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On the points of bifurcation along the sequence of rotating axisymmetric masses with magnetic fields

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Abstract. It is shown that the points of bifurcation belonging to the third harmonics along the sequence of Maclaurin spheroids viewed from an inertial frame are distinct from the corresponding points along the Maclaurin sequence considered stationary in a rotating frame and occur at eccentricity $e = 0 \cdot 73113$ and $e = 0 \cdot 99608$; the Maclaurin spheroids having become dynamically unstable before the second point is reached. A toroidal magnetic field leaves these points uneffected, while a general poloidal field may either raise or lower these points of bifurcation.

Keywords: Rotation ; magnetic fields

1. Introduction

The virial equations of various orders provide a very elegant and powerful method for investigating the equilibrium and the stability of rotating masses. *See* Chandrasekhar (1969 hereafter referred to as EFE; and the references therein) for their application to rotating liquid masses. The virial method has also been used to investigate the oscillations and the stability of differentially rotating gaseous masses in the presence of magnetic fields (Nakagawa and Trehan 1970; Kochhar and Trehan 1971, 1973, 1974; Kochhar 1974). An interesting outcome of these studies is the isolation of the points of bifurcation belonging to the second and the third harmonics from an investigation of the complete frequency spectra. However, it is possible to isolate the point of bifurcation from a consideration of the integral properties provided by the virial equations various orders (cf. EFE § 34).

The point of bifurcation, belonging to the second harmonics, where the Jacobian and the Dedekind sequence of ellipsoids branch off from the sequence of Maclaurin spheroids in the presence of toroidal magnetic fields has been isolated by Trehan and Singh (1975) using second order virial equations. In this paper, we use third order virial equations to isolate the points of bifurcation, belonging to the third harmonics, along the sequence of Maclaurin spheroids viewed from an inertial frame, and discuss the effects of a torodial and a general magnetic field on these points of bifurcation.

2. The equilibrium configuration

We consider a homogeneous, axisymmetric, self-gravitating gaseous mass rotating with an angular velocity $\Omega(x)$ and pervaded by a magnetic field $H(x)$ which vanishes at the surface. Both $\Omega(x)$ and $H(x)$ are assumed to be symmetric about the z-axis, which is taken to be the axis of rotation. We can express the velocity and the magnetic field vectors as

$$u = \omega\Omega 1_\phi \tag{1}$$

and

$$H = -\omega \frac{\partial P}{\partial Z} 1_\omega + \omega T 1_\phi + \frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2 P)\, 1_z. \tag{2}$$

Here ω, ϕ, z define a system of cylindrical polar coordinates; 1_ω, 1_ϕ, 1_z are the unit vectors along the three principal directions and the defining scalars Ω, P and T are azimuth independent.

The third order virial equations governing the equilibrium configuration are

$$2T_{ij;k} + 2T_{ik;j} - 2M_{ij;k} - 2M_{ik;j} + W_{ij;k} + W_{ik;j} + \delta_{ij}(\Pi_k + M_{ss;k}) + \delta_{ik}(\Pi_j + M_{ss;j}) = 0, \tag{3}$$

where

$$T_{ij;k} = \tfrac{1}{2}\int \rho u_i u_j x_k \, dx, \tag{4}$$

$$M_{ij;k} = \frac{1}{8\pi}\int H_i H_j x_k \, dx, \tag{5}$$

$$\Pi_k = \int p x_k dx, \tag{6}$$

$$W_{ij;k} = -\tfrac{1}{2}\int \rho v_{ij} x_k dx. \tag{7}$$

In equation (7), $v_{ij}(x)$ is the tensor potential at x due to a mass distribution $\rho(x')$ at x':

$$v_{ij}(x) = G\int \frac{\rho(x')(x_i - x_i')(x_j - x_j')\, dx'}{|x - x'|^3}. \tag{8}$$

The equilibrium conditions governing the system are obtained from the steady state virial equations of the second and the fourth orders; the former yield (Kochhar and Trehan 1973)

$$T_{11} - M_{11} + M_{33} = FI_{11} \tag{9}$$

where

$$F = A_1 - A_3(1 - e^2) = e^2 B_{13} \tag{10}$$

and the remaining symbols have their usual meaning. In equation (10), $A_{ij}\ldots$ and $B_{ij}\ldots$ are the index symbols defined in EFE (§ 21), and e is the eccentricity given by

$$a_3^2 = a_1^2(1 - e^2), \tag{11}$$

$a_1 = a_2 > a_3$ being the three semi axes.

