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COMOVING FRAME CALCULATIONS WITH LORENTZ PROFILES IN RADIALLY EXPANDING MEDIA

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ABSTRACT

A computationally simple method for solving the line transfer in comoving frame has been described for radially expanding spherical medium. We have assumed a radially increasing velocity distribution and a non LTE two level atom. In discrete space theory, the comoving terms

$$\left\{ (1-\mu^2)\frac{v(r)}{r} + \mu^2 \frac{dv(r)}{dr} \right\} \frac{\partial I(r,\mu,\nu)}{\partial \nu}$$

(where V (r) is the velocity at the radial point r in mean thermal units, $\cos^{-1}\mu$ is the angle made by the ray with specific intensity $I(r,\mu,\nu)$ of frequency ν, with the radius vector at r) will reduce to a single tridiagonal matrix. The boundary conditions for the frequency derivative can be introduced through the elements of the first and the last rows. The method is quite stable for any velocity law and any type of inhomogeneous media. We have considered the Lorentz profiles for various maximum velocities V_{max} = 0, 10, 30 and 60. Line profiles transformed into the observer's line of sight have been presented.

Key Words : comoving frame—Lorentz profiles—expanding media

1. Introduction

It is a well known fact that the matter in the outer layers of Wolf-Rayet stars, quasars, novae, P Cygni stars is flowing radially outwards with high velocities. There have been several attempts to compute spectral lines in such media. Kunasz and Hummer (1974), Peraiah and Wehrse (1978) and others have attempted at solving line transfer in expanding spherically symmetric media. However, all these techniques use the star's rest frame and one cannot consider beyond 2 or 3 mean thermal units of the gas velocity as the frequency-angle mesh becomes unmanageably large. The alternative approach to computing lines in rapidly expanding media is to work in comoving frame. In this frame the observer moves with gas and, therefore, notices no Doppler shifts which directly effect the absorption coeffiient and one can use the profile functions corresponding to those of static media. This will be quite helpful particularly when we consider angle dependent partial frequency redistribution. Simmoneau (1973) worked on the problem of comoving frame but it has restricted utility considering the few velocity laws for which the method is valid. Recently, Mihalas *et al.* (1975) developed solution of line transfer in comoving frame. This method seems to be quite versatile in many respects. However, one has to make adjustments when one tries to use different kinds of velocity laws.

We shall present a method of calculating the solution of line transfer in comoving frame which can accept arbitrary velocity laws in spherical medium. Here, we shall assume Lorentz profiles mainly to find out how photons diffuse into the outer layers of the stellar atmospheres in motion though the line wings. We shall develop the solution on the lines of Grant and Peraiah (1972) by invoking the flux conservation condition in a conservative case, for the integration of the comoving terms.

2. Solution of Line Transfer in the Comoving Frame

We shall consider the formation of spectral line with Non-LTE two level atom approximation, in a spherical medium expanding with velocity

$$V(r) = V_A + \frac{V_B - V_A}{B - A}(r - A) \tag{1}$$

where V_A, V_B and V (r) are the velocities at the inner radius A, outer radius B and at the radial point r, respectively in units of mean thermal velocity of the gas. We shall measure the line frequency points in terms of some standard frequency interval Δs by defining the quantity

$$X_i = \frac{\nu_i - \nu_0}{\Delta s} \tag{2}$$

where ν_0 is the central frequency of the line. We shall select the Lorentz profile function given by

$$\phi(x) = \frac{1}{\pi}\frac{1}{1+X^2} \tag{3}$$

which is normalized such that

$$\int_{-\infty}^{+\infty} \phi(x)\, dx = 1 \tag{4}$$

We shall consider a beam of radiation with intensities I (r, μ_j, x) where μ_j's are the angles made by the rays with the radius vector at the radial point r. We express this in the form of a vector

$$\begin{bmatrix} I(r, \mu_1, x) \\ I(r, \mu_2, x) \\ \vdots \\ I(r, \mu_J, x) \end{bmatrix} \tag{5}$$

where $\mu_1, \mu_2 \cdots \mu_J$ are the roots and weights of a polynomial suitable for evaluating the integrals. Similarly, an oppositely directed beam will have the intensity vector

$$\begin{bmatrix} I(r, -\mu_1, x) \\ I(r, -\mu_2, x) \\ \vdots \\ I(r, -\mu_J, x) \end{bmatrix} \tag{6}$$

