applied to calculate above emission line profiles of the K1 line formed in the circumstellar envelope of the N-type star R Scl. The input data was taken from GEKOO regarding expansion velocity, mass loss rate, radii etc. The equation of line transfer in spherically symmetric approximation in a comoving frame is applied to obtain the radial distribution of the source functions. Using these source functions, the emergent intensities along the line of sight are calculated by using the formal solution of radiative transfer. At different radii the theoretical line profiles are generated. The velocity of expansion with $V_B = 5$ mean thermal units (mtu) resembles the observed profiles. It also important to calculate the equivalent withs of the line and compare with the observations which is under progress and the detailed calculations are given in forth coming paper.

6 Merits and Demerits of the method:

- The DST method can be generalized for more complicated physical situations like, 2D and 3D-geometry Accuracy and stability of the numerical solution can be checked to the machine accuracy Self consistency checks like reflection and transmission matrices, flux conservation is used to ensure the correct coding and accuracy of the discrete representation of the method. The algorithm of DST is written in such a way that the code can be written for parametric study. While changing the required parameters one can compare the results with different geometries of the problem like plane parallel to spherical and comoving frame to rest frame etc.

⊕ Application to the realistic atmospheric modeling work, leads to increased dimensionality of the matrices in the algorithm heading to large memory and CPU time requirement.

7 Conclusion:

The Discrete Space theory method is a finite difference, discrete ordinance method is presented. It is very accurate method and can be generalized to different physical problems. At present only two applications of the method is discussed 1) reflection effect in close binaries, 2) K1 emission lines from circumstellar shells of N type stars. Many more applications are left because of constrains on size of the article.

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