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SOLUTION OF RADIATIVE TRANSFER EQUATION IN SPHERICALLY SYMMETRIC MEDIA WITH SPHERICAL HARMONIC APPROXIMATION

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ABSTRACT

A numerical method for obtaining solution of radiative transfer equation in spherically symmetric media with spherical harmonic approximation is presented. The angle derivative is approximated by an orthonormal polynomial and this is represented by a matrix called curvature matrix, for a given beam of rays. An error analysis of the curvature matrix and results using the solution for few representative cases have been presented.

Key words: spherical symmetry—radiative transfer equation—spherical harmonic approximation

1. Introduction

Recently, several techniques for getting solution of radiative transfer in spherically symmetric media have appeared in the literature (see Hummer and Rybicki 1971, Peraiah and Grant 1973 and others). However, each technique has its own advantage and disadvantage. Moreover, these techniques have to be modified according to the given physical situation. In this respect, the discrete space theory (Peraiah and Grant 1973) is one method which needs the least modification. The average specific intensities in the "cell" are approximated by the "diamond" scheme which proved to be the best choice (Wiscombe 1976). The *transmission* and reflection characteristics of the medium are directly dependant on its physics. These are treated as operators subjected to numerical analysis. The main condition for these operators is that they should be non-negative. For this condition to be satisfied in spherical geometry, we have to select a small "cell" *critical* thickness because of the fact that the two curvature matrices are asymmetric and large in their absolute values. To avoid this difficulty, we have tried to obtain the curvature matrix by the spherical harmonic method.

2. Development of the Method

2.1 *Derivation of Transmission and Reflection Operators*

The equation of radiative transfer in spherical symmetry is written as,

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}\left\{r^2 I(r,\mu)\right\} + \frac{1}{r}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\, I(r,\mu)\right\} + K(r)\, I(r,\mu) = K(r)\left\{[1-\omega(r)]\, B(r) + \tfrac{1}{2}\,\omega(r)\int_{-1}^{+1} P(r,\mu,\mu')\, I(r,\mu')\, d\mu'\right\} \qquad (1)$$

where K (r) is the absorption coefficient and $K(r) \geq 0$ and the $\omega(r)$ is the albedo for single scattering $0 \leq \omega(r) \leq 1$. $I(r,\mu)$ is the monochromatic *specific intensity* of the ray making an angle $\cos^{-1}\mu$ with the radius vector at the radial point r. B (r) is the Planck function at r and $P(r,\mu,\mu')$ is the phase function and we shall choose phase function for isotropic scattering in these calculations. The angles are discretized such that $0 < \mu_1 < \mu_2 < \dots < \mu_m \leq 1$. We shall write,

$$I_j^+ (r) = I (r, \mu_j), I^-_j (r) = I (r, -\mu_j) \tag{2}$$

and

$$M = [\mu_i \delta_{ij}] \tag{3}$$

We have to approximate $\frac{\partial I}{\partial \mu}$ within this discretization of the angular coordinate μ. Let $P_m (\mu)$, $\mu_j^{(M)}$, $C_j^{(M)}$ be orthonormal polynomials, abscissae and weights associated with M^{th} order Gauss rule on [0, 1]. We shall expand $I (\mu)$ as,

$$I (\mu) = \sum_{m=0}^{M} a_m P_m (\mu) \tag{4}$$

and the (m+1) coefficients are chosen so that

$$I (\mu_j) = \sum_{m=0}^{M} a_m P_m (\mu_j), \; j = 0, 1, \ldots, M \tag{5}$$

Since, the $P_m (\mu)$ are assumed orthonormal

$$\sum_{j=0}^{M} P_k (\mu_j) P_m (\mu_j) C_j = \delta_{k,m} \tag{6}$$

From equation (5), we have

$$a_m = \sum_{j=0}^{M} P_m (\mu_{j'}) C_{j'} I (\mu_{j'}) \tag{7}$$

(see Abramowitz and Stegun, 1964).