J being the total number of rays in the beam. Similarly, if we are considering a polychromatic problem, then for each frequency we shall have to attribute a beam of radiation with intensities given by

$$\begin{bmatrix} I(r, \mu_1, x_1) \\ I(r, \mu_2, x_1) \\ \vdots \\ I(r, \mu_J, x_1) \\ \vdots \\ I(r, \mu_j, x_i) \\ I(r, \mu_{j+1}, x_i) \\ \vdots \\ I(r, \mu_J, x_I) \end{bmatrix} \tag{7}$$

Therefore, the total number of rays will be JxI where I is the total number of frequency points. In polychromatic problems, we have to consider the scattering integral

$$\frac{1}{2}\int_{-1}^{+1}\int_{-\infty}^{+\infty} \phi(x)\, I(x, \mu, r)\, d\mu\, dx \tag{8}$$

because this will contribute photons differently into different frequencies. Now, write the equation of transfer in spherical symmetry for a two-level atom in comoving frame as

$$\mu \frac{\partial I(x, \mu, r)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(x, \mu, r)}{\partial \mu} = -k_L(r)\, I(x, \mu, r)\left(\beta + \phi(x)\right)$$

$$+ \frac{1}{2}(1-\epsilon)\,\phi(x)\int_{-1}^{+1}\int_{-\infty}^{+\infty} \phi(x)\, I(x, \mu, r)\, dx\, d\mu + \phi(x)\,\epsilon B(r) + \beta\phi(x)\,\rho(r)\, B(r)$$

$$+ \left[(1-\mu^2)\frac{V(r)}{r} + \mu^2\frac{dv(r)}{dr}\right]\frac{\partial I(x, \mu, r)}{\partial x} \tag{9}$$

and for the oppositely directed beam of radiation,

$$-\mu\frac{\partial I(x, -\mu, r)}{\partial r} - \frac{1-\mu^2}{r}\frac{\partial I(x, -\mu, r)}{\partial \mu} = -k_L(r)\, I(x, -\mu, r)\left(\beta + \phi(x)\right)$$

$$+ \frac{1}{2}(1-\epsilon)\,\phi(x)\int_{-1}^{+1}\int_{-\infty}^{+\infty} \phi(r)\, I(x, -\mu, r)\, dx,\, d\mu + \phi(x)\,\epsilon B(r) + \beta\,\phi(x)\,\rho(r)\, B(r) \tag{10}$$

$$+ \left[(1-\mu^2)\frac{V(r)}{r} + \mu^2\frac{dv(r)}{dr}\right]\frac{\partial I(x, -\mu, r)}{\partial x}$$

Here $K_L(r)$ is the absorption coefficient given by

$$K_L(r) = \frac{h\nu_0}{4\pi\Delta}\left(N_1 B_{12} - N_2 B_{21}\right) \tag{11}$$

N_1 and N_2 being the population densities of levels 1 and 2 and B's are the Einstien coefficients. We shall specify the absorption coefficient in terms of the optical depth τ. The quantity β is the ratio K_c/K_L of obsorption per unit frequency interval, in the continuum to that in the centre of the line. B (r) is the Planck function at the radial point r. The quantity ϵ is the probability per scatter that a photon is destroyed by collisional de-excitation. We shall specify ϵ, B, β and ρ (r) variation of Planck function, will be specified in advance. Equations (9) and (10) can be solved by integration as described in Grant and Peraiah (1972). The integrals are replaced by the appropriate quadrature formulae and the differentials by the weighted differences. For example we write for frequency discretization

$$\int_{-\infty}^{+\infty} \phi(x) f(x)\, dx \simeq \sum_{i=-I}^{I} a_i f(x_i), \quad \sum_{i=-I} a_i = 1 \tag{12}$$

$$\text{and} \quad a_i = \phi_i A_i \Big/ \sum_{i=-I}^{I} A_i \phi_i \tag{13}$$

where X_i and A_i are the roots and weights of a suitable quadrature formula on $[-1, +1]$. Similarly, the angle discretization is done by writing

$$\int_0^1 f(\mu)\, d\mu \simeq \sum_{j=1}^{m} b_j f(\mu_j), \quad \sum_{j=1}^{m} b_j = 1 \tag{14}$$

μ's and b's being the abscissae and weights of a quadrature formula on [0, 1]. We have to solve a large system of equations corresponding to J number of angles and 2I number of frequency points. We shall define

$$b = [b_j \delta_{ij}], \quad M_m = [\mu_j \delta_{ij}]$$

and

$$u^{+}_{i,n} = 4\pi r^2_n \begin{bmatrix} I(\tau_n, \mu_1, x_i) \\ I(\tau_n, \mu_2, x_i) \\ \vdots \\ I(\tau_n, \mu_j, x_i) \end{bmatrix} \tag{15}$$

and

$$U_n = [U^{\pm}_{1,n}, U^{\pm}_{2,n}, U^{\pm}_{3,n}, \ldots\ldots U^{\pm}_{i,n} \ldots\ldots U^{\pm}_{I,n}]^T \tag{16}$$

T represents the transpose.

$$\text{and} \qquad B'(r) = 4\pi r_n^2 B(r) \tag{17}$$

B (r) being the Planck function at r.

