Indian Journal of Pure & Applied Physics
Vol. 19, March, 1981, pp. 262-268

Holographic Cylindrical Transmission Grating in Divergent Illumination : Part I—Grating Rulings Perpendicular to Axis of Cylindrical Surface

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Received 27 *February* 1980

The theory of a cylindrical transmission grating prepared holographically has been developed. The condition and recording parameters for minimum aberration have been derived. The aberrations of the spectral images have been discussed in detail.

1. Introduction

Mechanically ruled cylindrical gratings with circular grooves have been considered theoretically by Singh and Majumdar.[1] They have described two types of gratings: (i) in which the rulings are parallel and (ii) in which the rulings are perpendicular to the axis of the cylindrical surface. They found that the second type of grating behaves as a plane grating.

In the present paper, we have considered the second type of cylindrical grating prepared holographically.

2. Theoretical Considerations

Let GHIJ represent a cylindrical grating shown in Fig. 1. The centre of the grating rulings 'O' is assumed to be the origin of the coordinate system, the axis of the cylindrical surface being C′ K. Let OZ, which is perpendicular to C′ K, be the Z-axis and the normal at O, i.e. OC′ X, be the X-axis and the Y-axis is shown in Fig. 1. Thus $OC' = R$ is the radius of the cylinder. Let us consider any point P on the grating given by the coordinate (u, w, l) and A (x, y, z) and B (x', y', z') be a point source and a focal point respectively.

For the present case, the equation of the cylindrical surface may be written as

$$u^2 + l^2 = 2\,uR \qquad \ldots(1)$$

Therefore, u can be written as

$$u = \frac{l^2}{2R} + \frac{l^4}{8R^3} + \frac{l^6}{16\,R^5} + \cdots \qquad \ldots(2)$$

Let us assume C $(x_C, y_C, 0)$ and D $(x_D, y_D, 0)$ as the recording sources. We further assume the distances OC and OD to be integral multiples of λ_0, the wavelength of the recording laser light and that the zeroth groove passes through O. Thus the nth groove is formed according to Noda *et al.*[2] at a distance given by

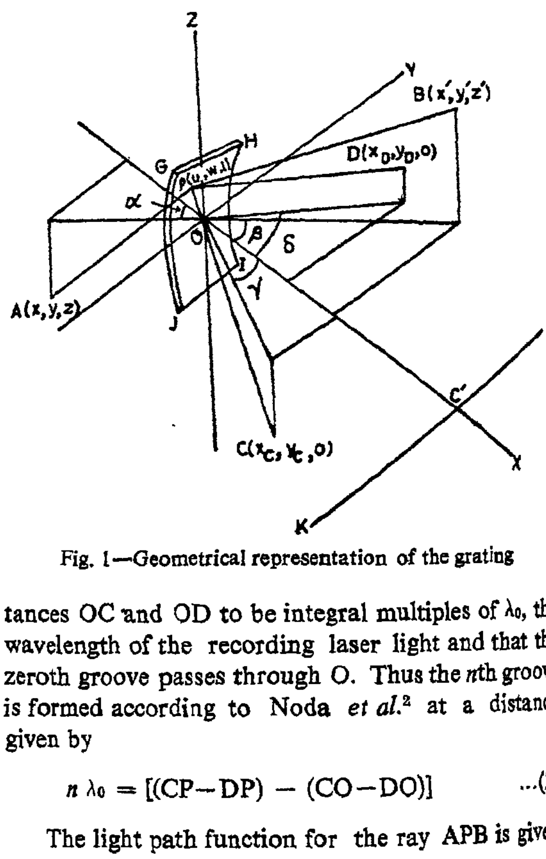

Fig. 1—Geometrical representation of the grating

$$n\,\lambda_0 = [(CP-DP) - (CO-DO)] \qquad \ldots(3)$$

The light path function for the ray APB is given by

$$F = (AP) + (PB) + \frac{m\,\lambda}{\lambda_0}\Big[(CP-DP)-(CO-DO)\Big] \qquad \ldots(4)$$

where m = order of the spectra, λ = wavelength of light source.

$$(AP)^2 = (-x-u)^2 + (y-w)^2 + (z-l)^2$$
$$(PB)^2 = (x'-u)^2 + (y'-w)^2 + (z'-l)^2$$
$$(CP)^2 = (x_C-u)^2 + (y_C-w)^2 + l^2$$
$$(DP)^2 = (x_D-u)^2 + (y_D-w)^2 + l^2$$
$$(CO)^2 = (x_C^2 + y_C^2)$$
$$(DO)^2 = (x_D^2 + y_D^2)$$

