Scattering of Light by Rough Surfaces: High and Low Roughness Approximations

S Chatterjee and Ravinder Kumar Banyal

Indian Institute of Astrophysics,
Bangalore -560034, India
Email:chat@iiap.res.in

Abstract. Scattering of light by rough surface is considered in the Kirchoff approximation. Analytical expressions are presented for the scattered intensity by considering the elevations in the z-direction, $\zeta(x, y)$ at any point (x, y) on the surface to be a zero mean, correlated Gaussian, stationary random variable, such that $< \zeta(x,y) > = 0$ and $< \zeta(x_1,y_1) \zeta(x_2,y_2) > = \sigma^2 g(r)$, where $r = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$ with $g(0) = 1$ and $g(r) \to 0$ for $r \gg l$. In the forgoing assumptions σ gives the measure of the height of the 'grooves' on the random surface, $g(r)$ is the correlation function and 'l' is the correlation length of the randomness. The correlation function is considered to be of the form of $g(r) = \exp[-(r/l)^\beta]$ where $1 \leq \beta \leq 2$. We present analytical expressions, for the values of the β in the above given range. In the above expression β is related to the fractal dimension of the surface. Special distinctions are made for $\alpha \ll 1$ and $\alpha \gg 1$, where $\alpha - \sigma/\lambda$ is a dimensionless quantity which measures the depth of the grooves w.r.t. the wavelength λ of the light. Several representative cases are considered, with reference to potentials applications.

1 Introduction

Scattering of light forms an important tool for non-destructive testing, particularly for characterization of surface profiles, as is required in fabrication of important optical elements like, mirrors, light diffusers, light deflectors, reflection gratings etc. For a perfectly smooth plane surface, the scattered light is expected to be confined in the specular direction; deviation for this condition, i.e. distribution of light intensity in speckles, carries information of the scattering properties of the roughness and on inversion, about the roughness itself. The problem of inversion is known to be a formidable one and mathematically intricate. The approach that we follow here is to describe the randomness of the surface by few parameters and examine the direct problem of light scattering by a model rough surface, thus parameterized. By fitting the parameterized equations to the data, the roughness properties of the surface can be extracted.

The direct problem of rough surface scattering is addressed in the Kirchoff approximation in our present paper, i.e., it is considered that the local radius of curvature $r_c(x, y) = (\zeta_{xx} + \zeta_{yy})^{-1} \gg \lambda$, the wavelength of the incident light,

where $\zeta(x, y)$ is the elevation in the z-direction at any point (x, y) on the surface. As the convention of the notation in literature, we designate as $\mathbf{v}(v_x, v_y, v_z)$ the change in the wavevector of the light upon scattering and calculate the intensity of light that is scattered along any arbitrary $\mathbf{v}$. As is known, the scattered intensity is dependent on the nature of the randomness, which in fact contributes to the phase consonance and dissonance of the radiation on scattering. We parameterize the disorder with a correlation function that has three independent parameters, σ which describes the typical depth of the grooves on the surface, l the correlation length of $\zeta(x, y)$ and an exponent β, such that structure factor $\mathcal{D}_{\zeta\zeta}(x_1, y_1; x_2, y_2) = \sigma^2[1 - \exp(-r/l)^\beta]$ where $r = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$. We give the scattered intensity as series expansion in (σ/λ) (which scales the groove height with respect to the wavelength of the light) and $(|\mathbf{v}|l)$ which scales the wavevector of scattering, v with respect to a typical wavevector $1/l$, in the system).
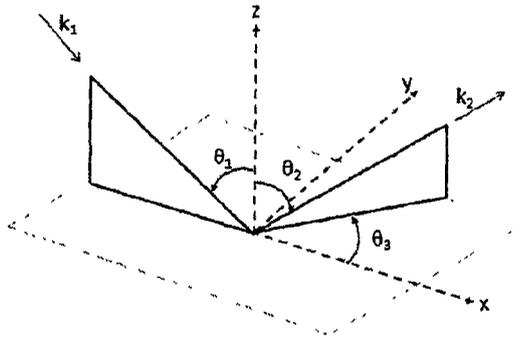

Fig. 1. Orientation of coordinate system and scattering geometry showing the propagation direction of incident wave and scattered wave, respectively along k_1 and k_2.