With these definitions, we shall integrate equations (9) and (10) and write their corresponding discrete equivalents as,

$$M[U^{+}_{n+1} - U^{+}_{n}] + \rho_c [\Lambda^{+} U^{+}_{n+\frac{1}{2}} + \Lambda^{-} U^{-}_{n+\frac{1}{2}}] + \tau_{n+\frac{1}{2}} \Phi^{+}_{n+\frac{1}{2}} U^{+}_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}} S^{+}_{n+\frac{1}{2}} + \tfrac{1}{2}(1-\epsilon)\, \tau_{n+\frac{1}{2}} [\Phi \Phi^T W] [U^{+} + U^{-}]_{n+\frac{1}{2}} + M_1\, d\, U^{+}_{n+\frac{1}{2}} \tag{18}$$

and

$$M\,[U^-_n - U^-_{n+1}] - \rho_c\,[\Lambda^+ U^-_{n+\frac{1}{2}} + \Lambda^- U^+_{n+\frac{1}{2}}] + \tau_{n+\frac{1}{2}}\,\Phi^-_{n+\frac{1}{2}}\,U^-_{n+\frac{1}{2}} = \tau_{n+\frac{1}{2}}\,S^-_{n+\frac{1}{2}} + \tfrac{1}{2}\,(1-\epsilon)$$

$$\tau_{n+\frac{1}{2}}\,[\Phi\,\Phi^T\,W]\,[U^+ + U^-]_{n+\frac{1}{2}} + M_l\,d\,U^-_{n+\frac{1}{2}} \qquad (19)$$

Here the quantities with the subscript $n+\frac{1}{2}$ reptesents the average value of the quantity over the cell bounded by radii r_n and r_{n+1}. And define

$$\overset{\pm}{\Phi}_{n+\frac{1}{2}} = \left[\Phi_{kk'}\right]_{n+\frac{1}{2}} = \left[\beta + \overset{\pm}{\phi}_k\right]_{n+\frac{1}{2}}^{\delta_{kk'}}$$

$$k = j + (i-1)\,J,\ 1 \leqslant K \leqslant K = IJ$$

$$\overset{\pm}{\phi}_k = \phi\,(x_i, \pm\mu_j, r_{n+\frac{1}{2}}) \qquad (21)$$

$$\overset{\pm}{S}_{n+\frac{1}{2}} = \left(\rho\,\beta + \epsilon\,\overset{\pm}{\phi}_k\right) B'_{n+\frac{1}{2}}\,\delta_{kk'}$$

$$\phi_i W_k = a_i\,C_j$$

with a_i being defined as in equation (13), and C's are the weights of the angle quadrature, and

$$M = [M_m\,\delta_{ij}]$$

$$M_m = [\mu_j\,\delta_{ij}]$$

$$\overset{\pm}{\Lambda} = [\overset{\pm}{\Lambda}_m\,\delta_{ij}]$$

where $\overset{\pm}{\Lambda}_m$ are the curvature matrices defined in Peraiah (1978a).

The curvature factor ρ_c is defined as

$$\rho_c = \Delta r / r_{n+\frac{1}{2}} \qquad (22)$$

and

$$\tau_{n+\frac{1}{2}} = K_{L,\,n+\frac{1}{2}}\,\Delta r \qquad (23)$$

Δr being the geometrical thickness of the cell.

