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Theory of Holographic Plane Gratings

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Theory of holographically recorded plane diffraction gratings (HRPDG) has been presented. In general HRPDG will have curved grooves with variable spacing. The self focussing property of HRPDG has been studied in detail. The curved form of the grooves is helpful in reducing the spectral image aberrations. Design parameters and aberrations are discussed in detail. It is found that by proper choice of design parameters, the aberrations can be reduced considerably and resolution can be improved.

1. Introduction

Holographically recorded diffraction gratings (HRDG) have opened a potential field for instruments designers. The attractive features of a holographic grating are large size, the number and pattern of grooves, low scattered light and possibility of aberration correction by controlling design parameters easily. The efficiency of the HRDG is not as good as that of the ruled grating but it is sufficient for various applications. One may improve the existing grating instruments in various fields such as astronomy, high resolution spectroscopy, information processing, etc. by the use of HRDG in place of conventional ruled gratings.

Various types of mechanically ruled diffraction gratings, studied theoretically and experimentally, have been reviewed by Namioka.[1] The theory of HRDG on spherical blanks and their applications to spectrographs and monochromators have been studied by Cordelle *et al.*,[2] Murty and Das,[3,4] Hayat and Pieuchard,[5] Namioka *et al.*[6,7] and Pouey.[8] A general geometric theory of HRDG, ray tracing through HRDG and application to Seya-Namioka monochromator have been studied recently by Noda *et al.*[9]

Murty *et al.*[3] have given the theory of certain diffraction gratings produced by holographic methods. They have described self-focussing plane gratings also, but their treatment cannot be considered a general one, covering all aspects.

Plane diffraction gratings suitable for spectrographic applications have been successfully produced first by Rudolph and Schmall[10] and by a group at Jobin-Yvon Inc. on photoresist coated optical surfaces. It has been shown by Laberie[2] that, holographically, by using a convergent beam, a self-focussing plane grating can be constructed.

The object of the present paper is to develop a general theory for self-focussing holographic plane diffraction gratings and to investigate, based on this theory, applications of these gratings for designing spectrographs and monochromators. The procedure followed is that of Noda *et al.*[9] for concave diffraction gratings.

2. Groove Patterns and Grating Constant of HRPDG

Plane holographic gratings are made by recording interference pattern on a plane grating blank coated with a suitable photoresist and using two monochromatic coherent beams. Let us take the origin of a rectangular coordinate system at the centre of the grating blank surface at O (Fig. 1), the x-axis being normal to the blank surface and the y- and the z-axis as shown in Fig. 1. $C(x_C, y_C, z_C)$ and $D(x_D, y_D, z_D)$

