THE ALFVÉN–CARLQUIST DOUBLE-LAYER THEORY
OF SOLAR FLARES

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Abstract. We use the Vlasov equations for ions and electrons to develop a theory of a double layer in which there are both free and trapped electrons and ions. We find the equations which replace the Langmuir condition and the Bohm conditions and by numerically solving the resultant differential equation we find for particular choices of distribution functions the potential distribution in the layer. We discuss the applicability of this theory to solar flares, and show that conditions in solar flares may be such that double layers can exist for which the free particles have a power-law energy distribution. These particles will be accelerated in a double layer and may in this way account for the production of high-energy particles during the impulsive phase of solar flares.

1. Introduction

Let us briefly summarize the main relevant data about solar flares. Many flares exhibit a period of rapid acceleration in which very energetic particles are produced (de Jager, 1969). The first phase of this process, the so-called impulsive phase, is characterized by the acceleration of electrons to energies which are typically of the order of 20 to 100 keV during a period of 1 to 2 minutes. With the possible exception of the electrons responsible for type III bursts – which may or may not be generically related to flares, as they are not always seen (Švestka, 1975) – these electrons are hardly detected in space during this stage so that they remain presumably confined to their respective loops. Their presence is inferred from the impulsive radiation consisting of hard X-rays, during flares. In small flares the energy in the accelerated electrons can constitute a major fraction of the flare energy (Lin and Hudson, 1971) so that an efficient acceleration mechanism is required. Although we have no direct observational data about the fate of the protons during this phase, it seems reasonable to assume that they will also be accelerated.

The second phase occurs only in large flares and gives rise to relativistic electrons and to protons with energies in excess of 10 MeV. Many of these particles are ejected into interplanetary space and have been detected.

It is not known whether the two phases are part of a single mechanism, but we shall concern ourselves here only with the first phase. In fact, the difficulty with flare theory has been to find a mechanism for the initial acceleration – which we shall discuss here – while for the subsequent acceleration several mechanisms have

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been proposed. Our suggestion is that the initial acceleration takes place in a double layer of the kind originally suggested by Alfvén and Carlquist (1967); cf. also Carlquist (1969, 1972).

From the observations, especially of hard X-rays, one can infer the electron spectrum, provided one can estimate the effect of various processes occurring during the propagation of the electrons from the flare region to the region where the X-rays are produced. If one uses the radio-emission data, one must correct for (unknown) magnetic field inhomogeneities and anisotropy in pitch-angle distributions, so that we shall for the purposes of the present paper use the X-ray data. These indicate the existence of two components (Kane and Anderson, 1970) – a soft gradual part with energies of 1 to 10 keV which can be fitted to a thermal spectrum (Kahler *et al.*, 1970) and a hard, impulsive, non-thermal part with energies from 20 keV upwards. The spectral intensity $J(\varepsilon)$ of the hard X-rays is well described by a power-law expression

$$J(\varepsilon) \propto \varepsilon^{-\gamma}, \tag{1.1}$$

where the photon energy ε lies in the range 20 to 100 keV and where the spectral index γ lies typically in the range 3.5 to 5.5 (Datlowe, 1975).

If one accepts that the X-rays are produced by bremsstrahlung (Korchak, 1971), one can use the technique developed by Brown (1971; see also Lin, 1975) to determine the energy distribution function $n(E)$ of the electrons producing the X-rays. If $J(\varepsilon)$ is given by (1.1) we have

$$n(E) = AE^{-\delta'}; \qquad \delta' = \gamma - \tfrac{1}{2}. \tag{1.2}$$

If, as is possible, the regions where the X-rays are produced are different from those where the high-energy particles originate, one must still relate $n(E)$ to the distribution function $f(E)$ of the electrons emerging from the acceleration region. This can be done by solving an equation of continuity in energy space (Lin, 1975; Brown and Melrose, 1976; Hasan, 1977). In the case of a stationary and homogeneous situation one finally ends up with

$$f(E) \propto E^{-\delta}, \tag{1.3}$$

where the value of δ depends on whether one is dealing with a thick target, when $\delta = \gamma + 1$, or with a thin target, when $\delta = \gamma - 1$. In order to produce the observed X-ray spectrum, one needs therefore a high-energy particle distribution with δ lying between 2.5 and 6.5.

We have already noted that it is difficult to say much about the proton spectrum in the impulsive phase; in the second phase, the proton spectrum can often be approximated by a power-law expression with index ~ 3 (Verzariu and Krimigis, 1972; Ramaty and Lingenfelter, 1975).

If we concentrate on the impulsive phase of solar flares, there are two important

problems to be solved: (i) how are the electrons (and protons) accelerated, and (ii) how are the power-law distributions produced? In the present paper we shall postulate that there are power-law distributions and concentrate on the first problem. We shall indicate that conditions in an active region (electron density $N_e \sim 10^{11} \, \mathrm{cm}^{-3}$, temperature $T_e \sim 10^5 \, \mathrm{K}$) may be such that a double layer may be set up. In such a double layer particles can be accelerated to sufficiently high energies. In the next section we discuss the basic features of the Alfvén–Carlquist double-layer model. In Section 3 we develop the basic theory for the case where the free particles have a delta-function distribution. This simple choice enables us to study the behaviour of the double-layer model in some detail. However, this model may be too simplified and we discuss therefore in Section 4, in less detail, a model in which the free particles have a power-law distribution. We find that under solar conditions a double layer can, indeed, be produced. In Section 5 we discuss our results with special reference to the solar situation.

2. The Double-Layer Model

In 1967, Alfvén and Carlquist (1967; see also Carlquist, 1969, 1972) suggested that as a consequence of an MHD instability a double layer might be formed in a flare loop. They invoked an analogy with a low-pressure discharge tube. The flare energy is released in this model through the interruption of the current. According to Carlquist (1969), voltage drops up to 10^{10} V in large flares can be generated which will accelerate electrons and protons. Smith and Priest (1972) criticized this model, pointing out that the instability proposed by Alfvén and Carlquist has the same threshold as the Buneman instability (1959) and as that instability has been observed to occur, while the Alfvén–Carlquist instability has not been observed, they suggested that the Buneman instability is the one more likely to develop. From the experimental data of Hamberger and Friedman (1968) it appears to follow that the Buneman instability will lead to turbulence and hence to an anomalous resistivity which will inhibit the fast interruption of the current needed for a successful theory of particle acceleration. On the other hand, computer experiments (Berk and Roberts, 1967; Goertz and Joyce, 1975) seem to indicate that double layers of the form involved in the Alfvén–Carlquist model will, in fact, be produced and there seems to be some experimental support for this (Babić and Torvén, 1972; Quon and Wong, 1976). We therefore feel that there are sufficient reasons to present a qualitative analysis of double layers which might be relevant to solar flare conditions. We give here such an analysis, and apply it to the solar flare problem.