The last terms on the R. H. S. in equations (18) and (19) are the discretized equivalents of the comoving terms appearing in the equations (9) and (10). The quantities M_l and d are given by

$$M_l = [M^1\,\Delta V_{n+\frac{1}{2}} + M^2\,\rho_c\,V_{n+\frac{1}{2}}] \qquad (24)$$

$$M^1 = \begin{bmatrix} M^1_m & & & \\ & M^1_m & & \\ & & \ddots & \\ & & & M^1_m \end{bmatrix} \qquad (25)$$

$$M^2 = \begin{bmatrix} M^2_m & & & \\ & M^2_m & & \\ & & \ddots & \\ & & & M^2_m \end{bmatrix} \qquad (26)$$

$$M^1{}_m = [\mu_j^2 \delta_{jk}] \qquad j, k = 1, 2, \quad J$$

$$M^2{}_m = [(1 - \mu_j^2) \delta_{jk}] \tag{27}$$

$$\Delta V_{n+\frac{1}{2}} = (V_{n+1} - V_n) \tag{28}$$

and $V_{n+\frac{1}{2}}$ is the average velocity of the shell bounded by radii r_n and r_{n+1}. The matrix **d** is determined (see Appendix I) from the condition of flux conservation and is given by

$$d = \begin{bmatrix} -d_1 & d_1 & & & & \\ -d_2 & 0 & d_2 & & & \\ & -d_3 & 0 & d_3 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot \quad \cdot & d_{I-1} \\ & & & & \cdot & \cdot \\ & & & & -d_I & d_I \end{bmatrix} \tag{29}$$

where

$$d_i = (X_{i+1} - X_{i-1})^{-1} \text{ for } i = 2, 3, \ldots, I-1$$

$$d_1 = (X_2 - X_1)^{-1} \text{ and } d_I = (X_I - X_{I-1})^{-1} \tag{30}$$

We shall discuss the boundary conditions for frequency and radiation field in the next section.

The average intensities $U_{n+\frac{1}{2}}$ are approximated by the diamond scheme given by (see Wiscombe 1976)

$$(I - t_{n+\frac{1}{2}}) U^+{}_n + t_{n+\frac{1}{2}} U^+{}_{n+1} = U^+{}_{n+\frac{1}{2}}$$

$$(I - t_{n+\frac{1}{2}}) U^-{}_{n+1} + t_{n+\frac{1}{2}} U^-{}_n = U^-{}_{n+\frac{1}{2}} \tag{31}$$

Wiscombe has shown that diamond scheme given by

$$t_{n+\frac{1}{2}} = \tfrac{1}{2} I \tag{32}$$

gives the best approximation.

On substituting equations (31) and (32) into equations (18) and (19) and by comparing these with the canonical equations of the interaction principle (Eq. 2.24 of Grant and Peraiah 1972), we obtain the expressions for the reflection and transmission operators for the "cell". These are given in Appendix II.

We shall now turn our attention to study the step size problems. Consider the matrices Δ^+ and Δ^- we must have these matrices with non-negative elements. This requires that the diagonal elements of $(\Delta\pm)^{-1}$ should be positive and the off-diagonal elements should be negative. We shall make use of these conditions to obtain the critical optical depth for the "cell". Therefore, for the diagonal elements, we must have

$$\tau_{kk} < \left| \frac{2\mu_k \pm \rho_c \Lambda^+{}_{kk} - d_{kk} \{\mu_k^2 \Delta V + (1 - \mu_k^2) \rho_c V_{n+\frac{1}{2}}\}}{(\beta + \phi_k) - \frac{1}{2}(1 - \epsilon)(\phi\phi^T w)_{kk}} \right| \tag{33}$$

and in the case of off-diagonal elements, we have

$$\tau_{k,\,k+1} < \left| \frac{2\rho_c \Lambda^+{}_{k,\,k+1} - 2d_{k,\,k+1} \{\mu^2{}_{k,\,k+1} \Delta V + (1 - \mu^2{}_{k,\,k+1}) \rho_c V_{n+\frac{1}{2}}\}}{(1 - \epsilon)(\phi\phi^T w)_{k,\,k+1}} \right|$$

for the upper diagonal elements and

$$\tau_{k+1,k} < \left| \frac{2\rho_e \Lambda^{+}_{k+1,k} + 2d_{k+1} \{\mu^2_{k+k}\Delta V + (1-\mu^2_{k+1k})\ \rho_0 V_{n+k}\}}{(1-\epsilon)\ (\phi\phi^{TW})_{k+1,k}} \right| \tag{35}$$

for lower diagonal elements.

$$\tau_{crit} = \min \{\tau_{kk},\ \tau_{k,k+1},\ \tau_{k+1,k}\} \tag{36}$$

Therefore, τ_{crit} is given by the equations (33), (34) and (35),

It is interesting to notice that the step size is determined not only by the angles μ's, curvature factor and curvature matrices, but also by the elements of the frequency derivative matrix d and the velocity of the medium. The computational aspects are discussed in the next section.