We take for the cylindrical coordinates

$$x = r\cos\alpha \qquad y = r\sin\alpha$$
$$x = r'\cos\beta \qquad y = r'\sin\beta$$
$$x_C = r_C\cos\gamma \qquad x_D = r_D\cos\delta$$
$$y_C = r_C\sin\gamma \qquad y_D = r_D\sin\delta$$

where α, β are respectively, the angles of incidence and diffraction and γ and δ are the angles of the recording sources, all measured in the xy plane and are positive when measured anticlockwise from the grating normal towards the respective ray. The signs of α and β are opposite if points A and B lie on different sides of the xz plane. The same kind of sign rule holds good for γ and δ. The signs of α and β should be consistent with the signs of γ and δ.

By using binomial theorem and retaining up to 5th order terms in the approximate expressions for AP, PB, CP, DP, CO and DO and substituting these expressions in Eq. (3) through the use of Eq. (2), we get

$$\begin{aligned}
F = {} & r\left(1+\frac{z^2}{r^2}\right)^{1/2} + r'\left(1+\frac{z'^2}{r'^2}\right)^{1/2} \\
& - w\left[\left\{\sin\alpha\left(1+\frac{z^2}{r^2}\right)^{-1/2} + \sin\beta\left(1+\frac{z'^2}{r'^2}\right)^{-1/2}\right\}\right. \\
& \left. + \frac{m\lambda}{\lambda_0}\left(\sin\gamma - \sin\delta\right)\right] \\
& + \frac{w^2}{2}\left[\left(\frac{\cos^2\alpha}{r} + \frac{\cos^2\beta}{r'}\right) + \frac{m\lambda}{\lambda_0}\left(\frac{\cos^2\gamma}{r_C} - \frac{\cos^2\delta}{r_D}\right)\right] \\
& + \frac{l^2}{2}\left[\left(\frac{1}{r}+\frac{\cos\alpha}{R}\right) + \left(\frac{1}{r'} - \frac{\cos\beta}{R}\right) + \frac{m\lambda}{\lambda_0}\left\{\left(\frac{1}{r_C} - \frac{\cos\gamma}{R}\right) - \left(\frac{1}{r_D} - \frac{\cos\delta}{R}\right)\right\}\right] \\
& - l\left[\frac{z}{r}\left(1+\frac{z^2}{r^2}\right)^{-1/2} + \frac{z'}{r'}\left(1+\frac{z'^2}{r'^2}\right)^{-1/2}\right] \\
& + \frac{l^2 w}{2}\left[\frac{\sin\alpha}{r}\left(\frac{1}{r}+\frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{r'}\left(\frac{1}{r'} - \frac{\cos\beta}{R}\right)\right. \\
& \left. + \frac{m\lambda}{\lambda_0}\left\{\left(\frac{\sin\gamma}{r_C^2} - \frac{\sin\gamma\cos\gamma}{r_C R}\right) - \left(\frac{\sin\delta}{r_D^2} - \frac{\sin\delta\cos\delta}{r_D R}\right)\right\}\right] \\
& + \frac{w^2 l^2}{2}\left[\frac{\sin^2\alpha}{r^2}\left(\frac{1}{r}+\frac{\cos\alpha}{R}\right) + \frac{\sin^2\beta}{r'^2}\left(\frac{1}{r'} - \frac{\cos\beta}{R}\right)\right. \\
& \left. + \frac{m\lambda}{\lambda_0}\left\{\frac{\sin^2\gamma}{r_C^2}\left(\frac{1}{r_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin^2\delta}{r_D^2}\left(\frac{1}{r_D} - \frac{\cos\delta}{R}\right)\right\}\right] \\
& - w\,l\left[\frac{z}{r^2}\sin\alpha + \frac{z'}{r'^2}\sin\beta\right] \\
& - l\,w^2\left[\frac{z\sin^2\alpha}{r^3} + \frac{z'}{r'^3}\sin^2\beta\right] \\
& - \frac{w^4}{8}\left[\frac{\cos^4\alpha}{r^3} + \frac{\cos^4\beta}{r'^3}\right] \\
& - \frac{m\lambda\, w^4}{\lambda_0\ 8}\left[\frac{\cos^4\gamma}{r_C^3} - \frac{\cos^4\delta}{r_D^3}\right] \\
& + \frac{w^3}{2}\left[\left(\frac{\cos^2\alpha\sin\alpha}{r^2} + \frac{\cos^2\beta\sin\beta}{r'^2}\right)\right. \\
& \left. - \frac{m\lambda}{\lambda_0}\left(\frac{\cos^2\delta\sin\delta}{r_D^2} - \frac{\cos^2\gamma\sin\gamma}{r_C^2}\right)\right] \\
& + \dots\dots\dots\dots
\end{aligned} \qquad \dots(5)$$