In 1972, Block (1972) extended Langmuir's double-layer theory (1929) to ionospheric conditions. The model we shall describe here has many features in common with Block's model, but differs from it in that we shall use a kinetic approach rather than the two-fluid MHD approach used by Block. We treat the double layer as an essentially one-dimensional structure – a space-charge region separating two plasmas

in possibly widely differing physical states. We shall only consider a stationary double
layer with a current flowing along a magnetic field, assumed to be in the positive
x-direction. The double layer is assumed to extend from $x = -\infty$ to $x = +\infty$,
and there are no restrictions in the transverse direction, but all variables are assumed
to be functions of x only. The electric potential $\Phi(x)$ is assumed to decrease mono-
tonically from a value Ψ at $x = -\infty$ to a value 0 at $x = +\infty$, with vanishing
gradients – that is, vanishing electric fields – at infinity. The effective variation of the
potential occurs over a comparatively short distance, of the order of a few Debye
lengths which, for the lower corona ($n \sim 10^7\,\mathrm{cm}^{-3}$, $T \sim 10^6$ K), is of the order of
a few centimetres (cf. Kaplan and Tsytovich, 1972, p. 102). The thickness of the
double layer is much less than the mean free path for Coulomb collisions which,
for the lower corona, is of the order of 1000 km. In an active region with $n \sim 10^{11}$
cm^{-3} and $T \sim 10^5$ K, the Debye length is $\sim 10^{-2}$ cm and the Coulomb mean free
path ~ 10 cm. Electrons (from $x = +\infty$) and ions (from $x = -\infty$) are accelerated
into the layer and produce a current (which in our stationary model is constant).
There are also 'thermal' particles present in the layer which are reflected somewhere
in the layer by the potential. They contribute to the space charge within the layer.
We shall assume that we are dealing with a strong layer, that is, $e\Psi/T \gg 1$, where T
is the plasma temperature in energy units, so that the number of thermal particles
which will be able to cross the double layer is negligibly small. In the layer quasi-
neutrality is violated, although the total net space charge integrated over the double
layer is very small. Figure 1 sketches the behaviour of the potential in our model.
We denote by subscripts 0 and 1 quantities at $x = -\infty$ and $x = +\infty$, respectively.

Our model resembles a laminar electrostatic shock with a net current. Montgomery
and Joyce (1968) were the first to apply the Vlasov equation to this problem; they
showed that shock-like solutions were possible. Their work was generalized and
extended by Smith (1970a, b) and Schamel (1971) who tried to find a physically
justified distribution of reflected electrons which could support an electrostatic shock
without current. The problem we shall be concerned with is the slightly different one
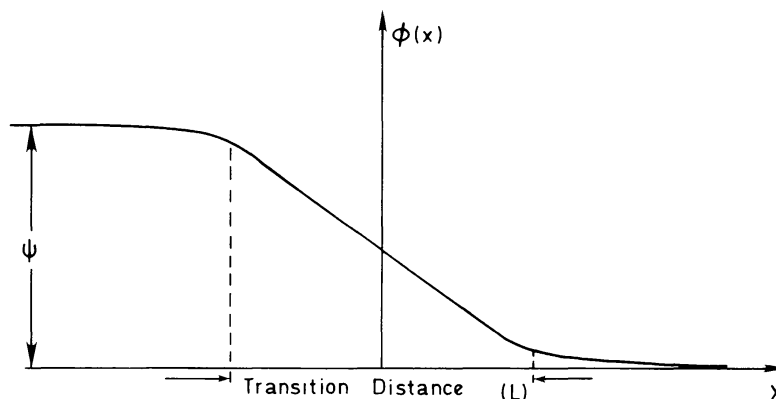

Fig. 1.   Sketch of the potential distribution in our double-layer model.

of determining the shock structure explicitly for given distributions of the free and trapped particles for the case of a strong, current-carrying shock. Schamel (1972) studied the associated problem of the existence of stationary ion-acoustic solitary wave solutions, but for a weak shock without a current. Our treatment is based on the technique developed by Bernstein *et al.* (1957).

3. Basic Theory

The basic equations for our stationary model are the time-independent collisionless Boltzmann equations (Vlasov equations) for the electrons and ions,

$$v \frac{\partial f_\sigma}{\partial x} - \frac{e_\sigma}{m_\sigma} \frac{\mathrm{d}\Phi}{\mathrm{d}x} \frac{\partial f_\sigma}{\partial v} = 0, \qquad \sigma = e, i, \tag{3.1}$$

where e_σ, m_σ and $f_\sigma(x, v)$ are, respectively, the charge, mass and distribution function of the σth kind of particle; the Poisson equation,

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = 4\pi e \int (f_e - f_i) \, \mathrm{d}v \, ; \tag{3.2}$$

and the expression for the total current density J,

$$J = e \int (f_i - f_e) v \, \mathrm{d}v. \tag{3.3}$$

One sees immediately that the energy $W_\sigma = \frac{1}{2} m_\sigma v^2 + e_0 \Phi$ is a constant of motion of Equation (3.1) so that any distribution function $f_\sigma(W_\sigma)$ will be a solution of Equation (3.1). Of course, these distribution functions must still satisfy Equation (3.2). For each species of particle there are some free particles – that is, particles which can move between $x = -\infty$ and $x = +\infty$ – and some reflected or 'trapped' particles. Whether a particle is free or trapped depends on the magnitude of W_σ. Electrons are trapped (free), if $W_e < 0$ ($W_e > 0$), and ions are trapped (free), if $W_i < e\Psi$ ($W_i > e\Psi$).

As mentioned earlier, we shall stipulate the distribution functions and then investigate the consequences. In order to keep the calculations manageable we assume in the present section the free particles to be mono-energetic and uni-directional and the trapped particles to have a 'water-bag' distribution. We thus stipulate the following distributions:

free electrons:

$$f_{ef} = N_{e1} \, \delta\left(\left[v^2 - \frac{2e\Phi}{m_e}\right]^{1/2} + V_1\right), \qquad v < -\left(\frac{2e\Phi}{m_e}\right)^{1/2}; \tag{3.4a}$$

$$= 0, \qquad\qquad\qquad v > -\left(\frac{2e\Phi}{m_e}\right)^{1/2}; \tag{3.4b}$$

free ions:

$$f_{ei} = N_{i0}\,\delta\left(\left[v^2 - \frac{2e}{m_i}(\Psi - \Phi)\right]^{1/2} - U_0\right), \qquad v > \left(\frac{2e(\Psi - \Phi)}{m_i}\right)^{1/2};$$

$$\tag{3.4c}$$

$$= 0, \qquad\qquad\qquad\qquad v < -\left[\frac{2e(\Psi - \Phi)}{m_i}\right]^{1/2};$$

$$\tag{3.4d}$$

trapped electrons:

$$f_{et} = N_{et}\frac{\theta(\Phi - \Psi_1)}{[2e(\Psi - \Psi_1)/m_e]^{1/2}}, \qquad -\left[\frac{2e(\Phi - \Psi_1)}{m_e}\right]^{1/2} < v < \left[\frac{2e(\Phi - \Psi_1)}{m_e}\right]^{1/2};$$

$$\tag{3.4e}$$

$$= 0, \qquad\qquad\text{otherwise}; \qquad\qquad\qquad \tag{3.4f}$$

trapped ions:

$$f_{it} = \tfrac{1}{2}N_{it}\frac{\theta(\Psi_2 - \Phi)}{[2e\Psi_2/m_i]^{1/2}}, \qquad -\left[\frac{2e(\Psi_2 - \Phi)}{m_i}\right]^{1/2} < v < \left[\frac{2e(\Psi_2 - \Phi)}{m_i}\right]^{1/2};$$

$$\tag{3.4g}$$

$$= 0, \qquad\qquad\text{otherwise}. \qquad\qquad\qquad \tag{3.4h}$$

In Equations (3.4), $\Psi_1 > 0$ and $\Psi > \Psi_2 > 0$; Ψ_1 and Ψ_2 can be associated with the temperature of the trapped particles; N_{e1}, N_{e0}, N_{et} and N_{it} are, respectively, the free electron density at $x = +\infty$, the free ion density at $x = -\infty$, the trapped electron density at $x = -\infty$ and the trapped ion density at $x = +\infty$; $\theta(x)$ is the Heaviside function,

$$\theta(x) = 1, \quad x > 0; \qquad \theta(x) = 0, \quad x < 0; \tag{3.5}$$

and U_0 and $-V_1$ are, respectively, the velocities of the free ions and the free electrons at $x = -\infty$ and $x = +\infty$.

Integration of the distribution functions over velocity leads to the following expressions for the number densities:

$$n_e(\Phi) = N_{e1}V_1\left[V_1^2 + \frac{2e\Phi}{n_e}\right]^{-1/2} + N_{et}\left[\frac{\Phi - \Psi_1}{\Psi - \Psi_1}\right]^{1/2}\theta(\Phi - \Psi_1), \tag{3.6a}$$

$$n_i(\Phi) = N_{i0}U_0\left[U_0^2 + \frac{2e(\Psi - \Phi)}{m_i}\right]^{-1/2} + N_{it}\left[\frac{\Psi_2 - \Phi}{\Psi_2}\right]^{1/2}\theta(\Psi_2 - \Phi). \tag{3.6b}$$

The trapped particle densities N_{et} and N_{it} can be expressed in terms of the free particle densities N_{e1} and N_{i0} by using the condition of charge neutrality at $x = \pm\infty$, as

$$N_{et} = N_{i0} - N_{e1}V_1\left[V_1^2 + \frac{2e\Psi}{m_e}\right]^{-1/2}; \tag{3.7a}$$

$$N_{it} = N_{e1} - N_{i0}U_0\left[U_0^2 + \frac{2e\Psi'}{m_i}\right]^{-1/2}$$ (3.7b)

If we define the temperature (in energy units) T_σ of the trapped particles as the average of $m_\sigma v_\sigma^2$ ($\sigma = e, i$) we find that

$$T_e = \tfrac{2}{3}e\left[\frac{(\Phi - \Psi'_1)^3}{\Psi' - \Psi'_1}\right]^{1/2}\theta(\Phi - \Psi'_1),$$ (3.8a)

$$T_i = \tfrac{2}{3}e\left[\frac{(\Psi'_2 - \Phi)^3}{\Psi'_2}\right]^{1/2}\theta(\Psi'_2 - \Phi).$$ (3.8b)

Equation (3.8) can be used to express Ψ'_1 and Ψ'_2 in terms of T_e and T_i.

We note for future reference that Equations (3.3) and (3.4) lead to

$$J = e(N_{i0}U_0 + N_{e1}V_1).$$ (3.9)

We now introduce an auxiliary function $V(\Phi)$ through the equation (cf. Bernstein *et al.*, 1957)

$$\frac{\mathrm{d}V}{\mathrm{d}\Phi} = 4\pi e(n_i - n_e).$$ (3.10)

Using Equations (6) for n_i and n_e and Equations (8) to eliminate Ψ'_1 and Ψ'_2, by integrating Equation (10) we get

$$\frac{1}{4\pi}[V(0) - V(\Phi)] = m_e N_{e1}V_1\left[\left(V_1^2 + \frac{2e\Phi}{m_e}\right)^{1/2} - V_1\right] + N_{et}T_e -$$
$$- m_i N_{i0}U_0\left\{\left(U_0^2 + \frac{2e\Psi'}{m_i}\right)^{1/2} -\right.$$
$$\left. - \left(U_0^2 + \frac{2e}{m_i}[\Psi' - \Phi]\right)^{1/2}\right\} -$$
$$- N_{it}(T_{i1} - T_i),$$ (3.11)

where T_{i1} is the value of T_i at $x = +\infty$.

This equation expresses the fact that at any point in the layer the electrical stress is balanced by the net mechanical stress.

If we substitute Equation (3.10) into the Poisson Equation (3.2), we get

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = -\frac{\mathrm{d}V}{\mathrm{d}\Phi};$$ (3.12)

or, integrating,

$$\frac{1}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right)^2 + V(\Phi) = V(0),$$ (3.13)

where we have used the fact that at $x = +\infty$, $d\Phi/dx = 0$. As $d\Phi/dx$ also vanishes at $x = -\infty$, we have

$$V(0) = V(\Psi),\tag{3.14}$$

and, hence, from Equation (3.11) on putting $\Phi = \Psi$,

$$m_e N_{e1} V_1\left[\left(V_1^2 + \frac{2e\Psi}{m_e}\right)^{1/2} - V_1\right] + N_{et}T_{e0} =$$

$$= m_i N_{i0} U_0\left[\left(U_0^2 + \frac{2e\Psi}{m_i}\right)^{1/2} - U_0\right] + N_{it}T_{i1},\tag{3.15}$$

where T_{e0} is the value of T_e at $x = -\infty$.

We can use Equations (3.15), (3.7) and (3.9) to express the electron and ion currents in terms of J, N_{e1}, T_{i1}, N_{i0} and T_{e0} and thus to obtain a generalization of the Langmuir condition (1929). The result is, for the case where the initial velocities of the particles are small compared to their final values – that is, $m_e V_1^2 \ll e\Psi$ and $m_i U_0^2 \ll e\Psi$,

$$J_e = eN_{e1}V_1 = \frac{J\left(1 - \dfrac{T_{i1}}{2e\Psi}\right) + \sqrt{\dfrac{e}{2\Psi m_i}}(N_{e1}T_{i1} - N_{i0}T_{e0})}{1 + \sqrt{\mu} - \dfrac{T_{i1}}{2e\Psi} - \sqrt{\mu}\,\dfrac{T_{e0}}{2e\Psi}},\tag{3.16a}$$

$$J_i = eN_{i0}U_0 = \frac{\sqrt{\mu}\,J\left(1 - \dfrac{T_{e0}}{2e\Psi}\right) + \sqrt{\dfrac{e}{2\Psi m_i}}(N_{i0}T_{e0} - N_{e1}T_{i1})}{1 + \sqrt{\mu} - \dfrac{T_{i1}}{2e\Psi} - \sqrt{\mu}\,\dfrac{T_{e0}}{2e\Psi}},\tag{3.16b}$$

where $\mu = m_e/m_i$. In the limit as $T_{i1}/2e\Psi \to 0$ and $T_{e0}/2e\Psi \to 0$, we get $J_e \to J$, $J_i \to \sqrt{\mu}\,J$. The ratio J_i/J_e is given by the equation

$$\frac{J_i}{J_e} = \sqrt{\mu}\left[\frac{J\left(1 - \dfrac{T_{e0}}{2e\Psi}\right) - e\left(\dfrac{2e\Psi}{m_e}\right)^{1/2}\left(N_{e1}\dfrac{T_{i1}}{2e\Psi} - N_{i0}\dfrac{T_{e0}}{2e\Psi}\right)}{J\left(1 - \dfrac{T_{i1}}{2e\Psi}\right) + e\left(\dfrac{2e\Psi}{m_i}\right)^{1/2}\left(N_{e1}\dfrac{T_{i1}}{2e\Psi} - N_{i0}\dfrac{T_{e0}}{2e\Psi}\right)}\right].\tag{3.17}$$