Following the series expansions calculated in this paper, we present the computational results for some typical cases of practical importance. It is suggested that confidence in surface profile characterization is greatly enhanced if measurements are done at various wavelengths. An obvious fallout of such experimental scheme is that a multiwavelength observation allows us to sample (σ/λ) directly. An impediment, however, arises due to the fact that the coherence length of light r_0 (arising due to random phase additions on reflection from different parts of the rough surface) is also dependent upon λ, for various value of β.

2   Theory

Consider a rough surface in $x - y$ plane. Let $\mathbf{k}_1$ and $\mathbf{k}_2$ define the direction of incident and scattering respectively, as shown in the Figure (1). Let the elevation in the z-direction at any point (x, y) be given by, $z = \zeta(x, y)$, where $\zeta(x, y)$ is a zero mean stationary random variable, such that,

$$\langle \zeta(x, y) \rangle = 0 \tag{1}$$

and the correlation function is considered to be

$$\langle \zeta(x_1, y_1)\, \zeta(x_2, y_2) \rangle = \sigma^2 g(r) \tag{2}$$

where

$$r = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^2 \quad \text{and} \quad g(r) = \exp\left[-(r/l)^\beta \right] \tag{3}$$

In the above expression l is the correlation length and $\beta > 0$ is the exponent of the correlation. It is clear that $g(0) = 1$ and $g(\infty) = 0$. The scattering geometry, following Beckmann and Spizzichino (1963) is given in figure 1, and we follow the symbolic notation given therein. We consider incident beam to be Gaussian, such that the electric field distribution across it varies as $\exp(-r^2/2d_0^2)$. Let $P(\omega)$ be the total power in the laser beam illuminating the surface. Then the total photon flux per unit solid angle (Ω), at large distances from the rough surface, is given by [1-4],

$$
\begin{aligned}
\dot{N}(\Omega) &\equiv \left(\frac{d}{d\Omega} \frac{d}{dt} N(\omega) \right) \\
&= \frac{1}{\lambda^2} \left(\frac{P(\omega)}{2\pi d_0^2 \hbar \omega} \right) \iiiint \exp\left[-\left(\frac{x_1^2 + x_2^2}{4d_1^2} \right) - \left(\frac{y_1^2 + y_2^2}{4d_0^2} \right) \right] \times \\
&\quad \langle \exp\{i v_z \left[\zeta(x_1, y_1) - \zeta(x_2, y_2) \right]\} \rangle \exp\{i \left[v_x(x_1 - x_2) + v_y(y_1 - y_2) \right]\} \times \\
&\quad dx_1 dx_2 dy_1 dy_2 \times \text{(certain geometrical factors involving } \theta_1, \theta_2, \theta_3)
\end{aligned}
\tag{4}
$$

where, the integrations are over the entire surface and

$$
\begin{aligned}
k &= 2\pi/\lambda, \quad d_1^2 = d_0^2 / \cos^2 \theta_1 \\
v_x &= k(\sin\theta_1 - \sin\theta_2 \cos\theta_3), \quad v_y = -k\sin\theta_2\sin\theta_3 \\
v_z &= -k(cos\theta_1 + cos\theta_2), \quad v_{xy}^2 = v_x^2 + v_y^2
\end{aligned}
\tag{5}
$$

The formula given eqs(4) can be easily simplified if we normalize as follows. Let

$$\langle \rho^* \rho \rangle = \frac{\dot{N}(\Omega)}{\dot{N}(\mathbf{v} = 0, \text{for a perfectly smooth surface})} \tag{6}$$

i.e. $\dot{N}(\Omega)$ for any direction $\mathbf{v}$ is normalized with respect to the $\dot{N}$ as seen in the specular direction for a perfectly smooth surface.