From equation (4), we have,

$$I' (\mu) \approx \sum_{m=0}^{M} a_m P_m' (\mu)$$

where the primes represent the derivatives with respect to μ. Therefore from equations (5) and (7), we obtain,

$$I' (\mu_j) \approx \sum_{j'=0}^{M} \sum_{m=0}^{M-1} Pm' (\mu_j) Cj, Pm (\mu_{j'}) I (\mu_{j'})$$

or

$$\frac{dI}{d\mu} = D I$$

$$D_{jj'} = \sum_{m=0}^{M} Pm' (\mu_j) C_{j'} Pm (\mu_{j'}) \tag{8}$$

Consequently,

$$-\frac{\Lambda}{2} \frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) I^+ (r,\mu) \right\} \Big| \mu = \mu_j = \mu_j I^+ (r,\mu_j) - \tfrac{1}{2} (1 - \mu_j^2) \frac{d I^+ (r,\mu_j)}{d\mu} \tag{9}$$

We shall introduce a matrix operator Λ called the "curvature scattering matrix" given by

$$\Lambda = [\Lambda_{jj'}] \tag{10}$$

and

$$\Lambda_{jj'} = \mu_j \delta_{jj'} - \tfrac{1}{2} (1 - \mu_j^2) D_{jj'} \tag{11}$$

Now, equation (9) can be rewritten as,

$$-\tfrac{1}{2}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\, I^+(r,\mu)\right\}\Big|_{\mu=\mu_j} = \Sigma\, \Lambda_{jj'}\, I^+(r,\mu_{j'}) \tag{12}$$

or alternatively,

$$-\tfrac{1}{2}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\, I^+(r,\mu)\right\} \approx \sum_{m=0}^{M} a_n\left\{-\tfrac{1}{2}\frac{\partial}{\partial\mu}[(1-\mu^2)\, P_m(\mu)]\right\}$$

so that

$$\Lambda_{jj'} = \sum_{m=0}^{M} -\tfrac{1}{2}\frac{d}{d\mu}[(1-\mu^2)\, P_m(\mu)]\Big|_{\mu=\mu_j} C_{j'}\, P_m(\mu_{j'}) \tag{13}$$

For computational purposes, we shall use the expression (11) instead of (13).

The equation of radiative transfer equation (eq 1) for a given beam of radiation is given by

$$M\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 I^+(r)\right\} - \frac{2}{r}\Lambda I^+(r) + K(r)\, I^+(r) = K(r)\left\{[1-\omega(r)]\, B^+(r) + \tfrac{1}{2}\omega(r)\, [P^{++}(r)\, C\, I^+(r) + P^{+-}(r)\, C\, I^-(r)]\right\}$$

and for the oppositely directed beam,

$$-M\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 I^-(r)\right\} - \frac{2}{r}\Lambda I^-(r) + K(r)\, I^-(r) = K(r)\left\{[1-\omega(r)]\, B(r) + \tfrac{1}{2}\omega(r)\, [P^{-+}(r)\, C\, I^+(r) + P^{-}(r)\, C\, I^-(r)]\right\} \tag{14}$$

We shall define

$$U(r) = A(r)\, I(r)$$
$$A(r) = 4\pi r^2,$$
$$b\pm(r) = A(r)\, B\pm(r) \tag{15}$$

By introducing (15) into (14), we obtain,

$$M\frac{dU(r)}{dr} + K(r)\, U^+(r) = K(r)\, [1-\omega(r)]\, b^+(r) + [\tfrac{1}{2}K(r)\,\omega(r)\, P^{++} C + \frac{2}{r}\Lambda]\, U^+(r) + \tfrac{1}{2}K(r)\,\omega(r)\, P^{+-} C\, U^-(r)$$

and

$$-M\frac{dU^-(r)}{dr} + K(r)\, U^-(r) = K(r)\, [1-\omega(r)]\, b^+(r) + \tfrac{1}{2}K(r)\,\omega(r)\, P^{-+} C\, U^+(r) + [\tfrac{1}{2}K(r)\,\omega(r)\, P^{--} C - \frac{2}{r}\Lambda]\, U^-(r) \tag{16}$$