Equation (3.17) is the modified Langmuir condition. We note that in the limit as $T_{i1}, T_{e0} \ll 2e\Psi$ we regain the Langmuir ratio $\sqrt{\mu}$, but that for finite values of $T_{i1}/2e\Psi$ and $T_{e0}/2e\Psi$ this ratio is modified by the expression in the square brackets.

In order that the solution we have found can survive, it must satisfy certain conditions which are similar to those derived by Bohm (1949) for wall sheaths. Consider thereto Equation (3.13). Because of charge neutrality at $x = +\infty$, where $\Phi = 0$, it follows from Equation (3.12) that $(dV/d\Phi)_{\Phi=0} = 0$, and expanding $V(\Phi) - V(0)$ in a Taylor series, we get

$$\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 \simeq -\left(\frac{d^2V}{d\Phi^2}\right)_{\Phi=0},\tag{3.18}$$

and as the left-hand side is positive definite, we find from Equations (3.18) and (3.10) the condition

$$\left[\frac{dn_e}{d\Phi} - \frac{dn_i}{d\Phi}\right]_{\Phi=0} \geqslant 0, \tag{3.19a}$$

and similarly

$$\left[\frac{d\dot{n_e}}{d\Phi} - \frac{dn_i}{d\Phi}\right]_{\Phi=\Psi} \geqslant 0. \tag{3.19b}$$

From Equations (3.19), (3.6), (3.7) and (3.8) we get rather complicated inequalities involving N_{i0}, N_{e1}, U_0, V_1, T_{ij}, T_{e0} and Ψ. In the limit of a strong layer ($e\Psi \gg m_i U_0^2$, $m_e V_1^2$) these inequalities reduce to

$$m_e V_1^2 \geqslant 3T_{ij} \tag{3.20a}$$

and

$$m_i U_0^2 \geqslant 3T_{e0}. \tag{3.20b}$$

These conditions are analogous to the necessary conditions derived by Bohm (1949). They differ from the Bohm conditions in an extra factor 3 on the right-hand sides, which is a consequence of our choice of water-bag distributions and our definition of T_{e0} and T_{i1}. Physically, the Bohm conditions fix the minimum current required in the layer to produce the necessary space charge for maintaining the double layer; that is, to counteract the neutralization near the boundaries by the trapped particles. Hasan (1977) has discussed the self-consistency conditions in more detail. Finite temperatures of the free particles will lower the minimum values of V_1 and U_0 below those given by Equations (3.20).

TABLE I

Variation of J_e and J_i with Ψ and T_e/T_i

(The currents are in units $eN_i\sqrt{T_e/m_i}$, and Ψ in units T_e/e)

Ψ	T_e/T_i	J_e	J_i	$\sqrt{\mu}\,J_e/J_i$
20	1.0	92.12	2.15	1.0
20	1.4	92.06	2.21	0.97
40	1.0	92.12	2.15	1.0
40	1.4	92.08	2.19	0.98
60	1.0	92.12	2.15	1.0
60	1.4	92.09	2.18	0.99
80	1.0	92.12	2.15	1.0
80	1.4	92.09	2.18	0.99
100	1.0	92.12	2.15	1.0
100	1.4	92.10	2.17	1.0

TABLE II

Variation of J_e and J_i with N_e and T_e/T_i
(Units as in Table I; the density is in units N_i)

N_e	T_e/T_i	J_e	J_i	$\sqrt{\mu}\,J_e/J_i$
1.2	1.0	92.1	2.17	0.99
1.2	1.4	92.04	2.22	0.96
1.0	1.0	92.12	2.15	1.00
1.0	1.4	92.10	2.17	0.99
0.8	1.0	92.2	2.08	1.04
0.8	1.4	92.17	2.10	1.03
0.6	1.0	92.31	1.95	1.1
0.6	1.4	92.30	1.97	1.10

Integrating Equation (3.13) we obtain the potential distribution in the layer

$$x - x_0 = -\frac{1}{\sqrt{2}}\int_{\Phi}^{\Psi} \frac{\mathrm{d}\Phi}{\sqrt{V(0) - V(\Phi)}}, \qquad (3.21)$$

where $V(0) - V(\Phi)$ is given by Equation (3.11). This equation was solved numerically; we discuss the results of that calculation in the next section.

In the numerical calculations, we first solved Equations (3.15) and (3.9) for V and U (we drop the indices 0 and 1 in what follows) for given values of N_e, T_e/T_i and Ψ and checked that Equations (3.20) were satisfied. In order to avoid slow convergence at $\Phi = 0$ and $\Phi = \Psi$, we defined the layer thickness as the distance over which Φ varied from $0.1 T_e/e$ to $\Psi - 0.1 T_e/e$. In the calculations we used dimensionless variables. The units of velocity, potential, number density, current and distance used were, respectively, $\sqrt{T_e/m_i}$, T_e/e, N_i, $J_s = eN_i\sqrt{T_e/m_i}$ and $\lambda_D = \sqrt{T_e/4\pi N_i e^2}$. With $T_e \sim 10^5$ K, $m_i = 1.6 \times 10^{-24}$ g and $N_i \sim 10^{11}$ cm^{-3} we have for these units, respectively, 30 km s^{-1}, 0.03 e.s.u. $= 10^{-4}$ V, 10^{11} cm^{-3}, 2×10 e.s.u. $= 4 \times 10^{11}$ A/m^2 and 10^{-2} cm. Table I gives the variation of J_e, J_i and $\sqrt{\mu}\,J_e/J_i$ with Ψ and T_e/T_i for $J = 94.3$, $n_e = 1.0$.