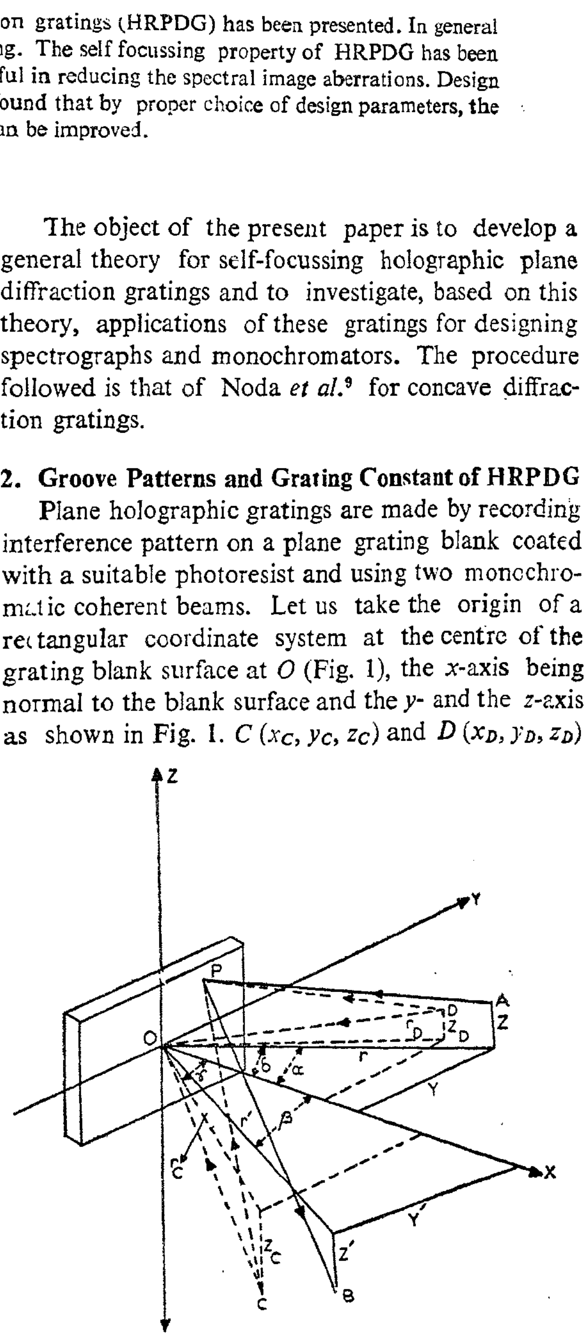

Fig. 1—Representation of the plane grating blank with respect to the rectangular coordinate system

are the recording sources. We assume the distances OC and OD to be integral multiples of λ_0, the wavelength of the recording laser light and that the zeroth groove passes through O. Then the nth groove is formed according to Ref. 9, at a distance

$$n\,\lambda_0 = [(CP) - (DP)] - [(CO) - (DO)] \quad ...(1)$$

where $P(0, w, l)$ is a point on the nth groove (see Ref. 9).

One can have specific desired form of the grooves by properly adjusting the design parameter, viz. (x_C, y_C) and (x_D, y_D). For obtaining straight grooves one should take $x_C, x_D \to \infty$ and $y_C, y_D \to \infty$. In general, the grating constant will be a function of both w and l. Let us take $\sigma(w, 0)$ grating constant, as defined by Noda *et al.*,[8] along the y-axis as

$$\sigma(w, 0) = \frac{w}{n} = \frac{w\,\lambda_0}{[(CP-DP)-(CO-DO)]_{l=0}} \quad ...(2)$$

The grating constant at the centre of the grating blank is given by $\sigma(0, 0)$ by evaluating Eq. (2) for $w = 0$. Using cylindrical coordinates $x_C = r_C \cos\gamma$, $y_C = r_C \sin\gamma$, $x_D = r_D \cos\delta$ and $y_D = r_D \sin\delta$, where $r_C^2 = x_C^2 + y_C^2$ and $r_D^2 = x_D^2 + y_D^2$, we get

$$\sigma(0, 0) = \frac{\lambda_0}{\left(1 + \frac{z_D^2}{r_D^2}\right)^{-1/2} \sin\delta - \left(1 + \frac{z_C^2}{r_C^2}\right)^{-1/2} \sin\gamma} \quad ...(3)$$

$$= \lambda_0/T$$

where T represents the denominator of RHS of Eq. (3). In order to keep σ_0 positive,

$$\sin\delta > \left(1 + \frac{z_C^2}{r_C^2}\right)^{-1/2} \left(1 + \frac{z_D^2}{r_D^2}\right)^{1/2} \sin\gamma \quad ...(4)$$

Now using Eqs. (2) and (3) we can write

$$\sigma(w, 0) = \sigma(0, 0) \frac{T}{\left\{\frac{1}{w}\left[(CP-DP) - (CO-DO)\right]_{l=0}\right\}} \quad ...(5)$$

and

$$n = [(CP - DP) - (CO - DO)]/\sigma(0, 0).\ T \quad ...(6)$$

3. Light Path Function

Let $A\,[x, y, z)$ and $B\,(x', y', z')$ be respectively a point light source and an image point as shown in Fig. 1. For the ray APB the light path function F is given by

$$F = (AP) + (PB) + nm\lambda \quad ...(7)$$

where n, m and λ are respectively the number of the grooves counting from the centre of grating blank, order of the spectrum and the wavelength of light. Eq. (7) can be transformed in terms of the cylindrical coordinates of the points $A\,(r, \alpha, z)$ and $B\,(r', \beta, z')$ with α and β respectively the angles of incidence and that of diffraction measured in the xy-plane.