By applying Fermat's principle for the perfect image, we obtain the following conditions:

$$\left(1+\frac{z^2}{r^2}\right)^{-1/2}\left(\sin\alpha + \sin\beta_0\right) = \frac{m\lambda}{\sigma_0} \qquad \dots(6)$$

$$\frac{z}{r} = -\frac{z_0'}{r_0'} \qquad \dots(7)$$

$$\frac{\cos^2\alpha}{r} + \frac{\cos^2\beta}{r'} + \frac{m\lambda}{\lambda_0}\left(\frac{\cos^2\gamma}{r_C} - \frac{\cos^2\delta}{r_D}\right) = 0 \qquad \dots(8)$$

$$\left(\frac{1}{r}+\frac{\cos\alpha}{R}\right) + \left(\frac{1}{r'} - \frac{\cos\beta}{R}\right) + \frac{m\lambda}{\lambda_0}\left\{\left(\frac{1}{r_C} - \frac{\cos\gamma}{R}\right) - \left(\frac{1}{r_D} - \frac{\cos\delta}{R}\right)\right\} = 0 \qquad \dots(9)$$

Let us take

$$\sigma_0 = \frac{\lambda_0}{\sin\delta - \sin\gamma} \qquad \dots(10)$$

$$R_e = \frac{\sin\delta - \sin\gamma}{(\cos^2\delta/r_D) - (\cos^2\gamma/r_C)} \quad ...(11)$$

$$\rho_0 = \frac{\sin\delta - \sin\gamma}{\left(\frac{1}{r_C} - \frac{\cos\gamma}{R} - \frac{1}{r_D} + \frac{\cos\delta}{R}\right)} \quad ...(12)$$

and substituting in Eqs. (8) and (9) we get

$$\frac{\cos^2\alpha}{r} - \frac{\cos^2\beta}{r'} - \frac{(\sin\alpha + \sin\beta)}{R_e} = 0 \quad ...(13)$$

$$\left(\frac{1}{r} + \frac{\cos\alpha}{R} + \frac{1}{r'} - \frac{\cos\beta}{R}\right) + \frac{(\sin\alpha + \sin\beta)}{\rho_0} = 0 \quad ...(14)$$

It is noted that the expressions for grating [Eq. (6)], magnification [Eq. (7)], and the horizontal focussing condition [Eq. (13)] are the same as those for a plane dffraction grating prepared holographically with variable spacing; however, Eq. (14) is different. The possible solutions of Eq. (13) for the finite value of r, are,

$$r = \frac{R_e\cos^2\alpha}{\sin\alpha}$$

$$r' = \frac{R_e\cos^2\beta}{\sin\beta} \quad ...(15)$$

and for $r = \infty$

$$r' = \frac{R_e\cos^2\beta}{(\sin\alpha + \sin\beta)} \quad (16)$$

It is noted from Eq. (15) that if the source is placed at a finite distance from the centre of the grating, the curves drawn for source point and focal point are represented by the same Eq. (15). In the case $r = \infty$, the focal curves are given by Eq. (16). In this paper we will discuss the first case only.

It can be seen from Eq. (13) that if R_e is infinite, the equation which we get is the same as that for an ordinary ruled plane diffraction grating with constant spacing and straight grooves. It has been shown[1,3] that circular grooves (curved grooves) can reduce the aberrations of the grating.

3. Aberrations

The amount of aberration in an image in a plane located at a distance r_0' from O and perpendicular to the diffracted principal ray can be easily computed from Gauss-Siedel theory, say up to O $(1/w^4)$ and the displacements $\Delta\beta$ and $\Delta z'$ in the horizontal and vertical directions respectively, from the position (r_0', β_0, z_0') specified by Eqs. (6) and (7) can be computed under the usual assumption $z \ll r, r_0'$.