TABLE III

Variation of the Bohm currents with Ψ and T_e/T
(Units as in Table I)

Ψ	T_e/T_i	J_{Be}	J_{Bi}	Ψ	T_e/T_i	J_{Be}	J_{Bi}
20	1.0	91.6	2.10	100	1.0	80.6	1.88
20	1.4	77.6	2.10	100	1.4	68.2	1.88
40	1.0	85.3	1.98	200	1.0	78.6	1.83
40	1.4	72.1	1.98	200	1.4	66.4	1.83
60	1.0	82.8	1.93	400	1.0	77.2	1.80
60	1.4	70.1	1.93	400	1.4	65.3	1.80
80	1.0	81.5	1.90				
80	1.4	68.9	1.90				

TABLE IV
Variation of the Bohm currents with N_e and T_e/T_i
(Units as in Table I)

N_e	T_e/T_i	J_{Be}	J_{Bi}	N_e	T_e/T_i	J_{Be}	J_{Bi}
1.2	1.0	95.5	1.88	0.8	1.0	65.74	1.88
1.2	1.4	80.73	1.88	0.8	1.4	55.59	1.88
1.0	1.0	80.6	1.88	0.6	1.0	50.8	1.88
1.0	1.4	68.2	1.88	0.6	1.4	43.0	1.88

We see that J_e and J_i are almost independent of Ψ, but that J_i increases and J_e decreases with increasing T_e/T_i. This behaviour can be understood from Equation (3.15): for large Ψ the pressure balance is dominated by the dynamic particle pressure, acquired as a result of acceleration through the layer. An increase in Ψ thus leads to an increase in the electron pressure, proportional to $\sqrt{\Psi}$, but, for large enough Ψ, this is compensated by an increase in the ion pressure which is also proportional to $\sqrt{\Psi}$. The decrease of J_e with increasing T_e/T_i is also easily understood
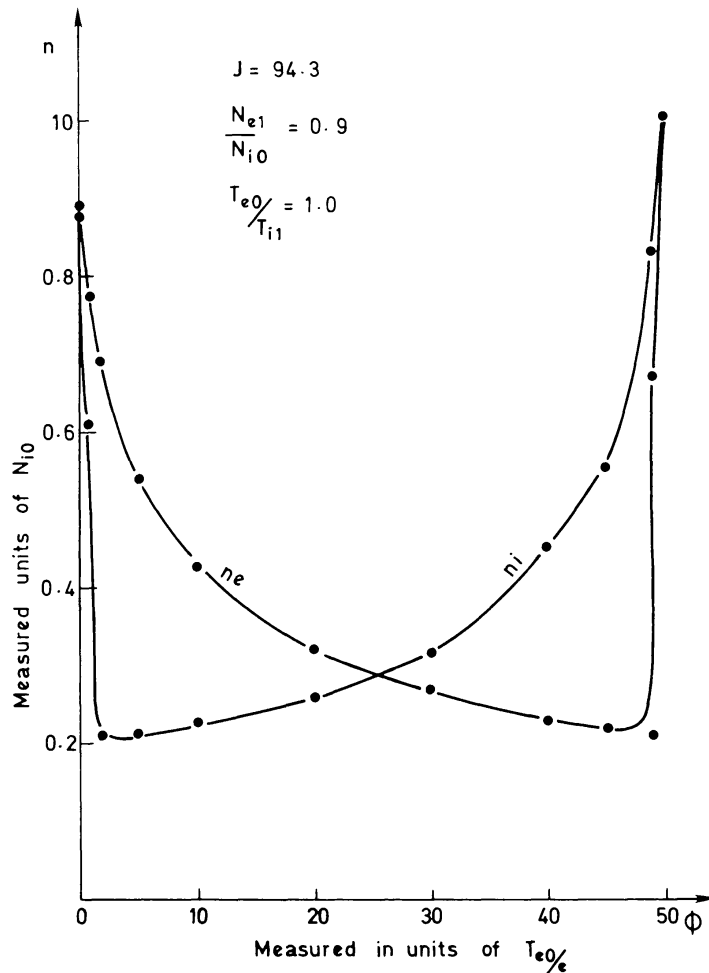

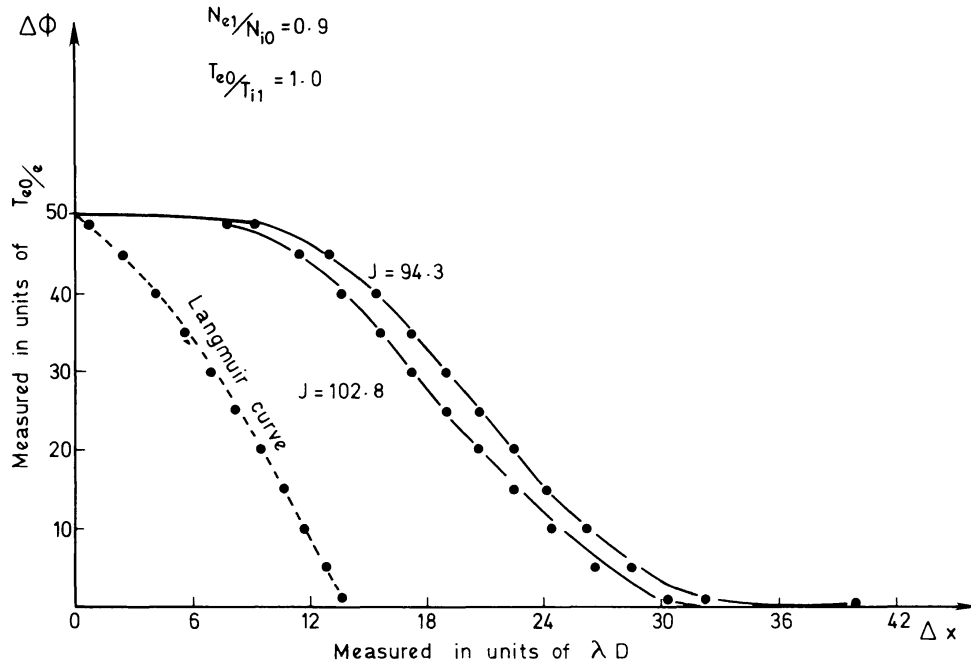Fig. 2. Electron and ion densities as functions of the potential Φ.

Fig. 3a.   Potential distribution in the layer: $N_e/N_i = 0.9$, $T_e/T_i = 1.0$, $J = 94.3$ and 102.8 $\Psi = 50$.
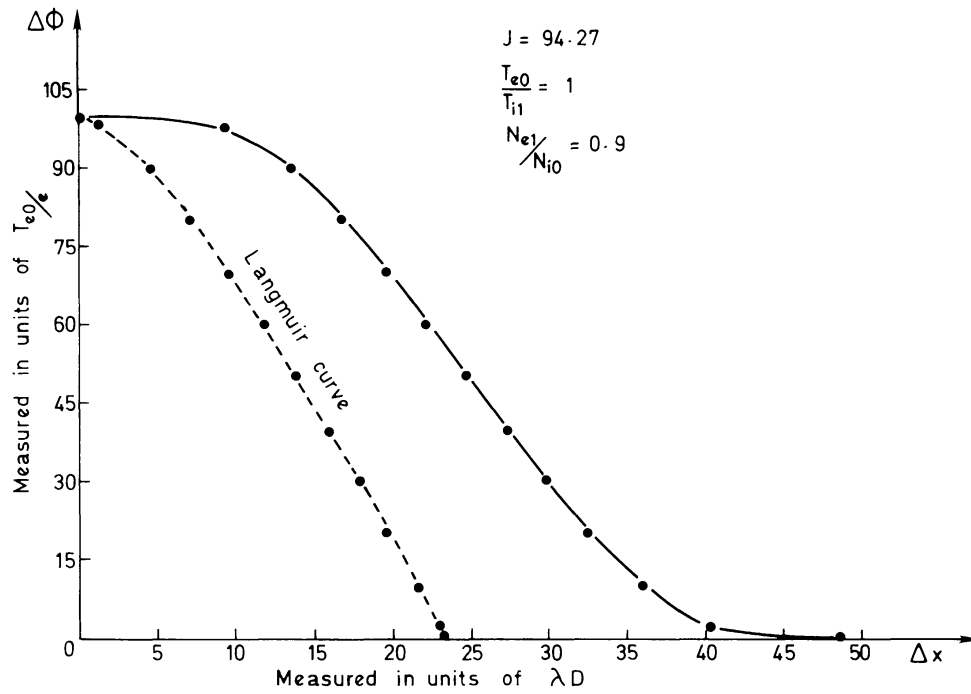