By the application of Fermat's principle, viz. $(\partial F/\partial w) = 0$ and $(\partial F/\partial l) = 0$, we can easily obtain [with the condition $(z_C/r_C) = (z_D/r_D)$], the grating equation, magnification equation, horizontal focal curve relation and vertical focal curve relation, as follows:

$$\left(1 + \frac{z^2}{r^2}\right)^{-1/2} (\sin\alpha + \sin\beta_0) = \frac{m\lambda}{\sigma_0} \quad ...(8)$$

$$\frac{z}{r} = -\frac{z'_0}{r'_0} \quad ...(9)$$

$$\frac{\cos^2\alpha}{r} + \frac{\cos^2\beta}{r'} - \frac{(\sin\alpha + \sin\beta)}{(\sin\delta - \sin\gamma)} \left(\frac{\cos^2\delta}{r_D} - \frac{\cos^2\gamma}{r_C}\right) = 0 \quad ...(10)$$

$$\frac{1}{r} + \frac{1}{r'} - \frac{\sin\alpha + \sin\beta}{\sin\delta - \sin\gamma} \left(\frac{1}{r_D} - \frac{1}{r_C}\right) = 0 \quad ...(11)$$

We define here

$$R_e = \frac{\sin\delta - \sin\gamma}{\frac{\cos^2\delta}{r_D} - \frac{\cos^2\gamma}{r_C}} \quad ...(12)$$

With this notation we get the following solutions for Eq. (10)

$$\left.\begin{aligned} r &= R_e \frac{\cos^2\alpha}{\sin\alpha} \\ r' &= R_e \frac{\cos^2\beta}{\sin\beta} \end{aligned}\right\} \quad ...(13)$$

and

$$r = \infty$$

$$r' = R_e \frac{\cos^2\beta}{\sin\alpha + \sin\beta} \quad ...(14)$$

The quantity R_e behaves as the radius of curvature. Obviously, for a self-focussing grating, R_e should be finite and positive.

If we take $R_e = \infty$, Eq. (10) reduces to

$$\frac{\cos^2\alpha}{r} + \frac{\cos^2\beta}{r'} = 0 \quad ...(15)$$

which is the well known relation for an ordinary ruled plane diffraction grating with constant spacing and straight grooves. Earlier work[11,12] on curved grooves shows that the self-focussing property is not the result of curved shape of the grooves which is only helpful in reducing aberrations.

It is evident from Eq. (13) that in this case the source curve and focal curve are the same, just as in the case of concave grating (Rowland circle). Similarly, Eq. (14) gives the focal curves as in the case of Wadsworth mounting. Hereafter we will refer to mounting given by Eq. (13) as type I and that given by Eq. (14) as type II mounting.

For type I mounting, we get the condition for zero astigmatism as

$$\frac{\frac{\cos^2 \delta}{r_D} - \frac{\cos^2 \gamma}{r_C}}{\frac{1}{r_D} - \frac{1}{r_C}} = \frac{\sin \alpha + \sin \beta}{\tan \alpha \sec \alpha + \tan \beta \sec \beta} = D_1 (\alpha, \beta) \quad ...(16)$$

For brevity, the LHS in Eq. (16) will hereafter be denoted by $f_1 (r_D, r_C, \delta, \gamma)$.

For type II mounting, we get the condition for zero astigmatism as

$$f_1 (r_D, r_C, \delta, \gamma) = \cos^2 \beta = D_1' \quad ...(17)$$

In this case, there will be two points on the focal curve for $\pm \beta$ at which astigmatism is zero except for the case $\beta = 0$, whereas in type I mounting there is only one point at which astigmatism is zero.