Discretization in the radial coordinate gives us,

$$M\, [U^+_{i+1} - U_i^+] + \tau_{i+\frac{1}{2}}\, U^+_{i+\frac{1}{2}} = \tau_{i+\frac{1}{2}}\, [(1-\omega_{i+\frac{1}{2}})\, b^+_{i+\frac{1}{2}} + Q^{++}_{i+\frac{1}{2}}\, U^+_{i+\frac{1}{2}} + Q^{+-}_{i+\frac{1}{2}}\, U^-_{i+\frac{1}{2}}] \tag{17}$$

and

$$M\, (U_i^- - U^-_{i+1}] + \tau_{i+\frac{1}{2}}\, U^-_{i+\frac{1}{2}} = \tau_{i+\frac{1}{2}}\, [(1-\omega_{i+\frac{1}{2}})\, b^-_{i+\frac{1}{2}} + Q^{-+}_{i+\frac{1}{2}}\, U^+_{i+\frac{1}{2}} + Q^{--}_{i+\frac{1}{2}}\, U^-_{i+\frac{1}{2}} \tag{18}$$

where

$$\tau_{i+\frac{1}{2}} = K_{i+\frac{1}{2}}\, (r_{i+1} - r_i) \tag{19}$$

$$
\begin{aligned}
Q^{++}_{i+\frac{1}{2}} &= \tfrac{1}{2}\,\omega_{i+\frac{1}{2}}\, P^{++}_{i+\frac{1}{2}}\, C + 2\,\frac{\rho}{\tau_{i+\frac{1}{2}}}\,\Lambda \\
Q^{--}_{i+\frac{1}{2}} &= \tfrac{1}{2}\,\omega_{i+\frac{1}{2}}\, P^{--}_{i+\frac{1}{2}}\, C - 2\,\frac{\rho}{\tau_{i+\frac{1}{2}}}\,\Lambda \\
Q^{+-}_{i+\frac{1}{2}} &= \tfrac{1}{2}\,\omega_{i+\frac{1}{2}}\, P^{+-}_{i+\frac{1}{2}}\, C \\
Q^{-+}_{i+\frac{1}{2}} &= \tfrac{1}{2}\,\omega_{i+\frac{1}{2}}\, P^{-+}_{i+\frac{1}{2}}\, C \\
U^{\pm}_{i+\frac{1}{2}} &= \tfrac{1}{2}\,(U^{\pm}_{i+1} + U^{\pm}_{i})
\end{aligned}
\tag{20}
$$

where ρ is the curvature factor given by

$$\rho = \Delta r/r$$

Substituting the last equation (20) in (17) and (18) and then writing these difference equations in the form of the interaction principle (see Peraiah and Grant 1973) would give us,

$$
\begin{bmatrix} M + \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{++}_{i+\frac{1}{2}}) & -\frac{1}{2}\tau_{i+\frac{1}{2}} Q^{+-}_{i+\frac{1}{2}} \\ -\frac{1}{2}\tau_{i+\frac{1}{2}} Q^{-+}_{i+\frac{1}{2}} & M + \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{--}_{i+\frac{1}{2}}) \end{bmatrix}
\begin{bmatrix} U^{+}_{i+1} \\ U^{-}_{i} \end{bmatrix}
+ \tau_{i+\frac{1}{2}}(1 - \omega_{i+\frac{1}{2}})
\begin{bmatrix} b^{+}_{i+\frac{1}{2}} \\ b_{i+\frac{1}{2}} \end{bmatrix} +
$$

$$
\begin{bmatrix} M - \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{++}_{i+\frac{1}{2}}) & \frac{1}{2}\tau_{i+\frac{1}{2}} Q^{+-}_{i+\frac{1}{2}} \\ \frac{1}{2}\tau_{i+\frac{1}{2}} Q^{-+}_{i+\frac{1}{2}} & M - \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{--}_{i+\frac{1}{2}}) \end{bmatrix}
\begin{bmatrix} U_{i}^{+} \\ U^{-}_{i+1} \end{bmatrix}
\tag{21}
$$

Here I is the identity matrix.