3. Computational Procedure and Discussion of the Results

By calculating the r and t operators (see Appendix II) in each "cell" subjected to the restriction (36), one can calculate the internal radiation field (see Peraiah 1978a) in the comoving frame. We have to transform this radiation field into those of star's rest frame and observer's frame at earth so that a comparison with observations can be made. In this paper, we have presented results in the observer's frame. The procedure is described in Peraiah (1978b) (see also Fig. 14.7 of Mihalas 1978). We shall now discuss the angle-frequency mesh to be employed in comoving frame calculations. Since in a comoving frame of reference there are no relative velocities between the observer and the moving gas, one can employ the profile function corresponding to that of a static medium. In such a situation, we can afford to employ a considerably small number of angle-frequency points, whereas in the star's rest frame calculations, we need to employ a large number of frequency-angle points We have tested the program with several sets of frequency points, the total number of frequency points being 9, 11, 13, 15 and 19 and odd number of frequency points are chosen so that the centre of the line is always included at x = 0 by choosing the trapizoidal equally spaced points. The frequency independent source function (S_9 for 9 points and S_{11} etc.) for various sets of frequency points have been compared. Very interestingly, only S_9 differs from others in 4th place and all others agree within 6th and 8th places. One must caution, however, that for substantially large velocities of the order of 90 or 100 mean thermal units the differences would occur even in 4th place for S_{13} to S_{19}. This again can be corrected either by increasing the number of frequency points or reducing the step-size τ_{crit} and correspondingly the velocity in each "cell" and then employing the doubling algorithm (Peraiah 1978). We have selected four angle points and we have used a Lorentz profile function given in equation (3).

We have treated two different physical situations : (1) $\epsilon = 10^{-4}$, $\beta = 0$ and (2) $\epsilon = \beta = 10^{-4}$, the first case represents media with line emission and the second case represents media with both line and continuum emission. The total optical depth is 10^4.

The problem is constrained to two kinds of boundary conditions given by :

1) the radiation incident on either side of the atmosphere and

2) the frequency derivative $\frac{\partial I}{\partial x}$ appearing in the comoving terms.

As there is already emission in the medium either line or line plus continuum, we have not given any external radiation incident on either side of the medium to see how the medium affects line formation. In this case, we have the boundary condition for the radiation field given by

$$\begin{aligned} U^{-}_{N+1}\ (x_i,\ \tau = T,\ \mu_j) &= 0 \\ U_1^{+}\ \ (x_i,\ \tau = 0,\ \mu_j) &= 0 \end{aligned} \tag{37}$$

And in the case of the frequency derivative, we shall set that in the continuum $\frac{\partial I}{\partial x} = 0$, or,

$$\left[d\ U^{\pm}_{n+1} \right] = 0 \quad \text{for } i = 1 \text{ and } I \tag{38}$$

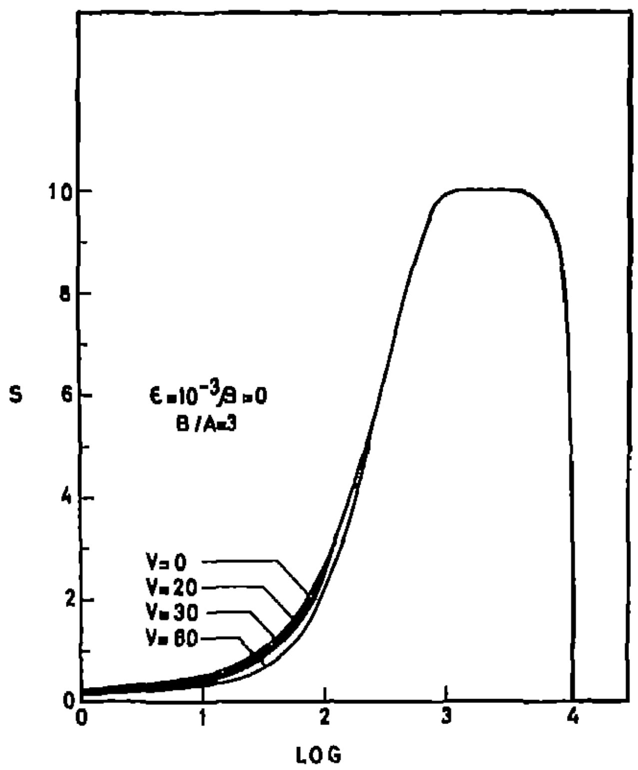

Fig. 1 Frequency integrated source functions are plotted with respect to optical depth for $\epsilon = 10^{-3}$, $\beta = 0$ and B/A = 3.