The steady state fourth-order virial equations have been written in Kochhar (1974, eq. [3]). Since the velocity field has been assumed to be toroidal and axisymmetric, we have

$$T_{33;ii} = T_{3i;3i} = 0; \quad T_{11;11} = -T_{12;12} = \tfrac{1}{3} T_{11;22}. \tag{12}$$

It should be noted that if the field is purely toroidal or if the poloidal scaler P is a function of ω alone, *i.e.*, the poloidal field has a non-vanishing component only along the Z-direction, the tensor $M_{ij;\,kl}$ obeys relations similar to the second one in equation (12).

With the help of the above relations and proceeding as in Kochhar (1974) we obtain the following equilibrium relations

$$T_{11;\,11} - M_{11;\,11} = FI_{1122} - M_{33;\,11} - 2M_{13;\,13} \tag{14}$$

and

$$T_{11;\,33} - M_{11;\,33} = FI_{1133} - M_{33;\,33} + 2M_{13;\,13}. \tag{15}$$

When rotation is uniform and the magnetic fields are absent, equations (14) and (15) reduce to equation (9). In this case thus one need not consider the fourth order virial equations. When the magnetic field is purely toroidal, we recover equilibrium relations of Kochhar (1974).

3. The points of bifurcation belonging to the third harmonics

Suppose that the system is now slightly perturbed from its equilibrium state and the resulting motions are characterized by a Lagrangian displacement $\xi\,(x, t)$. The existence of a point of bifurcation implies the existence of a non-trivial time-independent ξ which will deform the equilibrium configuration in such a manner that the sysem is carried over to another equilibrium configuration. In other words, at a point of bifurcation, not only should the equilibrium conditions be satisfied (identically or trivially) but the first variation of the integral properties due to the Lagrangian displacement ξ should also vanish non-trivially (EFE § 34). It is easy to see that the integral relations provided by the equation (3) are satisfied identically by virtue of the triplanar symmetry of the configuration. Thus a necessary and sufficient condition for the occurrence and the isolation of the points of bifurcation belonging to the third harmonics is that the first variation in equation (3) must vanish non-trivially:

$$\begin{aligned} &2\delta T_{ij;\,k} + 2\delta T_{ik;\,j} - 2\delta M_{ij;\,k} - 2\delta M_{ik;\,j} \\ &\quad + \delta W_{ij;\,k} + \delta W_{ik;\,j} + \delta_{ij}\,(\delta \Pi_k + \delta M_{ss;\,k}) + \delta_{ik}\,(\delta \Pi_j + \delta M_{ss;\,j}) = 0; \end{aligned} \tag{16}$$

here δA denotes the first variation in the quantity A brought about by the Lagrangian displacement $\xi\,(x)$.

We shall now consider two cases. First, when the rotation is uniform and the magnetic fields are absent, *i.e.*, the case of uniformly rotating Maclaurin spheroids viewed from an inertial frame. As we shall see, virial equations provide an exact solution in this case. In § 5, we consider the general case when differential rotation and a magnetic field, having both toroidal and poloidal components, are present.

4. The sequence of Maclaurin spheroids viewed from an inertial frame

We now consider the case of uniform rotation and no magnetic field. We set $\delta M_{ij;\,k} = 0$ in equation (16). Suspending the summation convention and letting $i \neq j \neq k$ denote distinct indices, we can group the eighteen relations contained in equation (16) (with $\delta M_{ij;k} = 0$) as follows:

$$4\delta T_{ii;i} + 2\delta W_{ii;i} + 2\delta \Pi_i = 0, \tag{17}$$

$$4\delta T_{jj;i} + 4\delta T_{ij;j} + 2\delta W_{jj;i} + 2\delta W_{ij;j} + 2\delta \Pi_i = 0, \tag{18}$$

$$4\delta T_{ij;j} + 2\delta W_{ij;j} = 0, \tag{19}$$

$$4\delta T_{ij;k} + 4\delta T_{ik;j} + 2\delta W_{ij;k} + 2\delta W_{ik;j} = 0. \tag{20}$$

Since the perturbations giving rise to the points of bifurcation are volume preserving, we can eliminate $\delta \Pi_i$ between equations (17) and (18). It is convenient to combine them in a particular manner and we get

$$4(\delta T_{ii;i} - \delta T_{jj;i} - 2\delta T_{ij;j}) + \delta S_{ijj} = 0, \tag{21}$$

where (EFE § 12, eq. [84])

$$\delta S_{ijj} = 2\delta W_{ii;i} - 2\delta W_{jj;i} - 4\delta W_{ij;j}. \tag{22}$$