3.1 Astigmatism and Calculation of Recording Parameters

Let L be the total length of a groove projected on the z-axis. From Eq. (14), we get for the length of astigmatic images due to a point source,

$$\left[z'\right]_{ast} = L\left[1 + \frac{\cos^2\beta.\sin\alpha}{\sin\beta.\cos^2\alpha} + \frac{R_e\cos^2\beta}{R\sin\beta}\left(\cos\alpha - \cos\beta\right) + \frac{R_e\cos^2\beta}{\rho_0\sin\beta}\left(\sin\alpha + \sin\beta\right)\right] \quad ...(17)$$

Eq. (17) reduces to the equation of a plane grating with straight grooves by taking $\rho_0 = R = \infty$. Let us take

$$f_1 = \frac{(R/r_D) - (R/r_C) + \cos\gamma - \cos\delta}{\sin\delta - \sin\gamma} \quad ...(18)$$

$$D_1 = \frac{\tan\alpha\sec\alpha + \tan\beta\sec\beta + \cos\alpha - \cos\beta}{(\sin\alpha + \sin\beta)} \quad ...(19)$$

$$R = R_e$$

Then Eq. (17) reduces to

$$\left[z'\right]_{ast} = \frac{L\cos^2\beta\,(\sin\alpha + \sin\beta)}{\sin\beta} \times \left[D_1(\alpha,\beta) - f_1(r_D, r_C, \delta, \gamma)\right] \quad ...(20)$$

It is evident from the Eq. (20) that if $D_1 = f_1$, the astigmatism will be zero and since f_1 depends on the recording parameters (r_C, r_D, γ and δ), one can make the astigmatism zero by choosing the appropriate values of these recording parameters.

In general, the recording parameters r_C and r_D for assumed values of γ and δ, for the case when the astigmatism is zero are given by

$$R/r_D = [(R/R_e)(\sin\delta - \sin\gamma) + \cos^2\gamma\,\{(\cos\gamma - \cos\delta) + R/\rho_0\,(\sin\delta - \sin\gamma)\}]/(\cos^2\delta - \cos^2\gamma) \quad ...(21)$$

$$\frac{R}{r_C} = \frac{R}{r_D} + (\cos\gamma - \cos\delta) + \frac{R}{\rho_0}(\sin\delta - \sin\gamma) \quad ...(22)$$

If we assume the value of $f_1(r_C, r_D, \gamma, \delta) = 2{\cdot}25$ which gives zero astigmatism at the value of $D_1(\alpha, \beta) = 2{\cdot}25$, this value of $D_1(\alpha, \beta)$ can be obtained at different sets of (α, β), i.e. wavelengths. By taking $\lambda_0 = 0{\cdot}4579\ \mu$m, the wavelength of the laser for recording the grating, $\sigma_0 = 0{\cdot}9158\ \mu$m, $\gamma = 0°$,

$\delta = 30°$, we arrive at $R/r_D = 1{\cdot}964$ and $R/r_C = 0{\cdot}973$ from Eqs. (21) and (22).

In Fig. 2, we have plotted the function $D_1(\alpha, \beta)$ at different wavelengths. The dotted lines represent the wavelengths at different angles of incidence and diffraction at $\sigma_0 = 1$ μm. Thus a proper value of $D_1(\alpha, \beta)$ to get zero astigmatism at a particular wavelength $\lambda(\alpha, \beta, \sigma_0)$ can be selected from Fig. 2. In Fig. 3, we have plotted the function $f_1(r_C, r_D, \gamma, \delta)$ at $r_C = 2{\cdot}0184\, r_D$ and different values of δ and γ. The values of f_1 are shown by dotted lines. The solid curves have been drawn for different values of $\sigma_0 = 0{\cdot}45$, $0{\cdot}5$, $0{\cdot}7$ and $1{\cdot}53$ μm respectively at different angles of δ and γ, for $\lambda_0 = 0{\cdot}4579$ μm. Fig. 3 will be very useful for selecting a different set of recording parameters for zero astigmatism and which can reduce the other aberrations also at the same time.