Therefore, the output intensities can be rewrttten in terms of the input intensities as,

$$
\begin{bmatrix} U^{+}_{i+1} \\ U^{-}_{i} \end{bmatrix}
= K^{-1}\tau_{i+\frac{1}{2}}(1 - \omega_{i+\frac{1}{2}})
\begin{bmatrix} b^{+}_{i+\frac{1}{2}} \\ b^{-}{}^{+}_{\frac{1}{2}} \end{bmatrix}
+ K^{-1}
\begin{bmatrix} M - \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{++}_{i+\frac{1}{2}}) & \frac{1}{2}\tau_{i+\frac{1}{2}} Q^{+-}_{i+\frac{1}{2}} \\ \frac{1}{2}\tau_{i+\frac{1}{2}} Q^{-+}_{i+\frac{1}{2}} & M - \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{--}_{i+\frac{1}{2}}) \end{bmatrix}
\begin{bmatrix} U^{+}_{i} \\ U^{-}_{i+1} \end{bmatrix}
\tag{22}
$$

where,

$$
K = \begin{bmatrix} M - \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{++}_{i+\frac{1}{2}}) & -\frac{1}{2}\tau_{i+\frac{1}{2}} Q^{+-}_{i+\frac{1}{2}} \\ -\frac{1}{2}\tau_{i+\frac{1}{2}} Q^{-+}_{i+\frac{1}{2}} & M + \frac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{--}_{i+\frac{1}{2}}) \end{bmatrix}
\tag{23}
$$

and

$$
K^{-1} = \begin{bmatrix} (I - r^{+-} r^{-+})^{-1}\Delta^{+} & (I - r^{+-} r^{-+})^{-1} r^{+-}\Delta^{-} \\ (I - r^{-+} r^{+-})^{-1} r^{-+}\Delta^{+} & (I - r^{-+} r^{+-})^{-1}\Delta^{-} \end{bmatrix}
\tag{24}
$$

where

$$
\begin{aligned}
\Delta^{+} &= [M + \tfrac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{++}_{i+\frac{1}{2}})]^{-1} \\
\Delta^{-} &= [M + \tfrac{1}{2}\tau_{i+\frac{1}{2}}(I - Q^{--}_{i+\frac{1}{2}})]^{-1} \\
r^{+-} &= \tfrac{1}{2}\tau_{i+\frac{1}{2}}\,\Delta^{+} Q^{+-}_{i+\frac{1}{2}} \\
r^{-+} &= \tfrac{1}{2}\tau_{i+\frac{1}{2}}\,\Delta^{-} Q^{-+}_{i+\frac{1}{2}}
\end{aligned}
\tag{25}
$$

compare equation (22) with the local interaction principle

$$\begin{bmatrix} U^{+}_{l+1} \\ U^{-}_{l} \end{bmatrix} = \begin{bmatrix} t(l+1,l) & r(l,l+1) \\ r(l+1,l) & t(l,l+1) \end{bmatrix} \begin{bmatrix} U^{+}_{l} \\ U^{-}_{l+1} \end{bmatrix} + \begin{bmatrix} \Sigma^{+}_{l+1} \\ \Sigma^{-}_{l+1} \end{bmatrix} \tag{26}$$

and setting

$$\begin{aligned} A^{++} &= M - \tfrac{1}{2}\tau_{l+1}(I - Q^{++}_{l+1}) \\ A^{--} &= M - \tfrac{1}{2}\tau_{l+1}(I - Q^{--}_{l+1}) \\ B^{+-} &= \tfrac{1}{2}\tau_{l+1} Q^{+-}_{l+1} \\ B^{-+} &= \tfrac{1}{2}\tau_{l+1} Q^{-+}_{l+1} \end{aligned} \tag{27}$$