It is not necessary to consider all the relations contained in equations (19)–(22). Following Chandrasekhar's arguments (EFE § 40), we need consider only the following relations which have an odd parity with respect to the index 1:

$$4\delta T_{12;2} + 2\delta W_{12;2} = 0, \tag{23}$$

$$4\delta T_{13;3} + 2\delta W_{13;3} = 0, \tag{24}$$

$$4(\delta T_{11;1} - \delta T_{22;1} - 2\delta T_{12;2}) + \delta S_{122} = 0, \tag{25}$$

$$4(\delta T_{11;1} - 2\delta T_{13;3}) + \delta S_{133} = 0. \tag{26}$$

The quantities $\delta T_{ij;k}$ and $\delta W_{ij;k}$ can be expressed in terms of the virial

$$V_{i;jk} = \int \rho \xi_i x_j x_k dx. \tag{27}$$

It is convenient to define the symmetrized virial

$$V_{ijk} = V_{i;jk} + V_{j;ki} + V_{k;ij}. \tag{28}$$

From the defining relation (4) we obtain

$$2\delta T_{ij;k} = \int \rho dx\, [-\xi_i (\boldsymbol{u} \cdot \nabla) u_j x_k - \xi_j (\boldsymbol{u} \cdot \nabla) u_i x_k + u_i u_j \xi_k - u_j u_k \xi_i - u_k u_i \xi_j]. \tag{29}$$

Substituting for μ_i from equation (1) and keeping in mind that Ω is now a constant we can distinguish the following cases

(*a*) $i \neq 3,\ j \neq 3$

$$2\delta T_{ij;k} = \Omega^2 \{V_{i;jk} + V_{j;ki} + (-)^{i+j} V_{k;i^*j^*} - (-)^{j+k} V_{i;j^*k^*} - (-)^{k+i} V_{j;k^*i^*}\} \tag{30}$$

(*b*) $i = 3,\ j \neq 3$

$$2\delta T_{3;jk} = \Omega^2 \{V_{3;jk} - (-)^{j+k} (1 - \delta_{k3}) V_{3;j^*k^*}\} \tag{31}$$

(*c*) $\delta T_{33;j} = 0. \qquad (32)$

Here, j^* is 2 if j is 1 and *vice versa*; and $3^* = 3$.

With the help of equations (30) and (31), we can express the various $\delta T_{ij;k}$ in terms of the $V_{i;jk}$. We shall now assume

$$\int \rho \xi_i dx = 0 \tag{33}$$

which ensures that the centre of gravity remains stationary. In view of equation (33), we can express $\delta W_{ij;k}$ in terms of the $V_{i;jk}$ (EFE § 23).

We consider equation (25) first. Making use of equation (30) and relations in § 23 of EFE, we obtain

$$(\Omega^2 - B_{11;1})(V_{1;11} - V_{122}) = 0, \tag{34}$$

where

$$B_{ij;k} = B_{ij} + a_k^2 B_{ijk}. \tag{35}$$

Accordingly a neutral point occurs when

$$\Omega^2 = B_{11} + a_1^2 B_{111} \tag{36}$$

for a displacement for which $V_{1;11} \neq V_{122}$.
Equation (36) is equivalent to

$$\frac{\sin^{-1} e}{e} = \frac{15 - 5e^2 + 118e^4 - 128e^6}{3(1-e^2)^{\frac{1}{2}}(5 + 40e^4 - 32e^6)} \tag{37}$$

and is satisfied when

$$e = 0{\cdot}73113 \quad \text{and} \quad \Omega^2 = 0{\cdot}30331. \tag{38}$$

The point of bifurcation (38) belonging to the third harmonics was obtained from an investigation of the complete frequency spectrum in Kochhar and Trehan (1974). Further, it should be noted that the corresponding point of bifurcation for the sequence of Maclaurin spheroids considered stationary in a rotating frame occurs when

$$\Omega^2 = 2(B_{11} + a_1^2 B_{111}) \tag{39}$$

which corresponds to

$$e = 0{\cdot}89926 \quad \text{and} \quad \Omega^2 = 0{\cdot}44015 \tag{40}$$

(EFE § 41).