Following Beckmann and Spizzichino (1963)and Beckman (1967) we find,

$$\langle \rho^*\rho \rangle = \mathcal{B}(\{\theta_i\}) \iiiint \exp\left[-\left(\frac{x_1^2 + x_2^2}{4d_1^2}\right) - \left(\frac{y_1^2 + y_2^2}{4d_0^2}\right)\right] \times$$
$$E_0(x_1, y_1)E_0(x_2, y_2) \langle \exp\{iv_z [\zeta(x_1, y_1) - \zeta(x_2, y_2)]\} \rangle \times$$
$$\exp\{i [v_x(x_1 - x_2) + v_y(y_1 - y_2)] dx_1 dx_2 dy_1 dy_2 \quad (7)$$

with

$$\mathcal{B}(\{\theta_i\}) = |\mathcal{F}_3(\{\theta_i\})|^2 \mathcal{S}(\{\theta_i\}) \quad (8)$$

$$\mathcal{F}_3(\{\theta_i\}) = \frac{1 + \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \sin\theta_3}{\cos\theta_1 (\cos\theta_1 + \cos\theta_2)} \quad (9)$$

$$\mathcal{S}(\{\theta_i\}) = \mathcal{S}(\theta_1) \mathcal{S}(\theta_2) \quad (10)$$

$$\mathcal{S}(\theta) = x \exp\left[(-1/4) \tan\theta \operatorname{erfc}(K \cot\theta)\right] \quad (11)$$

$$K^2 = 4(\sigma/l)^2 \quad (12)$$

where $\mathcal{F}_3(\{\theta_i\})$ is a geometric factor and $\mathcal{S}(\theta_1)\mathcal{S}(\theta_2)$ are the shadowing effects due to elevation, as introduced by Beckman (1967).

2.1 The model of the disorder and the average $< \cdots >$ over the randomness

We consider the model of the randomness as described in the eqs (1-3). For such a randomness, assumed to be correlated Gaussian zero mean, it is well known that

$$\langle \exp\{iv_z [\zeta(x_1, y_1) - \zeta(x_2, y_2)]\} \rangle = \exp\{-(1/2) \langle v_z^2 [\zeta(x_1, y_1) - \zeta(x_2, y_2)]^2 \rangle\}$$
$$= \exp\left[-\alpha^2 (1 - g(r))\right] \quad (13)$$

where $\alpha^2 = \sigma^2 v_z^2$

Performing the calculations given in (7), we find,

$$\langle \rho^*\rho \rangle = (4\pi d_0 d_1) \iint dq_x dq_y \exp\left[-(v_x^2 + q_x^2)d_1^2 - (v_y^2 + q_y^2)d_0^2\right] f(q_x, q_y) \quad (14)$$

where

$$f(q_x, q_y) = \iint \exp\left[-\alpha^2(1 - g(r))\right] \exp\left[-i(q_x x + q_y y)\right] dx dy \quad (15)$$

$$= 2\pi \exp(-\alpha^2) \sum_{n=0}^{\infty} \frac{(\alpha^2)^n}{n!} \mathcal{F}(n, q_{xy}) \quad (16)$$

$$= 2\pi \exp(-\alpha^2)S \quad (17)$$

$$\text{where} \quad q_{xy}^2 = q_x^2 + q_y^2 \quad \text{and} \quad \mathcal{F}(n, q_{xy}) = \int_0^\infty [g(r)]^n J_0(q_{xy} r) r dr \quad (18)$$

As is anticipated, $\langle \rho^*\rho \rangle$ as given in eqs(14) appears as a convolution, of which $f(q_x, q_y)$ is given by the summation in eqs(16), in which $\mathcal{F}(n, q_{xy})$ is to be determined from eqs(17).