4. Aberrations

The amount of aberration in an image in a plane located at a distance r_0' from O and perpendicular to the diffracted principal ray can be easily computed from Gaus-Seidel theory, say up to O $(1/w^4)$ and the displacements $\Delta\beta$ and $\Delta z'$ in the horizontal and vertical directions respectively, from the position (r_0', β_0, z_0') specified by Eqs. (8) and (9) can be computed under the usual assumption $z \ll r, r_0'$.

4.1 Type I Mounting

Let L be the total length of a groove projected on the z-axis. From Eq. (11), we get for the length of astigmatic images due to a point source,

$$[z']_{ast} = L\left[1 - \frac{\cos^2 \beta}{f_1} \left\{1 + \left(1 - \frac{f_1}{\cos^2 \alpha}\right) \frac{\sin \alpha}{\sin \beta}\right\}\right] \quad ..(18)$$

From Eq. (18), $[z']_{ast}$ depends upon r_D, r_C, δ and γ recording parameters. So astigmatism can be minimized by proper choice of these parameters. Fig. 2 represents the values of function $D_1 (\alpha, \beta)$ at different wavelengths. In Fig. 2 the solid curves represent $D_1 (\alpha, \beta)$ and the dashed lines, wavelengths in μm at different angles of incidence and diffraction. A proper value of $D_1 (\alpha, \beta)$ to get zero astigmatism at a particular wavelength can be selected from Fig. 2. Fig. 3 represents $f_1 (r_D, r_C, \delta, \gamma)$ at different values of δ and γ, when $r_C = 2 r_D$. Putting the value of $f_1 = D_1$, from Fig. 3, one can get the values of δ and γ at different values of σ_0, taking $\lambda_0 = 0.6328$ μm (—·—·—) curve) and $\lambda_0 = 0.4579$ μm (— — — curve). In this way one can determine the set of recording parameters for zero astigmatism at a particular wavelength. Thus it is obvious that there exists a set of parameters for which the grating will be better as compared to other sets at a particular wavelength.

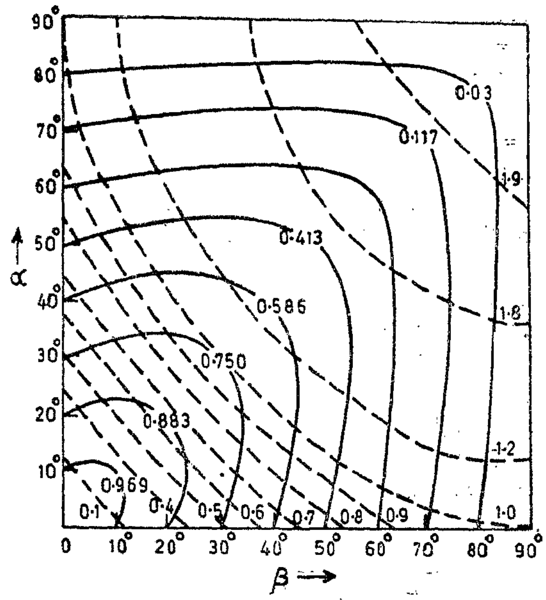

Fig. 2—D_1 (α, β) at different wavelengths (——) and λ(α,β) (– – –) for different α and β. wavelength in μm (σ_0=1μm)