We now consider equations (23), (24) and (26) which must be supplemented with the condition expressing the solenoidal nature of the Lagrangian displacement

$$\frac{1}{a_1^2}(V_{111} + V_{122}) + \frac{1}{a_3^2} V_{133} = 0. \tag{41}$$

To be consistent with equation (25) and exclude the point of bifurcation (36), we set

$$V_{1;11} = V_{122} = V_{1;22} + 2V_{2;12}. \tag{42}$$

In view of equation (42), equation (41) takes the form

$$\frac{4}{a_1^2} V_{1;11} + \frac{1}{a_3^2} V_{133} = 0 \tag{43}$$

Substituting the expressions for the relevant $\delta T_{ij;k}$ and $\delta W_{ij;k}$ in equations (23) and (24), we find that

$$2\Omega^2 V_{2;12} + 2(B_{11} + 3a_1^2 B_{111} - 2a_3^2 B_{113}) V_{1;11} = 0, \tag{44}$$

$$2\Omega^2 V_{3;13} - (2B_{13} + 3a_3^2 B_{133} - a_1^2 B_{113}) V_{133} = 0. \tag{45}$$

By virtue of the relations between the index symbols, we can easily verify that

$$2(B_{11} + 3a_1^2 B_{111} - 2a_3^2 B_{113}) = 2B_{13} + 3a_3^2 B_{133} - a_1^2 B_{113} = B\,(\text{say}) \tag{46}$$

Making use of equation (43) we can rewrite the two equations (44) and (45) as

$$8\Omega^2 (1 - e^2) V_{2;12} - BV_{133} = 0, \tag{47}$$

$$2\Omega^2 V_{3;13} - BV_{133} = 0. \tag{48}$$

from which we conclude that

$$V_{3;13} = 4(1 - e^2) V_{2;12}. \tag{49}$$

We now turn our attention to equation (26). It is convenient to combine it with equation (23) and write

$$4(\delta T_{11;1} - 3\delta T_{12;2} - 2\delta T_{13;3}) + \delta S_{133} - 6\delta W_{12;2} = 0 \tag{50}$$

which leads to

$$4\Omega^2 (1 - e^2)(3 - 8e^2) V_{2;12} + [\Omega^2 - 2(1 - e^2) C] V_{133} = 0, \tag{51}$$

where

$$C = 3[B_{33} + B_{13} + (a_1^2 + 2a_3^2) B_{133}] - \frac{a_1^2}{a_3^2}(2a_1^2 + a_3^2) B_{113}. \tag{52}$$

In arriving at equation (51) explicit use has been made of equations (41), (42) and (49).

Equations (47) and (51) lead to the following condition for the occurrence of a further neutral point

$$\Omega^2 = 2(1 - e^2) C - \tfrac{1}{2}(3 - 8e^2) B; \tag{53}$$

this is equivalent to

$$\frac{\sin^{-1} e}{e} = \frac{225 - 351 e^2 + 134 e^4 - 8e^6}{(1 - e^2)^{\frac{1}{2}} (225 - 276 e^2 + 72 e^4 - 16 e^6)} \tag{54}$$

and is met then

$$e = 0{\cdot}99608 \quad \text{and} \quad \Omega^2 = 0{\cdot}22209. \tag{55}$$

This point could not be isolated from the small perturbation analysis of Kochhar and Trehan (1974). It should be noted that long before this point of bifurcation is reached, the Maclaurin spheroids become dynamically unstable to both second and third harmonics. In the case of Maclaurin spheroids considered stationary in a rotating frame of reference, the corresponding point of bifurcation occurs at $e = 0{\cdot}96937$ where $\Omega^2 = 0{\cdot}4141$ [EFE § 41, eq. (68)].

5. Rotating axisymmetric masses with magnetic fields

We now consider the general case of a sequence of rotating axisymmetric masses with magnetic fields. We return to equation (16) and assume the following form for the Lagrangian displacement (EFE § 33 *d*)

$$\xi_i = L_{i;mn} x_m x_n - \tfrac{1}{3} L_{i;ss} a_s^2 \tag{56}$$

The particular form of the constant term here ensures a stationary centre of mass. The assumed solenoidal character of the Lagrangian displacement imposes the following restriction on the constants $L_{i;\,jk}$:

$$\sum_j L_{j;\,ji} = 0 \quad , \quad i = 1, 2, 3. \tag{57}$$

We should now express the various quantities occurring in equation (16) in terms of the $L_{i;\,jk}$. Substituting for ξ_i in equation (29), we obtain

$$\begin{aligned} ST_{ij;\,k} = {} & T_{ij;\,mn} L_{k;\,mn} - (T_{jk;\,mn} + Z_{j;\,kmn}) L_{i;\,mn} \\ & - (T_{ik;\,mn} + Z_{i;\,kmn}) L_{j;\,mn} - \tfrac{1}{5} T_{ij}\, a_s^2 L_{k;\,ss} \end{aligned} \tag{58}$$

where

$$Z_{i;\,jmn} = \tfrac{1}{2} \int \rho u_l \frac{\partial u_i}{\partial x_l} x_j x_m x_n \, dx. \tag{59}$$