2.2 Approximation for $g(r)$

We consider $g(r)$ to follow,

$$g(r) = \exp\left[-(r/l)^\beta\right] \tag{19}$$

which shows that $g(0) = 1$ and $g(r) \to 0$ for $(r/l) \to \infty$, i.e., l is the distance beyond which correlation in $\zeta(x, y)$ is lost. It is important to note that the exponent β determines many of the properties of the randomness, both in the short range $(r/l) \ll 1$ and in the long range limit. This is easily understood by refering to the structure function,

$$\begin{aligned}\mathcal{D}_{\zeta\zeta}(\mathbf{r}) &= \langle [\zeta(0) - \zeta(\mathbf{r})]^2 \rangle \\ &= 2\sigma^2 \left[1 - \exp(-r/l)^\beta\right]\end{aligned} \tag{20}$$

which shows that in the limit $r/l \to \infty$, $\mathcal{D}_{\zeta\zeta}(\mathbf{r}) = 2\sigma^2$, i.e. it saturates. Also

$$\lim_{(r/l) \to 0} \mathcal{D}_{\zeta\zeta}(\mathbf{r}) = 2\sigma^2 \left(\frac{r}{l}\right)^\beta \tag{21}$$

so that on identifying $\mathcal{D}_{\zeta\zeta}(\mathbf{r}) = \left\langle (\partial\zeta/\partial r)^2 \right\rangle r^2$, we find that $\left\langle (\partial\zeta/\partial r)^2 \right\rangle \sim 2\sigma^2 r^{\beta-2} l^{-\beta}$. This shows that the $\zeta(x, y)$ is non differentiable for $\beta < 2$. This implies that the roughness consists of jagged steps and the exponent β is related to the *fractal* nature of the roughness. In what follows, we confine ourselves in the range $1 \le \beta \le 2$. The case $\beta = 2$ brings it to the case of differentiable roughness, while $\beta = 1$ describes the case where the random elevation $\zeta(x, y)$ look like random telegraph signals, there being $\nu \sim 1/\pi l^2$ grooves per unit area of the surface.

2.3 Evaluation of the sum $\mathcal{S}$

We have from eqs(16)

$$\mathcal{S} = \sum_{n=0}^{\infty} t_n = t_0 + \sum_{n=1}^{0} t_n \quad \text{where} \tag{22}$$

$$t_n = \frac{\alpha^{2n}}{n!} \mathcal{F}(n, \beta, q_{xy}) \tag{23}$$

Using eqs(19) i.e. $g(r) = \exp\left[-(r/l)^\beta\right]$, we have from eqs(18)

$$\mathcal{F}(n, \beta, q_{xy}) = \int_0^R \exp\left[-n\left(\frac{r}{l}\right)^\beta\right] J_0(q_{xy}, r) \; r \, dr \tag{24}$$

$$= \int_0^R \exp\left[-\left(\frac{r}{l(n)}\right)^\beta\right] J_0(q_{xy}, r) r \, dr \quad \text{with} \tag{25}$$

$$l(n) = \frac{l}{n^{1/\beta}} \tag{26}$$

i.e the form of $\mathcal{F}(n,\beta,q_{xy})$ for all n's remains the same, but with decreasing values of $l(n)$, n increases as can be seen from eqs(26). It is thus seen [5,6]
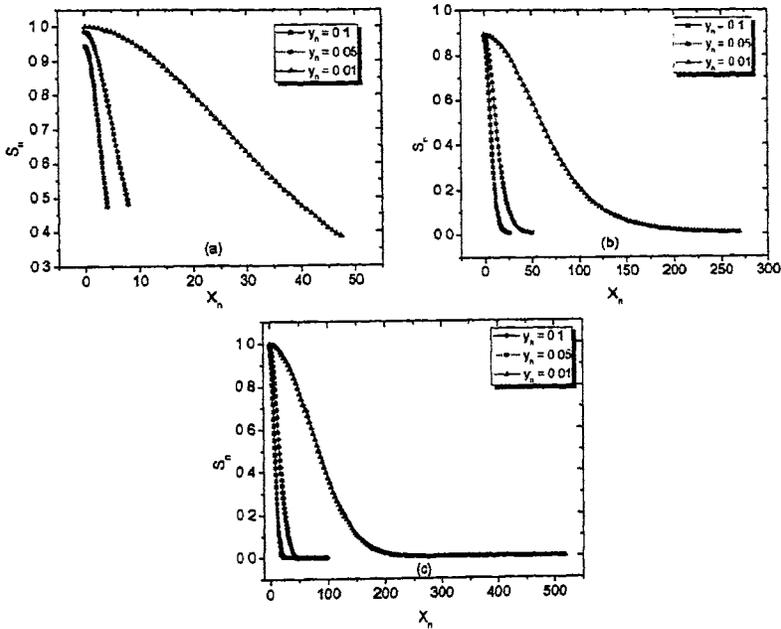
$$\mathcal{F}(0,\beta,q_{xy}) = \int_0^R J_0(q_{xy},r)\,r\,dr$$
$$= R^2\left[\frac{J_1(q_{xy},R)}{q_{xy}R}\right] \tag{27}$$