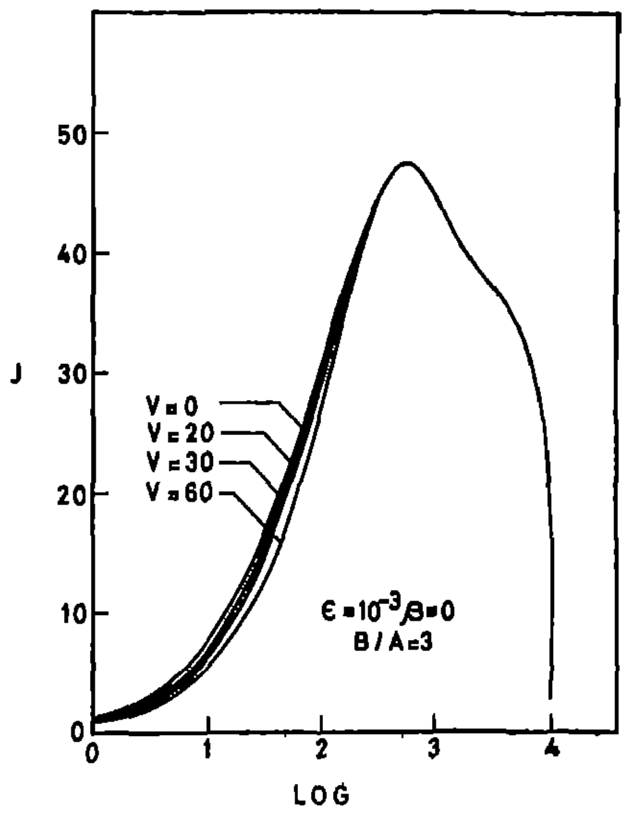

Fig. 2. Frequency integrated mean intensities are given for $\epsilon = 10^{-3}$, $\beta = 0$ and B/A = 3.

Equations (37) and (38) will specify our problem. In addition to these boundary conditions, we have to choose a velocity law. This is given in equation (1). We have set v (R) = 0, 20, 30 and 60, and the ratio of outer to inner radii B/A equal to 3 and 9 and a linear velocity law has been used given by equation (1).

We have presented the results in Figures 1 – 6. The frequency integrated source functions

$$S = \Sigma A_i \Sigma S(x_i, \mu_j, \tau_r) C_i$$

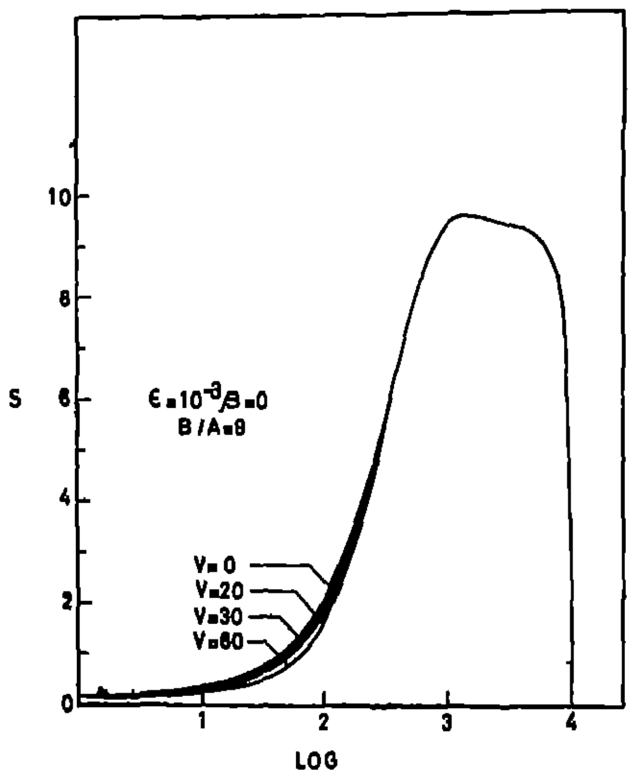

Fig. 3. Same as in Fig. 1 with B/A = 9.