By virtue of the axisymmetry of the configuration, the non-vanishing components of $Z_{i;\,jmn}$ satisfy

$$Z_{\omega;\,\omega jj} = - T_{\omega^*\omega^*;\,jj} \tag{60}$$

where ω and ω^* take the values 1 and 2, ω^* being 1 if ω is 2 and *vice versa*.

The expression for $\delta M_{ij;\,k}$ in terms of the $L_{i;\,jk}$ is [Kochhar 1974, eq. (18)].

$$\begin{aligned} \delta M_{ij;\,k} = {} & 2M_{js;\,mk} L_{i;\,ms} + 2M_{is;\,mk} L_{j;\,ms} + \\ & + M_{ij;\,mn} L_{k;\,mn} - 1/5\, M_{ij} a_s^2 L_{k;\,ss}. \end{aligned} \tag{61}$$

In writing equation (61) explicit use has been made of equation (57). The expressions for the $\delta W_{ij;\,k}$ are available in terms of the $V_{i;\,jk}$ which are related to the $L_{i;\,jk}$ as follows

$$V_{i;\,jk} = L_{i;\,mn} I_{mnjk} - 1/5\, L_{i;\,ss}\, a_s^2 I_{jk}. \tag{62}$$

Proceeding as in § 4 we need consider only the following equations

$$4\,(\delta T_{12;\,2} - \delta M_{12;\,2}) + 2\delta W_{12;\,2} = 0, \tag{63}$$

$$4\,(\delta T_{13;\,3} - \delta M_{13;\,3}) + 2\delta W_{13;\,3} = 0, \tag{64}$$

$$\begin{aligned} & 4\,(\delta T_{11;\,1} - \delta T_{22;\,1} - 2\delta T_{12;\,2}) - 4\,(\delta M_{11;\,1} - \delta M_{22;\,1} - 2\delta M_{12;\,2}) \\ & \quad + \delta S_{122} = 0, \end{aligned} \tag{65}$$

$$4\,(\delta T_{11;\,1} - 2\delta T_{13;\,3}) - 4\,(\delta M_{11;\,1} - \delta M_{33;\,1} - 2\delta M_{13;\,3}) + \delta S_{133} = 0. \tag{66}$$

The simultaneous (non-trivial) solution of these equations will lead to the necessary and sufficient conditions for the existence of points of bifurcation belonging to the third harmonics. Equation (65) leads to

$$\begin{aligned} & (6FI_{1122} - 3B_{11;\,1} I_{1122} + 2M_{11;\,11} + 2M_{12;\,12} - \\ & \quad - 6M_{22;\,11} - 12\, M_{12;\,12})\,(L_{1;\,11} - L_{1;\,22} - 2L_{2;\,12}) = 0. \end{aligned} \tag{67}$$

Thus a neutral point occurs when

$$3\left(B_{11}+a_1^2 B_{111}-2F\right) I_{1122}-2 M_{11;\,11}-$$
$$-2M_{12;\,12}+6M_{33;\,11}+12M_{13;\,13}=0 \tag{68}$$

for displacements for which $L_{1;\,11} \neq L_{1;\,22}+2L_{2;\,12} \neq 0$ and the remaining $L_{i;\,jk}$ vanish.

When rotation is uniform and magnetic fields are absent, equation (68) reduces to

$$2F=B_{11}+a_1^2 B_{111}, \tag{69}$$

which is equivalent to equation (36) by virtue of equation (9). We thus note that the trial function (56) is exact in the case of pure uniform rotation. When the magnetic field is purely toroidal, equation (68) again reduces to equation (69) (see remarks following equation (13)). Thus we conclude that the point of bifurcation remains unaffected by the presence of a toroidal magnetic field.