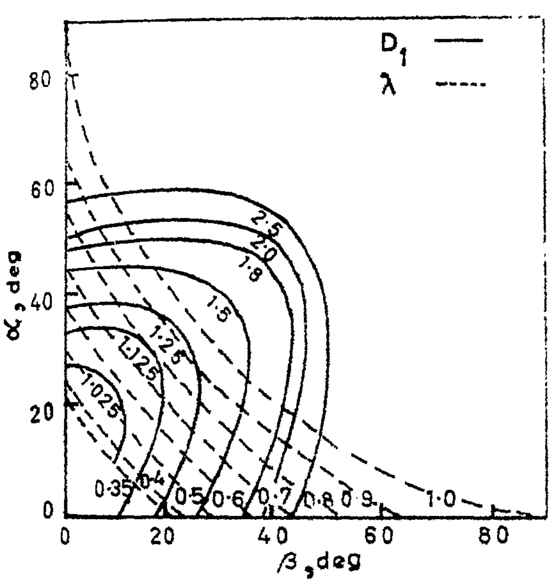

Fig. 2—$D_1(\alpha, \beta)$ (—) against β and $\lambda(\alpha, \beta)$ (- - - -) against β for different values of α

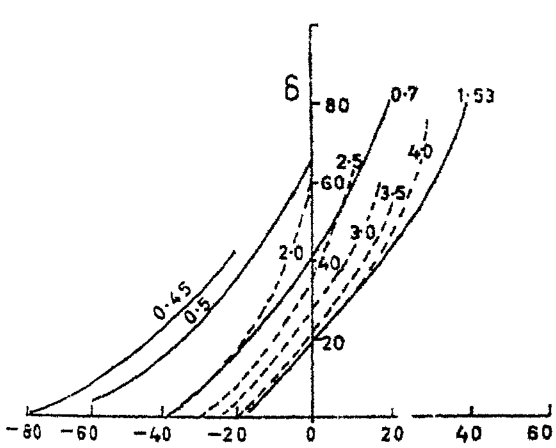

Fig. 3—$f_1(r_D, r_C, \delta, \gamma)$ (— — —) and $\sigma_0(\delta, \gamma)$ (—) for different values of δ versus γ values

The values of astigmatism per unit groove length $[z']_{ast}/L$ at different angles of incidence and diffraction are shown in Fig. 4, for the assumed recording parameters as given earlier. This graph shows that the astigmatism is zero only at one wavelength in each case.

3.2 Coma

For a point source, the coma is given, to a first approximation, by

$$\Delta p_c = \frac{l^2}{2R_e}\frac{\sqrt{3+\operatorname{cosec}^2\beta}}{\sin\beta}\left[\left\{\frac{\sin^3\alpha}{\cos^4\alpha} + \frac{\sin^3\beta}{\cos^4\beta} + \frac{R_e}{R}\left(\frac{\sin^2\alpha}{\cos^3\alpha} - \frac{\sin^2\beta}{\cos^3\beta}\right)\right\} + R_e^2\left(\frac{\sin\alpha+\sin\beta}{\sin\delta-\sin\gamma}\right)\left(\frac{\sin\gamma}{r_C^2} - \frac{\sin\delta}{r_D^2} - \frac{\sin\gamma\cos\gamma}{r_C R} + \frac{\sin\delta\cos\delta}{r_D R}\right)\right] \quad \ldots(23)$$

If we assume $R_e = R$, and rearrange the terms we get from Eq. (23),

$$\Delta p_c = \frac{l^2}{2R}\frac{\sqrt{3+\operatorname{cosec}^2\beta}\,(\sin\alpha+\sin\beta)}{\sin\beta} \times [D_2(\alpha,\beta) - f_2(r_C, r_D, \delta, \gamma)] \quad \ldots(24)$$

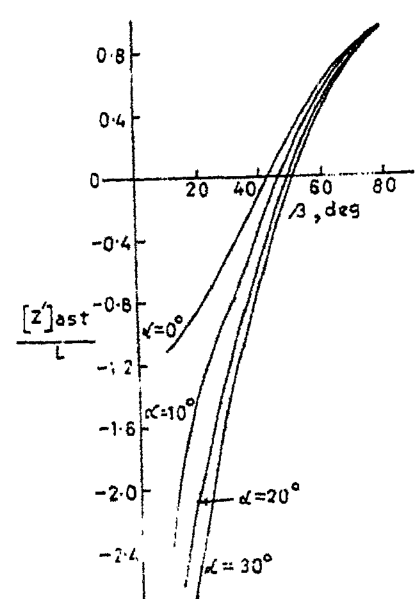

Fig. 4—$[Z']_{ast}/L$ versus β for different values of α

where

$$D_2(\alpha,\beta) = [\tan^3\alpha\sec\alpha+\tan^3\beta\sec\beta+\tan^2\alpha\sec\alpha - \tan^2\beta\sec\beta]/(\sin\alpha+\sin\beta)$$

$$f_2(r_C, r_D, \delta, \gamma) = [R^2(\sin\delta/r_D^2-\sin\gamma/r_C^2)-\{(R/r_D)\sin\delta\cos\delta - (R/r_C)\sin\gamma\cos\gamma\}]/(\sin\delta-\sin\gamma)$$