We obtain the transmission and reflection operators in the basic "cell" and they are given by,

$$\begin{aligned} t(l+1,l) &= R^{+-}[\Delta^{+}A^{++} + r^{+-}\Delta^{-}B^{-+}] \\ t(l,l+1) &= R^{-+}[\Delta^{-}A^{--} + r^{-+}\Delta^{+}B^{+-}] \\ r(l+1,l) &= R^{-+}[\Delta^{-}B^{-+} + r^{-+}\Delta^{+}A^{++}] \\ r(l,l+1) &= R^{+-}[\Delta^{+}B^{+-} + r^{+-}\Delta^{-}A^{--}] \end{aligned} \tag{28}$$

And the source vectors are given by,

$$\begin{aligned} \Sigma^{+}_{l+1} &= \tau_{l+1}(1-\omega_{l+1}) R^{+-}[\Delta^{+}b^{+} + r^{+-}\Delta^{-}b^{-}] \\ \Sigma^{-}_{l+1} &= \tau_{l+1}(1-\omega_{l+1}) R^{-+}[\Delta^{-}b^{-} + r^{-+}\Delta^{+}b^{+}] \end{aligned} \tag{29}$$

where

$$\begin{aligned} R^{+-} &= [I - r^{+-}r^{-+}]^{-1} \\ R^{-+} &= [I - r^{-+}r^{+-}]^{-1} \end{aligned} \tag{30}$$

Equations (28) to (30) will give us the operators in a "cell".

2.2 *Accuracy of the Orthonormal Approximation*

In the previous sub-section, we have approximated,

$$-\tfrac{1}{2}\frac{\partial}{\partial\mu}\{(1-\mu^2)\, I^{\pm}(r,\mu)\}\Big|_{\mu=\mu_j} \approx \sum_{j=1}^{M} \Lambda_{jj'}\, I^{\pm}_{j'}(r),\; j'=1,\;\ldots\ldots,m$$

Let

$$f(\mu_j) = -\tfrac{1}{2}\frac{\partial}{\partial\mu}\{(1-\mu^2)\, I(\mu)\}\Big|_{\mu=\mu_j} \tag{32}$$

So, we must calculate the error E involved

$$E = \Big| f(\mu_j) - \sum_{j'} \Lambda_{jj'}\, I(\mu_{j'}) \Big| \tag{33}$$

Let us make use of a simple expansion for the specific intensity as some power of μ given by,

$$I(\mu) = \mu^{n}\;,\quad n=0,1,2,\ldots\ldots,N \tag{34}$$

Substituting (34) into (32) and (33) we obtain,

$$E = \frac{(n+2)\,\mu_j^{n+1} - n\mu_j^{n-1}}{2} - \sum_{j'} \Lambda_{jj'}\, \mu_{j'}^{n} \tag{35}$$

The $\Lambda_{jj'}$ are evaluated by using the zeros and weights of Gauss-Legendre quadrature formula over (0,1) for μ's and C's. We have considered j, j' = 1, . . . J, J = 8. The orthonormal shifted Legendre polynomials are calculated by the recurrence relation

$$P_{m-1}(\mu_j) = \frac{2m-3}{m-1}(2\mu_j - 1)\,P_{m-2}(\mu_j) - \frac{m-2}{m-1}P_{m-3}(\mu_j)\,, \quad m = 3,4,\ldots,M \tag{36}$$

with

$$P_0(\mu_j) = 1, \text{ and } P_1(\mu_j) = (2\mu_j - 1)$$

The orthonormal condition requires that

$$P_{m-1}(\mu_j) = \sqrt{2m-1}\,P_{m-1}(\mu_j) \tag{37}$$

One can easily write the derivatives $P'_{m-1}(\mu_j)$ from the relation (36) and (37) The polynomials and their derivatives for J 8 are given in tables 1 and 2. The Λ matrices have been calculated for 4 and 8 angles. These are given in Tables 3 and 6. By making use of these, we evaluate E in equation (35) and is giveu by

$$E = \left| f(\mu_j) - \sum_{j'} \Lambda_{jj'}\, \mu_{j'}^{n} \right| < 10^{-8} \text{ for } n \leq M-1$$

and for n $\geq$ M, E increases gradually. For n> 8, we have plotted the error in figure 1. For n<8, the results seem to be almost exact.