and in the limit $R \to 0$, this leads us to $\mathcal{F}(0,\beta,q_{xy}) = \delta(q_{xy})$. For higher n values the integral (25) has to be found explicitly. And hence the integral over $q_x.q_y$ for the first term in (23) gives us

$$t_0 = \exp[-v_{xy}^2 d^2] \tag{28}$$

By expanding the Bessel function in the form of a series, we find, for $n! = 0$ [5, 6]

$$\mathcal{F}(n,\beta,q_{xy}) = [l(n)]^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} Z_n^{2m} \int_0^{\infty} x^{2m}\exp(-x^\beta)\,x\,dx$$

Fig. 2. Plot for (a) $\beta = 1$ (b) $\beta = 1.5$ and $\beta = 2.0$ at different values of correlation length l_n where $y_n = l_n/d$.

$$= \frac{[l(n)]^2}{\beta} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} Z_n^{2m} \Gamma\left(\frac{2m+2}{\beta}\right) \qquad (29)$$

where $Z_n = q_{xy}l(n)/2$.

On performing the q_x, q_y integral, and using (29), we find

$$\langle \rho^* \rho \rangle = (4\pi d^2)(2\pi e^{-\alpha^2}) + 4\pi d^2 (2\pi e^{-\alpha^2}) \sum_{n=1}^{\infty} \frac{(\alpha^2)^n}{n!} \frac{|l_n|^2}{\beta} S_n \qquad (30)$$

where

$$S_n = \sum_{m=0}^{\infty} \frac{(-1)^m}{[\Gamma(m+1)]^2} y_n^{2m} \sum_{p=0}^{m} \frac{[\Gamma(m+1)]^2 x_n^{2p}}{[\Gamma(p+1)]^2 \Gamma(m-p+1)} \qquad (31)$$

with $y_n = l_n/2d$ and $x_n = v_{xy}d$.

The first term in (30) for $\alpha = 0$ (i.e. smooth surface) would give a Gaussian profile and is decided by specular reflection of the Gaussian beam by a smooth surface. As $\alpha = \sigma/\lambda$ increases, the contribution from this part falls as $e^{-\alpha^2}$ and the contribution from the roughness term may increase as $e^{-\alpha^2}\alpha^n$ with n if $\alpha > 1$, while l_n^2 falls with increasing n. The other factor S_n broadens with increasing n as seen from the form of S_n in (31). Equation (31) can be simplified for $l_n/\lambda << 1$ ($y_n \ll 1$), $d/\lambda \gg 1$ ($x_n \gg 1$) to read approximately.

$$S_n = \sum_{m=0}^{\infty} \frac{(-1)^m}{[\Gamma(m+1)]^2} \Gamma\left(\frac{2m+2}{\beta}\right) x_n^{2m} \times$$

$$\left\{ 1 + \left[\frac{\Gamma(m+1)}{\Gamma(m)}\right]^2 \frac{1}{\Gamma(2)} x_n^{-2} \right.$$