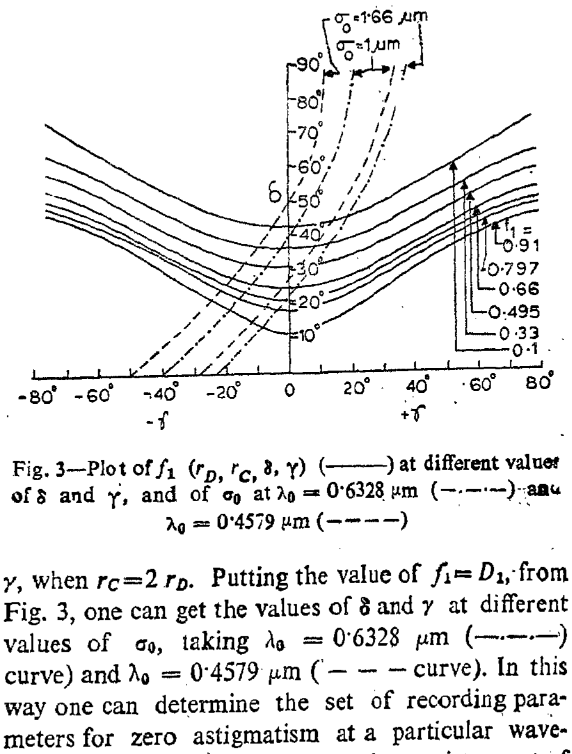

Fig. 3—Plot of f_1 (r_D, r_C, δ, γ) (——) at different values of δ and γ, and of σ_0 at λ_0 = 0·6328 μm (–·–·–) and λ_0 = 0·4579 μm (– – –)

Now as HRPDG can be equivalent to that of concave diffraction gratings, it will be worthwhile to compare the astigmatism of a HRPDG and a mechanically ruled concave diffraction grating with straight grooves and a holographically recorded concave diffraction grating (HRCDG) that is astigmatism corrected on a spherical surface. We assume $\sigma_0 = 1$ μm and $\alpha = 10°$. Let us also assume that by proper choice of the recording parameters in the case of HRPDG and HRCDG the astigmatism is zero at $\lambda = 0{\cdot}5156$ μm, i.e. $\beta = 20°$. Table 1 shows the values of astigmatism $[z'_{ast}]/L$ in the case of these three diffraction gratings. It can be seen from Table 1 that the astigmatism in the case of HRPDG is less as compared to that of mechanically ruled concave diffraction grating and greater than that of HRCDG. In this example, we have taken $r_C = 2r_D, f_1 = 0{\cdot}91$; and $\delta = 18°\,6'$, $\gamma = -18°\,47'$ when $\lambda_0 = 0{\cdot}6328$ μm and $\delta = 15°$, $\gamma = -11{\cdot}5°$ when $\lambda_0 = 0{\cdot}4579$ μm; $\alpha = 10°$, $\beta = 0°$ to $35°$.

The astigmatism cannot be made zero for $r_C = r_D$, because in this case $f_1 = \infty$ and

$$\frac{[z']_{ast}}{L} = \left[1 + \frac{\cos^2\beta \sin\alpha}{\cos^2\alpha \sin\beta}\right]$$

The most troublesome aberration of mechanically ruled plane diffraction gratings with constant spacing is coma. But in the case of HRPDG we can reduce or eliminate for a particular wavelength this type of aberration also. For a point source it is given, to a first approximation by

$$\Delta y_C = \frac{l^2}{2R_e}\,\frac{(1+3\sin^2\beta)^{1/2}}{\sin^2\beta}\left[\frac{\sin^3\alpha}{\cos^4\alpha} + \frac{\sin^3\beta}{\cos^4\beta} - \frac{R_e^2(\sin\alpha+\sin\beta)}{(\sin\delta-\sin\gamma)}\left(\frac{\sin\delta}{r_D^2} - \frac{\sin\gamma}{r_C^2}\right)\right] \quad ...(19)$$

For elimination of coma type aberration we have to choose recording parameters such that

$$D_2(\alpha, \beta) = f_2 \quad ...(20)$$

where

$$D_2(\alpha, \beta) = \frac{\tan^3\alpha \sec\alpha + \tan^3\beta \sec\beta}{\sin\alpha + \sin\beta}$$

Table 1—Values of Astigmatism in Case of Three Diffraction Gratings with Different Wavelengths ($\alpha = 10°$, $\sigma_0 = 1\mu$m)