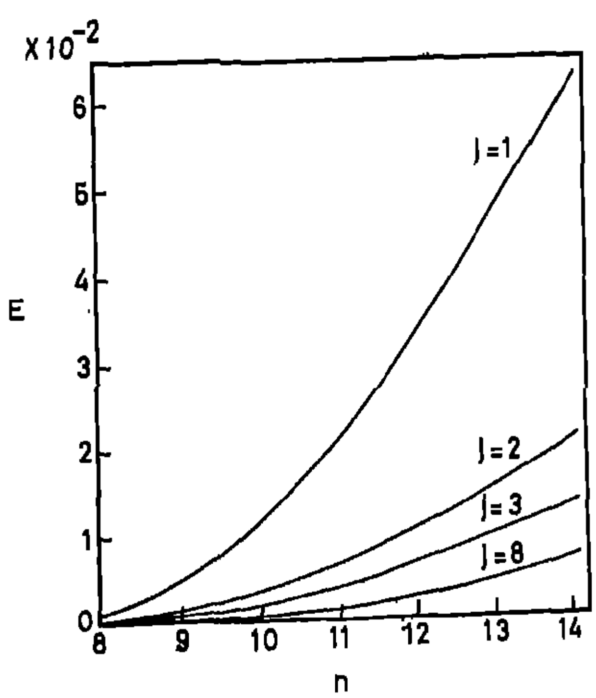

Fig. 1 The errors defined in equation (35) are plotted against n. For n=1 to 7, E=0 (10^-8).

Table 1. Orthonormal Polynomials for J=8

m \ j	1	2	3	4	5	6	7	8
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	−1.6632	−1.3798	−.9102	−.3177	.3177	.9102	1.3798	1.6632
2	1.9749	1.0107	−.1916	−1.0051	−1.0051	−.1916	1.0107	1.9749
3	−2.0462	−.1827	1.1256	.6871	−.6871	−1.1256	.1827	2.0462
4	1.9118	−.7281	−.9809	.7613	.7613	−.9809	−.7281	1.9118
5	−1.6014	1.3376	−.1028	−.9670	.9670	.1028	−1.3376	1.6014
6	1.1601	−1.3946	1.0902	−.4089	−.4089	1.0902	−1.3946	1.1604
7	−.6003	.8775	−1.0399	1.1175	−1.1175	1.0399	−.8775	.6003

Table 2. Derivatives of the orthonormal polynomials given in Table 1

m \ j	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0
1	3.4641	3.4641	3.4641	3.4641	3.4641	3.4641	3.4641	3.4641
2	−12.6836	−10.6884	−7.0507	−2.4610	2.4610	7.0507	10.6884	12.6836
3	26.6587	17.2507	3.0234	−6.6018	−6.6018	3.0234	17.2507	26.6597
4	−49.7684	−17.2407	8.4089	7.6064	−7.6064	−8.4088	17.2407	49.7684
5	73.9728	7.1346	−16.7300	6.8741	6.8741	−16.7300	7.1346	73.9728
6	−98.1147	11.2711	7.6463	−13.9859	13.9859	−7.6436	−11.2711	98.1147

Table 3. Curvature scattering matrices Λ_{ij}, for J=4

3.3863	−4.8367	2.0775	−0.6778
0.6750	0.8725	−1.3105	0.2929
−0.1811	0.8106	0.4661	−0.4174
0.0778	−0.2826	0.6514	0.4830

Table 4. Corrected Λ_{ij}, for J=4

2.2880	−4.2519	1.7946	−0.4959
0.0897	0.9644	−1.4619	0.3385
−0.7664	1.1224	0.3073	−0.3738
−1.0194	0.3020	0.3687	0.5658

Table 5. Correction matrix ϵ_{ij}, for J=4

1.0972	−0.5847	0.2827	−0.0818
0.5652	−0.3119	0.1508	−0.0436
0.5652	−0.3119	0.1508	−0.0436
1.0972	−0.5847	0.2827	−0.0818