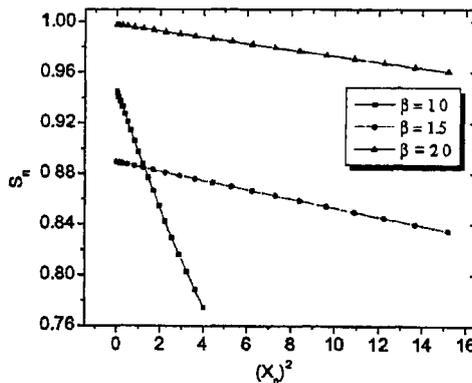$$\left. + \left[\frac{\Gamma(m+1)}{\Gamma(m-1)}\right]^2 \frac{1}{\Gamma(3)} x_n^{-4} \right.$$

Fig. 3. Linearity of the plots for $y_n = 0.05$

$$+ \left[\frac{\Gamma(m+1)}{\Gamma(m-2)} \right]^2 \frac{1}{\Gamma(4)} x_n^{-6} \} \tag{32}$$

where we have written $x_n = v_{xy} ln/2$. It is to be remembered that the above expressions are asymptotic expressions. The complete summation of the second term in (30) is not presented here. We largely concentrate on the term S_n, which describes the broadening of the profile.
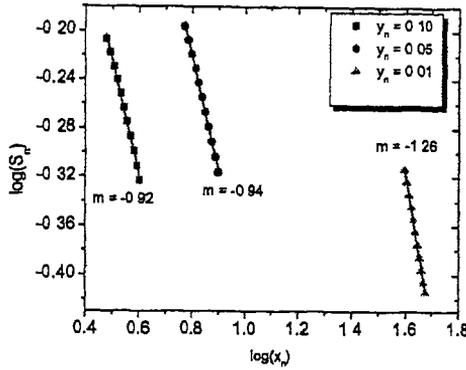

Fig. 4. Slope near the tail-end of each plot for $\beta = 1$.

3   Computational Results and discussions

The calculations given in the forging sections have been performed with an idea to present results that are amenable to simple computations, with adequate degree of accuracy. The change in the profile are largely contained in the expressions for S_n. Figure 2 thus shows some typical curves, in which the computations are performed with the limits of validity of asymptotic expansion.

Figure 3 and figure 4 demostrate the way the profile would change with x_n. As is seen in Figure 3, for smaller values of x_n, $S_n \sim a - b\, x_n^2$, where the coefficient a and b depend on y_n and β. This shows that profound signature of β is seen even in this range, i.e. at the low angle scattering. Full details of the roughness parameters can be obtained in a more comprehensive way if observations are extended to larger values of x_n, as can be seen from figure 4, in which the m gives the slope of the $\log S_n$ versus $\log x_n$ plot. In this paper we limit ourselves to the study of the function S_n alone. Since the intensity profile of the scattering radiations is simply a superposition of several such profiles of S_n type, their strength being decided by the quantity α. In conclusion, we state that the expressions given in this paper are amenable to simple calculations for wide range of roughness (in particular the case $\beta = 5/3$ corresponds to the phase dissonance

caused by Kolmogorove turbulence) and are to be applied to several areas of our ongoing projects, particularly the characterization of reflecting elements for the forthcoming National Large Solar Telescope project We are in the process of obtaining asymptotic expressions for large angle scattering also, which will render completeness to this approach of study, particularly for characterization through multi-wavelength studies.

References

1. P. Beckmann and A.Spizzichino, *The Scattering of Electromagnetic Waves from Rough Surfaces* (Artech House Radar Library, 1987).
2. P. Beckmann, "Scattering of light by rough surfaces", Progress in Optics, **VI**, Ed E. Wolf, North Holland (1967) 53-69.
3. R. L. Fante, "Wave propagation in random media: a system approach", Progress in Optics, **22**, Ed E. Wolf, North Holland (1986) 341-398.
4. J. A. Desanto and G. S. Brown, "Analytical techniques for multiple scattering from rough surfaces", Progress in Optics, **23**, Ed E Wolf, North Holland (1987) 1-62
5. Abramowitz and Stegun, *Handbook of Mathematical functions* (Dover, New York 1965).
6. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Harcourt India Private Limited, New Delhi 2000).