Fig. 3b.   Potential distribution in the layer: $N_e/N_i = 0.9$, $T_e/T_i = 1.0$, $J = 94.3$, $\Psi = 100$.

as being due to an increase in thermal pressure at the boundary. Finally, we note that the behaviour of $\sqrt{\mu}\,J_e/J_i$ is in accordance with Equation (3.17).

Table II gives the variation of J_e, J_i and $\sqrt{\mu}\,J_e/J_i$ with N_e and T_e/T_i for $J = 94.3$, $\Psi = 100$. The variation of J_e with N_e can be explained by noting that the dynamic electron pressure varies as $J_e[(J_e/N_e)^2 + (2\Psi/\mu)]^{1/2}$ near $\Phi = \Psi$. A decrease in N_e leads to a small increase in J_e to keep the pressure constant.

We note that for large Ψ, J_e and J_i are only very weakly dependent on Ψ, T_e/T_i and N_e and also that $\sqrt{\mu}\,J_e/J_i$ is almost equal to unity.

Tables III and IV show the variations of the Bohm currents J_{Be} $(= N_e V_{Be} = N_e\sqrt{3T_i/m_e})$ and J_{Bi} $(= U_{Bi} = N_i\sqrt{3T_e/m_i})$ with Ψ and T_e/T_i (Table III) or N_e and T_e/T_i (Table IV). We note that both J_{Be} and J_{Bi} decrease with increasing Ψ, asymptotically reaching the values $J_{Be} = (3\mu T_e/T_0)^{1/2}N_e$ and $J_{Bi} = \sqrt{3}$. This behaviour arises because of a decreasing contribution from accelerated particles from the opposite boundary to the space charge (cf. variation of n_e and n_i as functions of Φ in Figure 2). Table IV shows that V_{Be} and U_{Bi} depend very weakly on N_e – which follows for the same reasons.

In Figures 2, 3 and 4 we plot the various aspects of the potential distribution in the layer. From Figures 3a and 3b we note that our potential distribution is more gradual than the one which follows from Langmuir's formula (Langmuir, 1929). The difference arises from the inclusion of trapped particles which partially neutralize the space charge in the layer. We note also that, in fact, the effective variation of the potential is when Φ changes from Ψ -0.1 to 0.1. In Figures 4a and 4b we show the variation of Ψ with the transition distance L.
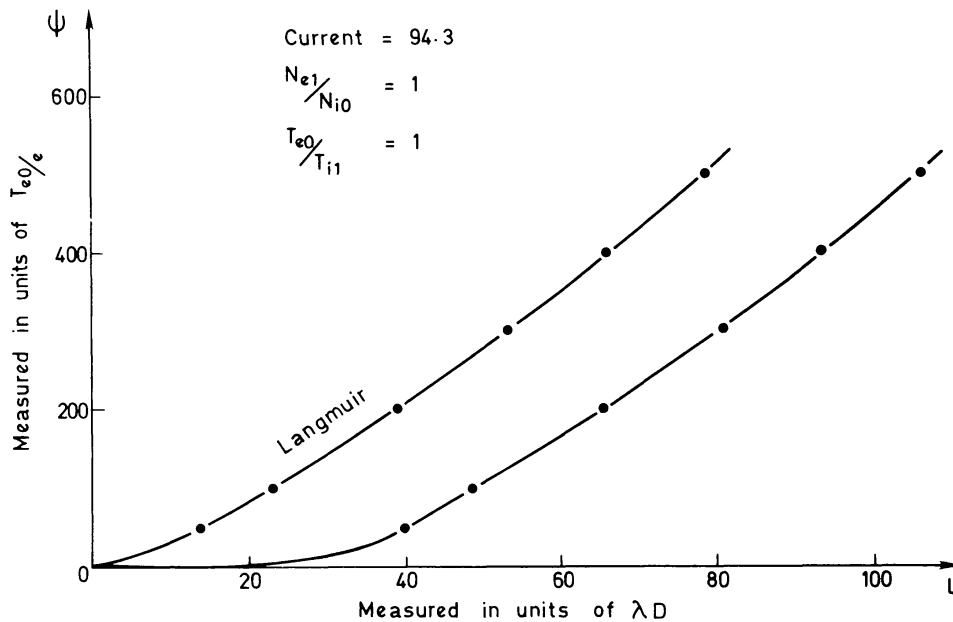

Fig. 4a. Variation of maximum potential drop Ψ with transition distance L: Ψ up to about 500.
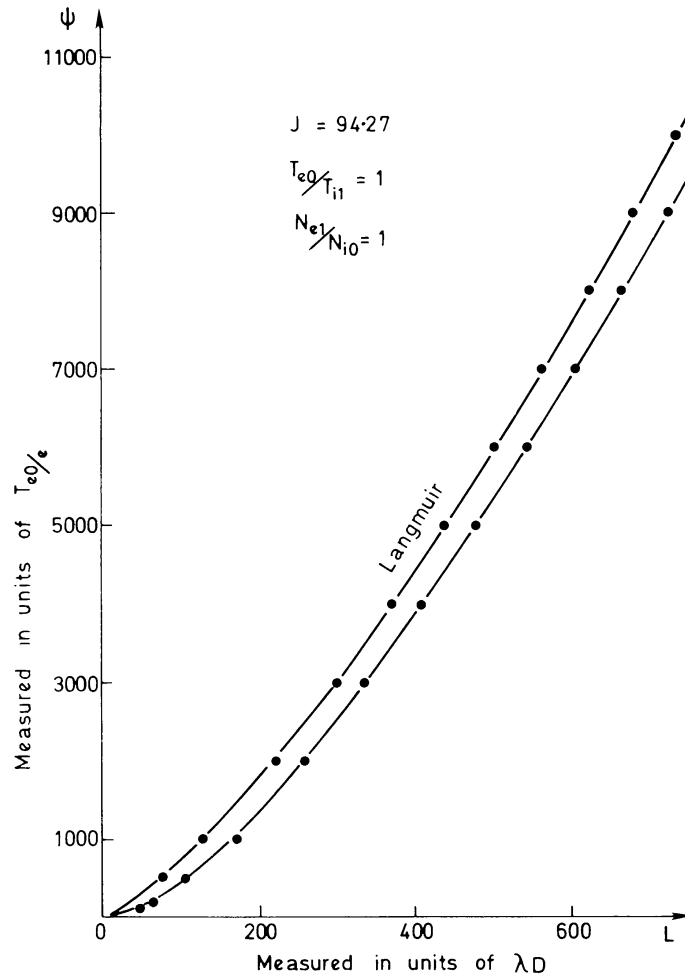
Fig. 4b.   Variation of maximum potential drop Ψ^* with transition distance L: very large Ψ^*.