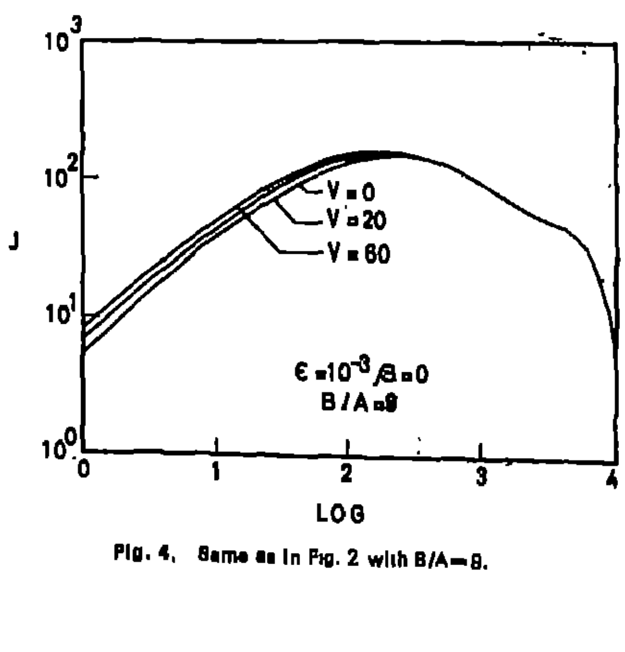

Fig. 4. Same as in Fig. 2 with B/A = 9.

are plotted in Fig. 1 for $\epsilon = 10^{-3}$, $\beta = 0$ and $B/A = 3$. We have used $V_a = 0, 10, 20, 30, 60$. The corresponding total mean intensities are plotted in Fig. 2. As there is no incident radiation on either side of the medium, we notice sharp drops at τ 0 and $\tau = \tau_{max}$. Both the source functions and mean intensities reach maximum inside the medium as there is emission in the line. S and J for B/A = 9 show similar tendencies. Line profiles at the observer's point, corresponding to the source functions given in Fig. 1 and 3 are presented in Fig. 5 and 6. Here Q(X) = Flux (X) / Flux (Xmax) One notices purely emission lines and these are very similar to those observed in quasars (see Baldwin 1975, Baldwin and Netzer 1978).

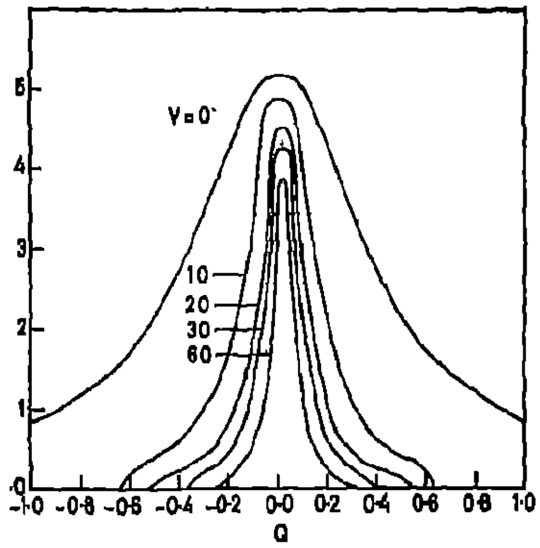

Fig. 5. Line profiles at observer's point corresponding to the source functions given in Fig. 1. Here Q=Flux (X)/Flux (Xmax)

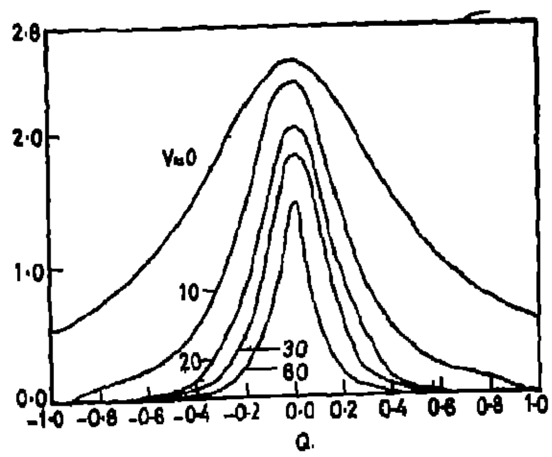

Fig. 6. Same as in Fig. 5, but with source functions given in Fig. 3.

4. Conclusions

A stable solution for the line transfer in comoving frame has been presented. This method can be used for any kind of arbitrary velocity law and physical inhomogeneities.