Table 6. Matrix for J=8

12.3511	−19.6159	12.3422	−8.5814	5.8287	−3.6144	1.8268	0.5204
1.6837	2.2598	−6.1022	3.3620	−2.0973	1.2513	−0.6211	0.1764
−0.4043	2.0819	0.9224	−3.4272	1.6537	−0.8978	0.4269	0.1183
0.1608	−0.6542	1.9808	0.5664	−2.2714	0.8460	−0.4083	0.1092
−0.0852	0.3192	−0.7378	1.7714	−0.4683	−1.5290	0.5102	0.1284
0.0824	−0.1891	0.3978	−0.7328	1.5167	0.4591	−0.9226	0.1791
−0.0342	0.1211	−0.2440	0.4090	−0.6536	1.1900	0.4774	0.3873
0.0204	−0.0716	0.1420	−0.2292	0.3375	−0.4854	0.7716	0.4961

2.3 Conservation of flux and non negativity of transmission and reflection matrices.

Any medium which neither creates nor absorbs radiation, must conserve flux. We shall apply this condition (see Grant and Hunt 1969 a, b) and arrive at the identity that

$$\sum_{j=1}^{J} C_j \Lambda_{jk} = 0 \tag{38}$$

However, when we apply (38) to the Λ matrices given in Tables (3) and (6), we find that this identity is not satisfied. We shall, therefore, derive corrections to the curvature matrices so that they always satisfy the flux condition. Let

$$\sum_{j=1}^{J} C_j \Lambda_{jk} = \epsilon'_k \qquad (k = 1, \ldots, J) \tag{39}$$

or

$$\sum_{j=1}^{J} C_j \left(\Lambda_{jk} - \frac{\epsilon'_k}{C_j}\right) = 0 \tag{40}$$

Let

$$\epsilon_{jj'} = \frac{\epsilon_{j'}}{C_j}$$

and

$$\Lambda'_{jj'} = \Lambda_{jj'} - \epsilon_{jj'} \tag{41}$$

then, we have

$$\sum_{j=1}^{J} C_j \Lambda'_{jj'} = 0, \; j' = 1, \ldots J \tag{42}$$

The matrices ϵ' and Λ' are given in Tables 5 and 8 and 4 and 7 for J = 4 and 8 respectively.

To obtain non-negative tranmission and reflection operators, we must have the critical thickness in the "cell" given by (for positive Δ^+ and Δ^- matrices)

$$\tau_{crit} = \min_j \left| \frac{\mu_j \pm \varrho \Lambda'_{jj}}{\frac{1}{8}\left(-\frac{1}{2}\omega P^{++}_{jj} C_j\right)} \right| \tag{43}$$

Results for few simple cases have been presented in the next section.

3. Results

We have used the method to calculate the emergent intensities in two cases. We have used J = 4 and B/A = 1.5, 2 and 2.5 and ω = 1 and τ = 10 and 100 the total optical depth, where B and A are the outer and inner radii of the atmosphere. In figures 2 and 3, we have given the angular distribution of the emergent radiation I (0, μ) (= U (0, U) /$4\pi B^2$). We notice that as B/A increases, the emergent intensities reduce. This is nothing but the dilution due to extendedness and curvature scattering. When the optical depth is increased from 10 to 100, we notice a considerable fall in the emergent intensities which is quite obvious on the physical grounds.