An interesting aspect of the theory is the existence of fairly stringent conditions for the existence of double-layer solutions. In Table V we show the variation of J_e and J_i with J and we compare them with the Bohm currents. We note that, in the case where $\Psi^* = 100$, $N_2 = 0.8$, $T_e/T_i = 1.4$, for $J \leqslant 80$ no double-layer solution exists, as $J_i < J_{Bi}$.

TABLE V

Ion and electron currents as functions of the total current
(Units as in Table I)

J	J_e	J_i	$\sqrt{\mu}\,J_e/J_i$	J_{Be}	J_{Bi}
94.3	92.2	2.1	1.03	55.6	1.88
85.7	83.8	1.92	1.02	55.0	1.86
77.1	75.4	1.73	1.01	54.6	1.85
68.6	67.0	1.55	1.01	54.0	1.84
51.4	50.2	1.18	0.99	53.0	1.80
42.8	41.9	0.99	0.98	52.6	1.79

4. Results for Power-Law Distributions of Free Particles

So far we have used delta-function distributions for the free electrons and ions. However, as we saw in Section 1, from the observational data one can infer power-law distributions. It is, therefore, of interest to look at the consequences of such distributions. As the calculations are much more complicated and not qualitatively different, we have not considered this case in as much detail as the delta-function case.

The analysis proceeds as in the preceding section, but for a few changes. Equations (3.4a) and (3.4c) are replaced by

free electrons:

$$f_e = A\left(v^2 - \frac{2e\Phi}{m_e}\right)^{-\delta_e}, \qquad v < -\left[\frac{2}{m_e}(e\Phi + B_1)\right]^{1/2}, \tag{4.1a}$$

where B_1 is the low-energy cut-off at $x = +\infty$; and

free ions:

$$f_i = C\left(v^2 - \frac{2e(\Psi - \Phi)}{m_i}\right)^{-\delta_e}, \qquad v > \left[\frac{2e(\Psi - \Phi)}{m_i} + D_1\right]^{1/2}, \tag{4.1b}$$

where D_1 is the low-energy cut-off at $x = -\infty$. A and C are normalization constants. In the first instance, we shall choose $\delta_e = \delta_i = \delta = 2$; we have also performed the analysis for $\delta = 3$. We note that B_1 and D_1 are related to V_1 and U_0 through the equations

$$V_1 = \frac{3}{2}\left(\frac{2B_1}{m_2}\right)^{1/2}, \qquad U_0 = \frac{3}{2}\left(\frac{2D_1}{m_1}\right)^{1/2}. \tag{4.2}$$
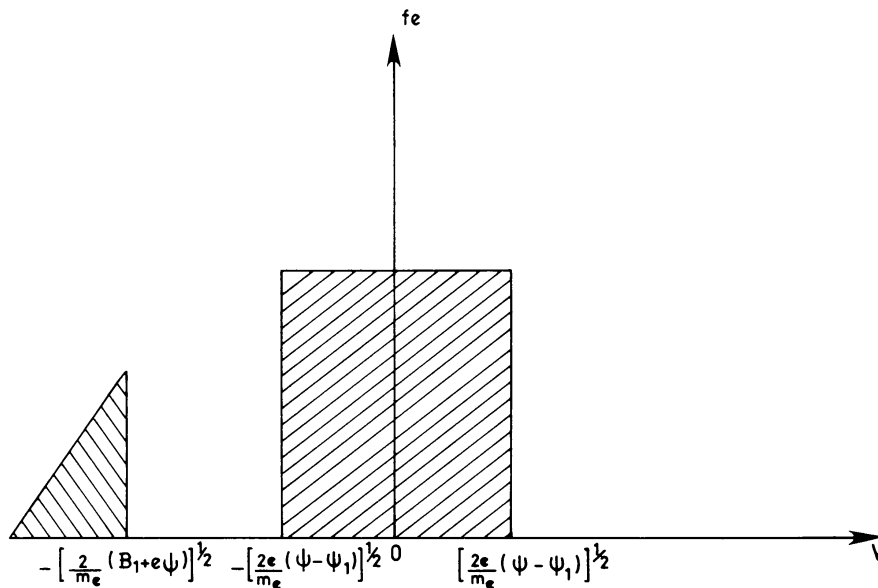

Fig. 5. Electron distribution function at $x = -\infty$. The free electrons have a power-law distribution.

The electron and proton distribution functions at $x = -\infty$ are shown in Figure 5.
Equations (3.6) are now replaced by

$$n_e(\Phi) = \tfrac{3}{2}N_{e1}y^{1/2}\left[(y + 1)^{1/2} - \tfrac{1}{2}y \ln \frac{(y + 1)^{1/2} + 1}{(y + 1)^{1/2} - 1}\right] +$$

$$+ N_{et}\left[\frac{\Phi - \Psi_1}{\Psi + \Psi_1}\right]^{1/2}\theta(\Phi - \Psi_1), \tag{4.3a}$$

$$n_i(\Phi) = \tfrac{3}{2}N_{i0}z^{1/2}\left[(z + 1)^{1/2} - \tfrac{1}{2}z \ln \frac{(z + 1)^{1/2} + 1}{(z + 1)^{1/2} - 1}\right] +$$

$$+ N_{it}\left[\frac{\Psi_2 - \Phi}{\Psi_2}\right]^{1/2}\theta(\Psi_2 - \Phi), \tag{4.3b}$$

where $y = B_1/e\Phi$ and $z = D_1/e(\Psi - \Phi)$. Similarly, Equations (3.7) are replaced by

$$N_{et} = N_{i0} - \tfrac{3}{2}N_{e1}y'\left[(y' + 1)^{1/2} - \tfrac{1}{2}y' \ln \frac{(y' + 1)^{1/2} + 1}{(y' + 1)^{1/2} - 1}\right], \tag{4.4a}$$

$$N_{it} = N_{e1} - \tfrac{3}{2}N_{i0}z'\left[(z' + 1)^{1/2} - \tfrac{1}{2}z' \ln \frac{(z' + 1)^{1/2} + 1}{(z' + 1)^{1/2} - 1}\right], \tag{4.4b}$$

where $y' = B_1/e\Psi$ and $z' = D_1/e\Psi$.
Proceeding as in Section 3, we find instead of Equations (3.11) and (3.15)

$$\frac{1}{4\pi}[V(0) - V(\Phi)] = 3\frac{B_1N_{e1}}{e}\left[\tfrac{1}{2}y^{1/2}\ln \frac{(y + 1)^{1/2} + 1}{(y + 1)^{1/2} - 1} + \left(\frac{y + 1}{y}\right)^{1/2} - 2\right] -$$

$$- \frac{3D_1N_{i0}}{e}\left[\tfrac{1}{2}z'^{1/2}\ln \frac{(z' + 1)^{1/2} + 1}{(z' + 1)^{1/2} - 1} - \right.$$