For elimination of coma, we have to choose the recording parameters such that

$$D_2(\alpha,\beta) = f_2(r_D, r_C, \delta, \gamma) \quad \ldots(25)$$

By calculating the values of $D_2(\alpha,\beta)$ for different sets of α and β, i. e. for different wavelengths, we can determine the values of $f_2(r_D, r_C, \delta, \gamma)$. These plots of $D_2(\alpha,\beta)$ versus α and β are given in Fig. 5. Similarly f_2 at different δ and γ values at $r_C = 2{\cdot}0184\ r_D$, is shown in Fig. 6. The dotted curves in Fig. 6 represent σ_0 for different δ and γ at $\lambda_0 = 0{\cdot}4579\ \mu$m. The solid curves represent variation of f_2 for different δ and γ ($r_C = 2{\cdot}0184\ r_D$). In Fig. 5, solid curves are drawn for different values of $D_2(\alpha,\beta)$ and dotted curves for different values of $\lambda(\alpha,\beta)$ for $\sigma_0 = 1\ \mu$m.

With the help of Figs. 5 and 6, one can find out the required recording parameters for elimination of coma at particular wavelengths. In Fig. 7, we have represented $\Delta p_c/(l^2/2R)$ at different angles of incidence and diffraction for the recording parameters given earlier in the text. From Fig. 7 we conclude that for these recording parameters, the grating is most suitable in the wavelength range corresponding to small values of α and $\beta = 0$ to $\approx 35°$, where the coma is zero at two wavelengths and reduced at others.

For elimination of astigmatism and coma simultaneously, one has to make use of the Figs. 2, 3, 5 and 6 for deciding the proper recording parameters for a particular wavelength.

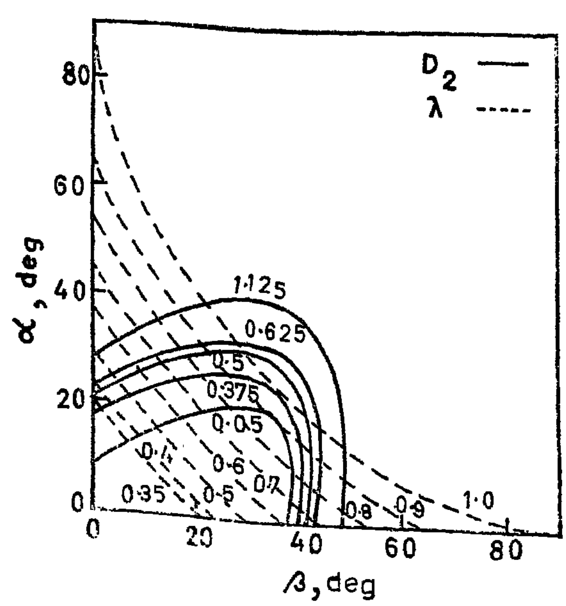

Fig. 5—$D_2(\alpha,\beta)$ (—) and $\lambda(\alpha,\beta)$ [(— — —) against β for different values of incidence α

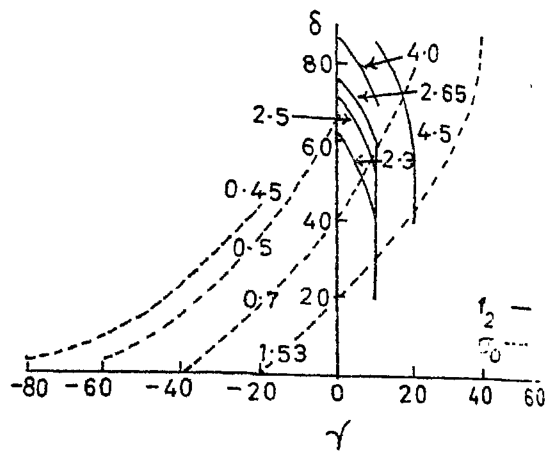

Fig. 6—$f_2(r_D, r_C, \delta, \gamma)$ (—) and σ_0 (— — —) for different values of δ versus angle γ