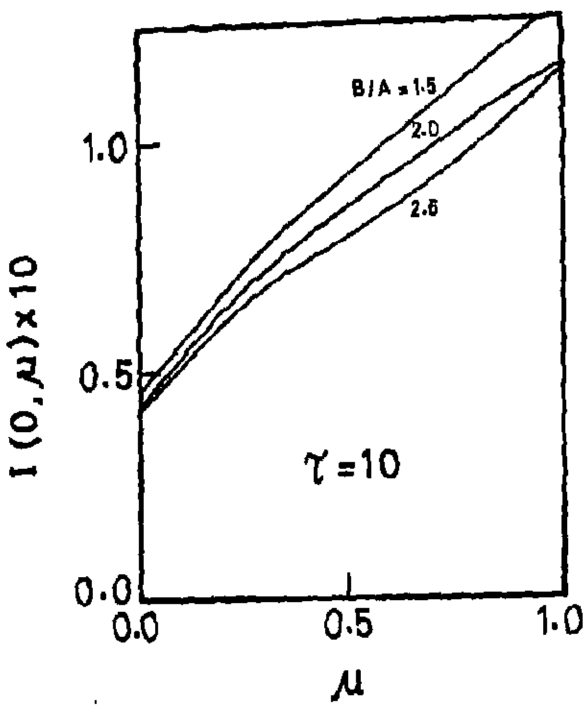

Fig. 2 Angular distribution of the emergent radiation for τ = 10.

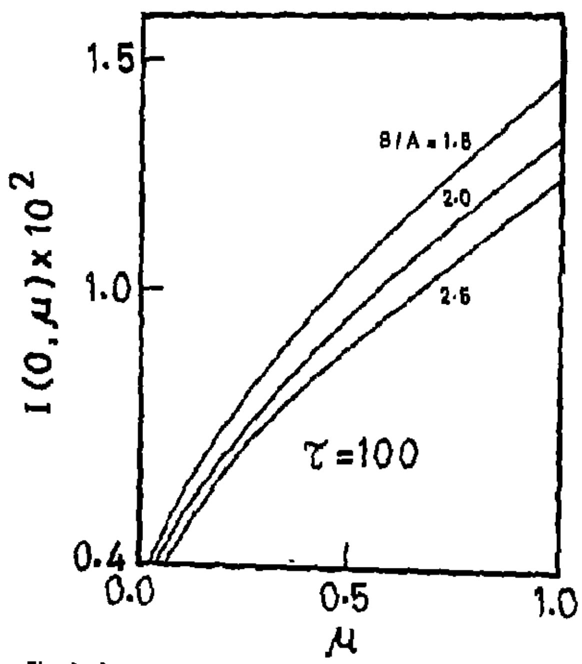

Fig. 3 Angular distribution of the emergent radiation for τ = 100.

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Table 7. Corrected JJ for J = 8

10.3865	− 18.3928	11.4667	−7.9490	5.3920	−3.3387	1.6886	−0.4809
0.9952	2 8166	−6.5006	3 6398	−2.2960	1.3764	−0.6840	0.1934
−1 0334	2.4767	0.6399	−3.2231	1.5128	−0.8098	0.3823	− 0.1066
−0.3842	0.3128	1.7163	0.7428	−2.3933	1.0221	−0.4479	0.1202
−0.6302	0.6606	−0.9821	1.8479	0.3464	1.4529	0.4716	−0.1144
−0.5777	0.2057	0.1153	−0.5287	1.3778	0 5471	−0 9671	0.1919
−0.9227	0.6778	−0.6424	0.6968	−0.8523	1.3141	0 4146	− 0.3493
−1.9322	1 1615	−0.7366	0.4032	−0.0092	−0.2127	0.6332	0 5346

Table 8. Correction matrix JJ for J = 8

1.9526	−1.2233	0.8765	−0.6324	0.4367	−0.2727	0.1383	−0 0396
0.8885	−0.5567	0 3984	−0.2878	0.1987	− 0.1241	0.0629	−0.0180
0.6301	0.3948	0.2825	−0.2041	0.1409	−0.0880	0 0446	− 0 0128
0.5450	− 0 3414	0.2443	−0.1765	0.1219	−0 0761	0.0386	−0 0110
0.5450	0 3414	0.2443	−0.1766	0.1219	−0 0761	0.0386	−0.0110
0.6301	−0.3948	0.2825	−0.2041	0.1409	−0 0880	0 0446	− 0.0128
0.8885	− 0.5567	0.3984	−0.2878	0.1987	− 0.1241	0.0629	−0.0180
1.9526	−1 2233	0.8765	−0.6324	0.4367	−0.2727	0.1383	−0.0396

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