$$- \tfrac{1}{2}z \ln \frac{(z + 1)^{1/2} + 1}{(z + 1)^{1/2} - 1} +$$

$$\left. + \left(\frac{z' + 1}{z'}\right)^{1/2} - \left(\frac{z + 1}{z}\right)^{1/2}\right] +$$

$$+ N_{et}T_e - N_{it}(T_{i1} - T_i), \tag{4.5}$$

and

$$3\frac{B_1N_{e1}}{e}\left[\tfrac{1}{2}y'^{1/2}\ln \frac{(y' + 1)^{1/2} + 1}{(y' + 1)^{1/2} - 1} + \left(\frac{y' + 1}{y'}\right)^{1/2} - 2\right] + N_{et}T_{e0} =$$

$$= \frac{3D_1N_{i0}}{e}\left[\tfrac{1}{2}z'^{1/2}\ln \frac{(z' + 1)^{1/2} + 1}{(z' + 1)^{1/2} - 1} + \left(\frac{z' + 1}{z'}\right)^{1/2} - 2\right] + n_{it}T_{i1}. \tag{4.6}$$

Finally, we must derive the Bohm conditions. Under the same conditions under which Equations (3.20) hold, we now have

$$m_eV_1^2 \geqslant 4T_{i1}, \qquad m_iU_0^2 \geqslant 4T_{e0}. \tag{4.7}$$

The analysis of the equations was again performed numerically, both for $\delta = 2$ and for $\delta = 3$, in which case Equations (4.2) to (4.7) are slightly changed, mainly

TABLE VI

Ion and electron currents as function of the spectral index

(Units as in Table I)

δ	J_e	J_i	$\sqrt{\mu}\,J_e/J_i$	J_{Be}	J_{Bi}
2	100.53	2.31	1.01	78.1	2.20
3	100.53	2.31	1.01	71.2	2.02
∞ (mono-energetic)	100.52	2.32	1.01	67.4	1.89

in the values of the numerical coefficients. The results are given in Table VI, which replaces Table V, and in Figure 6.

We notice that J_e and J_i are very insensitive to the value of δ; this is due to the fact that the main contribution to these currents comes from near the cut-off. However, J_{Be} and J_{Bi} decrease appreciably with increasing δ. This means that the harder the spectrum, the more difficult it is to construct a double layer.

5. Discussion

Let us first say a few words about the stability of double layers – without attempting a quantitative analysis. Although we have derived conditions for the existence of a double layer, there is no guarantee that these conditions are also sufficient. In fact, the delta-function distributions are likely to be unstable against plasma oscillations
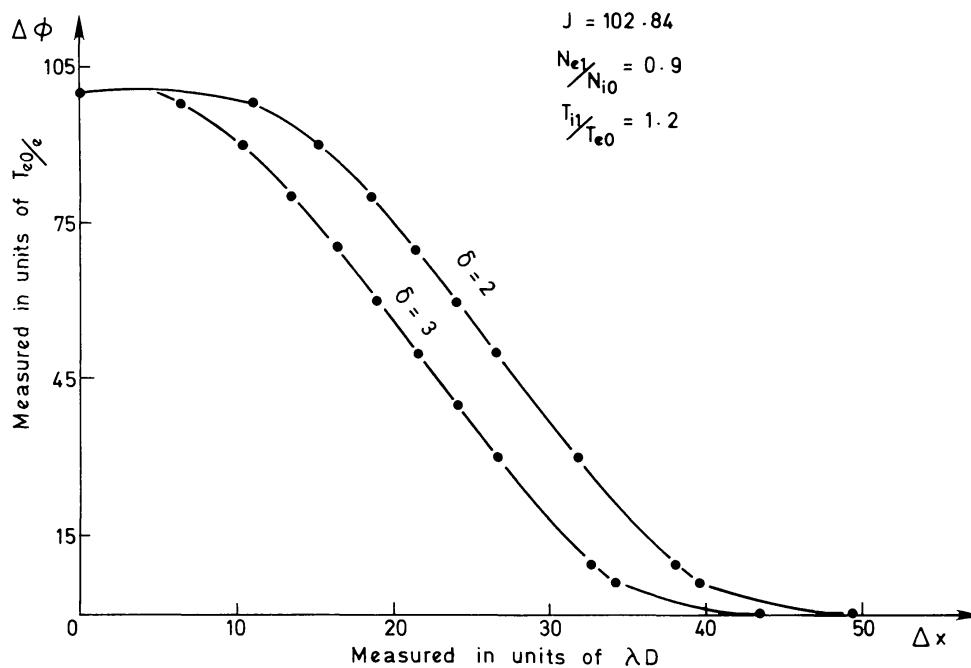

Fig. 6. Potential distribution in the layer for the case of power-law distributions. $J = 102.84$; $N_e/N_i = 0.9$; $T_e/T_i = 1.2$.

and this would make the layer short-lived as the current would rapidly be dissipated (Buneman, 1959). However, the distribution considered in Section 4 would be more stable and our results seem to be relatively insensitive to the choice of distribution functions so that there may well be distribution functions which would be stable, something which seems also to be indicated by computer experiments (Berk and Roberts, 1967; Goertz and Joyce, 1975).

If the double-layer theory discussed in this paper can be applied to describe a double layer in a solar current loop – as we are doing here – we must consider whether solar parameters can fit in with the picture presented. First of all, we note that the current will be dissipated on a time-scale of the order of the release time for flares (Alfvén and Carlquist, 1967; Carlquist, 1969) which for typical solar conditions is of the order of 100 s. This means that the layer can be considered to be stationary only for times less than this. Secondly, we note that for the existence of a double layer it is necessary that the current density must be sufficiently high so that J_e and J_i exceed the Bohm values. This corresponds to a critical current of the order of 80 to 100 in our units, or a few times about 10^9 stat amp cm^{-2} for typical solar flare parameters. This is rather higher than the current densities observed by Moreton and Severny (1968) and by Title and Andelin (1971); but perhaps the present observational limitations are such that larger current densities may exist on length scales of the size of double layers.

If we look at the margins to be acquired by the electrons when they are accelerated in the double layer, we see that the observed impulsive hard X-rays indicate electron energies of up to 100 keV (Kane and Anderson, 1970). For $T_e \sim 10^7$ K, a voltage drop of about $100 T_e/e$ will, indeed, accelerate electrons up to such energies over a distance of $\sim 50 \lambda_D$ which in solar conditions ($N_e \sim 10^{11}$ cm^{-3}) corresponds to about 10 cm. We see thus that the electrons can, indeed, acquire the necessary energies in the model considered by us.

Finally, the fact noticed in Section 4, that the harder the spectrum the more difficult it is to construct a double layer, is in agreement with the observational data about the relative infrequency of very hard spectra and their occurrence mainly in strong flares in which large current densities – possibly generated by sunspot motions – can exist and support hard spectral distributions.

Altogether, we feel that the double-layer model of solar flares probably deserves further study. Points to be studied are the initial development of power-law distributions and a more detailed study of the relation between initial and observed spectra indexes.

Acknowledgements

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