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Appendix I

In a purely scattering media, radiation is neither created nor destroyed. This condition can be used to calculate the d matrix of the frequency derivative appearing in equations (18) and (19). We follow Grant and Hunt (1969a, b) in deriving the nature of d matrix. In terms of the infinitesimal generators, we obtain the r and t operators as,

$$t(n+1,n) \quad I-\tau\gamma^{++} = I-\tau M^{-1}\left[\Phi - \tfrac{1}{2}\sigma(\phi\phi^{T}w) + \frac{\rho_0\Lambda^{+}}{\tau} - \frac{M_1 d}{\tau}\right] + O(\tau) \tag{A1}$$

$$r(n+1,n) = \tau\gamma^{-+} = \tau M^{-1}\left[\tfrac{1}{2}\sigma(\phi\phi^{T}w) + \frac{\rho_0\Lambda^{--}}{\tau}\right] + O(\tau)$$

where $\sigma = 1-\epsilon$

For a conservative case, we must have,

$$\| t(p+1,p) + r(p+1,p) \| = 1 + O(\tau) \tag{A2}$$

where $\| . \|$ is the radiometric flux norm defined (see Grant and Hunt 1969b).

$$\| A \| = \operatorname*{Max}_{k'} \sum_{k} \left| DAD^{-1} \right|_{kk'} \tag{A3}$$

where $D = 2\pi Mw$. From (A1), (A2) and (A3), we obtain for the flux norm,

$$\| t(n+1,n) + r(n+1,n) \| =$$

$$\operatorname*{Max}_{k'} \sum_{k=1}^{K} \left| 2\pi Mw \left\{ I - \tau M^{-1}\left[\Phi - \sigma(\phi\phi^{T}w) + \frac{\rho_0}{\tau}(\Lambda^{+} - \Lambda^{-}) - \frac{M_1 d}{\tau}\right\} (2\pi Mw)^{-1} \right|_{kk'} \tag{A4}$$

For a conservative case $\sigma = 1$, $\Phi = \phi$. Furthermore the normalization of the line profile is given by

$$\int_{-\infty}^{+\infty} \phi(x)\, dx = 1 \tag{A5}$$

or in its discrete form, we have

$$\Sigma\, \phi_k w_k = 1 \tag{A6}$$

The curvature factors $\Lambda\pm$ satisfy the inequality (Peraiah and Grant 1973)

$$\sum_{j=1}^{J} C_j\,(\Lambda^+_{ji} - \Lambda^-_{ji}) = 0,\ i = 1, 2, \ldots, J \tag{A7}$$

This identity satisfies for every frequency point in the line. By making use of (A6) and (A7), the relation (A4) is simplified into,

$$\| t\,(p+1, p) + r\,(p+1, p) \| = 1 + \Sigma d_i a_i - 0(\tau) \tag{A8}$$

where $a_i\,a_j = \phi_i w_i$

For the conservation then, we must have

$$\Sigma d_i a_i = 0 \tag{A9}$$

or the weighted column sums of the d matrix should identically be equal to zero.

Appendix II

$$\begin{aligned} t\,(n+1, n) &= a^{+-}\,[\Delta^+\,\Gamma^+ + \beta^{+-}\,\beta^{-+}] \\ t\,(n, n+1) &= a^{-+}\,[\Delta^-\,\Gamma^- + \beta^{-+}\,\beta^{+-}] \\ r\,(n+1, n) &= a^{-+}\beta^{-+}\,[I + \Delta^+\Gamma^+] \\ r\,(n, n+1) &= a^{+-}\beta^{+-}\,[I + \Delta^-\Gamma^-] \end{aligned} \tag{33}$$

and the cell source vectors are given by

$$\begin{aligned} \Sigma^{+}_{n+1} &= \tau a^{+-}\,[\Delta^+ S^+ + \beta^{+-}\Delta^- S^-] \\ \Sigma^{-}_{n+1} &= \tau a^{-+}\,[\Delta^- S^- + \beta^{-+}\Delta^+ S^+] \end{aligned} \tag{34}$$

where,

$$a^{+-} = [I - \beta^{+-}\beta^{-+}]^{-1},\ \beta^{+-} = \tfrac{1}{2}\Delta^+ Y_- \tag{35}$$

(similarly a^{-+} and β^{-+} are defined)

$$\Delta^{\pm} = [M + \tfrac{1}{2}\tau Z_\pm]^{-1},\ \Gamma^{\pm} = [M - \tfrac{1}{2}\tau Z\pm]$$

$$Z = \Phi - Y,\ Z_+ = Z + \frac{\rho_c \Lambda^+}{\tau} - \frac{M_1 d}{\tau}$$

$$Z_- = Z - \frac{\rho_c \Lambda^+}{\tau} - \frac{M_1 d}{\tau}$$

$$Y = \tfrac{1}{2}\,(1 - t)\,(\phi\,\phi^T w),\ Y_+ = Y + \frac{\rho_c \Lambda^-}{\tau},\ Y_- = Y - \frac{\rho_c \Lambda^-}{\tau}$$

$\Phi = \Phi\pm$ as we are considering a static profile function.