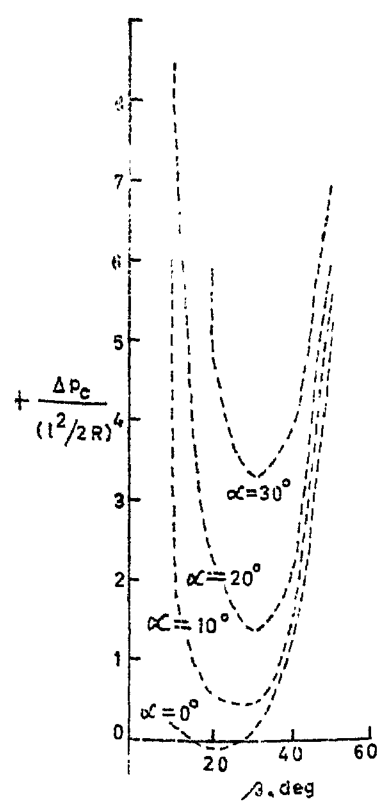

Fig. 7—Variation of $\Delta p_c/(l^2/2R)$ with β at different values of angle of incidence α

4. Secondary Focal Curves

The equation for secondary focal curve is given by

$$r_h^1 = -R \Big/ \left\{ \frac{R}{R_e} \frac{\sin\alpha}{\cos^2\alpha} + (\cos\alpha - \cos\beta) + \frac{R}{\rho_0}\left(\sin\alpha + \sin\beta\right)\right\} \quad ..(26)$$

We see from Fig. 8, that these secondary focal curves (drawn at $R/R_e = 1$) cut the primary focal curve only at one wavelength, i.e. zero astigmatism is achieved only at one wavelength. In Fig. 8, the dotted curves are the secondary focal curves at different angles of incidence.

5. Optimum Grating Width and Resolution

The optimum grating width is given by

$$W_{opt} = \left(\frac{\sigma_0 R^2}{2m}\right)^{1/3} \times \left[\frac{\sin\alpha + \sin\beta}{\tan^2\alpha \sin\alpha + \tan^2\beta \sin\beta - B(\sin\alpha + \sin\beta)}\right]^{1/3} \quad ...(27)$$

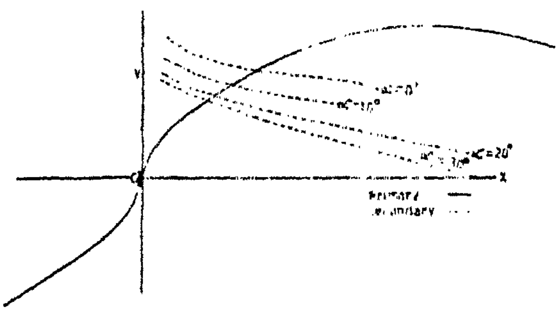

Fig. 8—Secondary focal curves (--- ---) for different values of α and primary focal curve (- -) at $R=10$

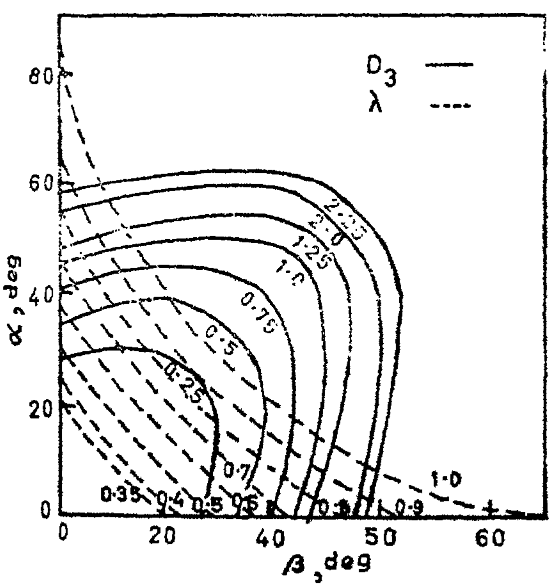

Fig. 9—$D_3(\alpha, \beta)$ (—) and $\lambda(\alpha, \beta)$ (— — —) against for different values of angle of incidence α

Let us assume $R_e = R$ and rearranging the terms we get

$$W_{opt} = \left(\frac{\sigma_0 R^2}{2m}\right)^{1/3} \left[\frac{1}{D_3(\alpha, \beta) - B}\right]^{1/3} \quad ...(28)$$

where

$$D_3(\alpha, \beta) = \frac{\tan^2\alpha \sin\alpha + \tan^2\beta \sin\beta}{\sin\alpha + \sin\beta} \quad ...(29)$$

$$B = R^2\left(\frac{\sin\delta \cos^2\delta}{r_D^2} - \frac{\sin\gamma \cos^2\gamma}{r_C^2}\right) \Big/ \left(\sin\delta - \sin\gamma\right) \quad ...(30)$$