Now considering the case when a general magnetic field, having both toroidal and poloidal components, is present, we first note that the first term in equation (67) approaches zero at $e=0.73113$ from positive values and is a decreasing function of e. Thus the point of bifurcation would be moved to higher (lower) values of eccentricity if the magnetic terms are postive (negative). If the poloidal field is along the axis of rotation, $M_{11;\,11}+M_{12;\,12}=0$ and the magnetic terms are positive. The point of bifurcation is then moved to higher values of eccentricity.

These results should be compared with the corresponding results for second harmonics, where the effect of a poloidal field is always to raise the point of bifurcation (Kochhar and Trehan 1973). Further while it is only the component of the poloidal field along the axis of rotation that affects the point of bifurcation belonging to the second harmonics, all the components of a poloidal magnetic field affect the point of bifurcation (55), belonging to the third harmonics. Equations (63) and (64) lead to

$$\left(FI_{1122}-M_{11;\,11}-M_{12;\,12}-M_{33;\,11}-2M_{13;\,13}-\frac{B}{20} I_{1122}\right) L_{2;\,12}$$
$$-\left(M_{11;\,11}+M_{12;\,12}+\frac{3B}{20} I_{1122}\right) L_{3;\,13}+\left(M_{13;\,13}-\frac{B}{20} I_{1133}\right) \times$$
$$\times L_{1;\,33}=0, \tag{70}$$

$$\left(4M_{13;\,13}-\frac{B}{5} I_{1133}\right) L_{2;\,12}+\left(FI_{1133}-\frac{3B}{5} I_{1133}-M_{33;\,33}+2M_{13,\,13}\right) \times$$
$$\times L_{3;\,13}-L_{1;\,33}\left(M_{33;\,33}+\frac{B}{15} I_{3333}\right)=0. \tag{71}$$

It is convenient to combine equation (66) with equation (63) as follows

$$4\left(\delta T_{11;\,1}-3\delta T_{12;\,2}-2\delta T_{13;\,3}\right)-$$
$$-4\left(\delta M_{11;\,1}-3\delta M_{12;\,2}-\delta M_{33;\,1}-2\delta M_{12;\,3}\right)+\delta S_{123}-6\delta W_{1;\,22}=0 \tag{72}$$

Equation (72) yields

$$\left(\frac{24}{5}FI_{1122}+\frac{C}{5}I_{1133}-12M_{13;\,13}-4M_{33;\,11}\right)L_{2;\,12}$$
$$-\left[\frac{1}{5}(13-10e^2)FI_{1122}-\frac{3C}{5}I_{1133}+2M_{11,\,11}+2M_{12;\,12}\right.$$
$$\left.-2M_{13;\,13}-2M_{33;\,33}-3M_{33;\,11}\right]L_{3;\,13}$$
$$-\left(\frac{F}{5}I_{1133}-\frac{C}{15}I_{3333}-2M_{13;\,13}-2M_{33;\,33}\right)L_{1;\,33}=0. \tag{73}$$

The solution of equations (70), (71) and (73) leads to an equation of the type

$$f+g=0 \tag{74}$$

where the function g includes all the magnetic terms and vanishes when the magnetic field is absent or is toroidal. In either case equation (74) becomes equivalent to equation (53). Since f is a monotonically decreasing function of e, the point of bifurcation will be raised to values higher (lower) than $e=0.99608$ [(eq. (55)] if g is positive (negative). At this stage it should be recalled (Kochhar and Trehan 1971; Kochhar 1974) that a sufficiently large magnetic field can suppress the instability due to the second and the third harmonics, and the point of bifurcation (74) may lie in the stable region.

6. Conclusion

The points of bifurcation belonging to the third harmonics along the sequence of Maclaurin spheroids viewed from an inertial frame are isolated using the integral relations provided by the virial equations of various orders. These points are distinct from their counterparts for Maclaurin spheroids considered stationary in a rotating frame of reference and occur at eccentricity $e=0.73113$ and $e=0.99608$. The effect of magnetic fields on these points is then considered. A toroidal magnetic field leaves them unaffected, while a poloidal field consisting of a component only along the axis of rotation raises the first point of bifurcation beyond $e=0.73113$. A general poloidal magnetic field may raise or lower the points of bifurcation. These results should be compared with the corresponding results for the point of bifurcation belonging to the second harmonics, which is always raised by a poloidal magnetic field to values higher than the one obtaining in the case of uniform rotation. Further, while the point of bifurcation belonging to the second harmonics depends only upon the axial component of the poloidal magnetic field, the points of bifurcation belonging to the third harmonics depend on all the components of the poloidal magnetic field.

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