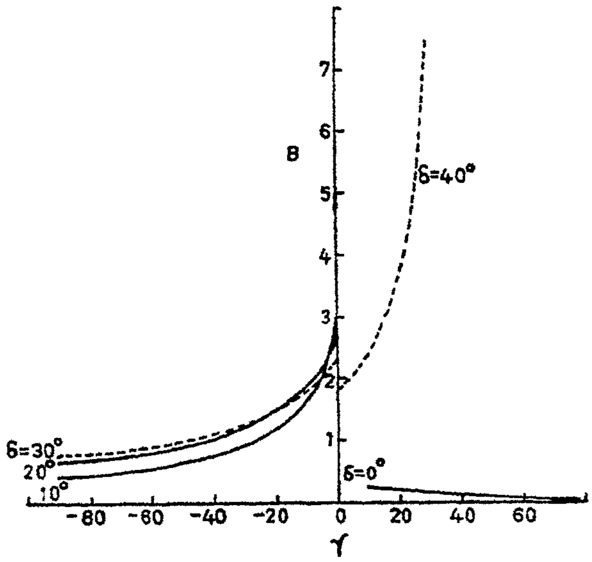

Fig. 10—$B(r_C, r_D, \delta, \gamma)$ against γ for different values of δ

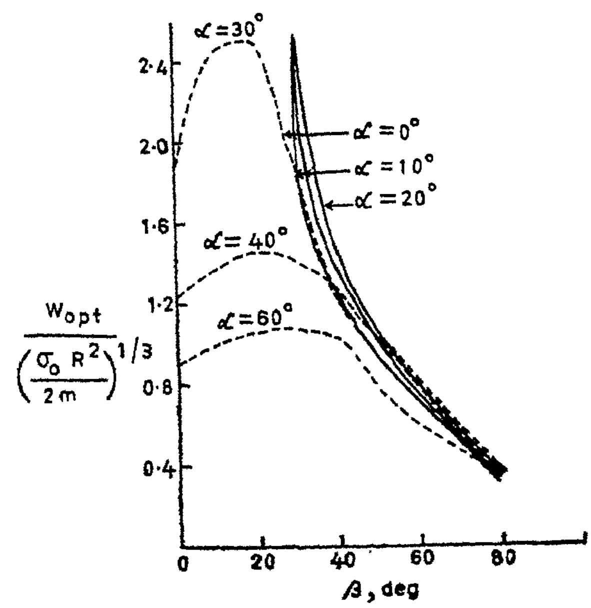

Fig. 11—Variation of $W_{opt}/(\sigma_0 R^2/2m)^{1/3}$ with β at different values of angle of incidence α

It is clear from the above result that by proper selection of the recording parameters, the resolving power, viz. 0·95 $(m/\sigma_0)\, w_{opt}$, of the grating can be considerably increased. For maximum resolution, one should have

$$D_3\,(\alpha,\,\beta) = B \qquad \ldots(31)$$

In Fig. 9, we have plotted $D_3\,(\alpha,\,\beta)$ values (solid curve) at different wavelengths. Fig. 10 shows the variation of $B\,(r_C,\,r_D,\,\delta,\,\gamma)$ with γ for different values of δ for $r_C = 2{\cdot}0184\, r_D$. With the help of Figs. 9 and 10 one can find recording parameters for maximum resolution at a given wavelength. Fig. 11, represents $w_{opt}/(\sigma_0 R^2/2\, m)^{1/3}$ at different angles of incidence and diffraction. It is evident from Fig. 11 that for the recording parameters given in the text earlier one can have better resolution only in the wavelength range corresponding to $\alpha = 0$ to $\approx 35°$ and $\beta = 0$ to $\approx 35°$.

6. Conclusion

It is clear from this study that a holographically recorded cylindrical transmission grating can be usefully exploited for spectroscopic purposes. In a way similar to those given above one can find out the various relations for type II mounting [obtained by using Eq. (16)] and hence the required design parameters for a particular problem.

Acknowledgement

One of the authors (SS) thankfully acknowledges the financial support from the University Grants Commission, New Delhi, and is thankful to Dr M K V Bappu, Director of Indian Institute of Astrophysics, Bangalore, for giving permission to carry out this work with the co-author Mahipal Singh.

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