β°	Wavelength, μm	Holographic plane grating	Mechanically ruled concave grating	Holographic concave grating
0	0·1736	—	0·0306	−0·0216
5	0·2608	−0·2240	0·0381	−0·0401
10	0·3472	−0·1320	0·0603	−0·0426
15	0·4324	−0·0670	0·0966	0·0292
20	0·5156	—	0·1458	—
25	0·5962	0·0751	0·2063	0·0436
30	0·6736	0·1590	0·2765	0·1009
35	0·7472	0·2491	0·3542	0·1699

and

$$f_2 = R_e^2\left(\frac{\sin\delta}{r_D^2} - \frac{\sin\gamma}{r_C^2}\right)\Big/(\sin\delta - \sin\gamma) \quad ...(21)$$

By calculating the values of the LHS of Eq. (20), for different sets of α and β, i.e. for different wavelengths, we can determine the right hand factor $f_2(r_D, r_C, \delta, \gamma, \lambda_0)$ which is a function of r_D, r_C, δ, γ and λ_0 parameters.

In an analogous manner by plotting $D_2(\alpha, \beta)$ at different wavelengths and also $f_2(r_D, \gamma_D, \delta, \gamma, \lambda_0)$ at different δ and γ values at $r_C = 2r_D$, one can obtain the set of parameters at different wavelengths for elimination of coma at these particular wavelengths.

For elimination of astigmatism and coma simultaneously, the following condition should be satisfied for selecting the recording parameters:

$$D_2\left[\cos^2\delta - \cos^2\gamma\left(\frac{D_1 - \cos^2\delta}{D_1 - \cos^2\gamma}\right)\right] = (\sin\delta - \sin\gamma)\left[\sin\delta - \sin\gamma\left(\frac{D_1 - \cos^2\delta}{D_1 - \cos^2\gamma}\right)^2\right] \quad ...(22)$$

The equation for the secondary focal curve is given by

$$r'_h = \cos^2\alpha \Big/ f_2\left[\frac{\cos^2\alpha(\sin\alpha + \sin\beta)}{f_1} - \sin\alpha\right] \quad ...(23)$$

We see that these secondary focal curves cut the primary focal curves only at one wavelength, i.e. zero astigmatism is achieved only at one wavelength.

The optimum grating width is given by

$$W_{opt} = \left(\frac{\sigma_0}{2m}\right)^{1/3}\left[\frac{R_e^2(\sin\alpha + \sin\beta)}{\tan^2\alpha \sin\alpha + \tan^2\beta \sin\beta - B(\sin\alpha + \sin\beta)}\right]^{1/3}$$

where

$$B = \left(\frac{\sin\delta\cos^2\delta}{r_D^2} - \frac{\sin\gamma\cos^2\gamma}{r_C^2}\right) R_e^2\Big/(\sin\delta - \sin\gamma) \quad ...(24)$$

It is evident from the above result that by proper selection of R_e and B, i.e. recording parameters, the resolving power, $0{\cdot}95\,(m/\sigma_0)\,W_{opt}$, of the HRPDG can be considerably increased.

For maximum resolving power one should have

$$D_3(\alpha, \beta) = B(r_C, r_D, \delta, \gamma) \quad ...(25)$$

where

$$D_3(\alpha, \beta) = \frac{(\tan^2\alpha \sin\alpha + \tan^2\beta \sin\beta)}{(\sin\alpha + \sin\beta)}$$

Fig. 4 is a plot of $D_3(\alpha, \beta)$ (solid curve) at different wavelengths (dotted curve). Fig. 5 shows the

Table 2—Parameters for the Design of a Typical HRPDG for Type I Mounting

r_C	$= 2 r_D$
δ	$0°$
γ	$-30°$
R_e	$0{\cdot}8\, r_D$
σ_0	(1) 1·2656 μm (λ_0=0·6328 μm)
	(2) 0·9158 μm (λ_0=0·4579 μm)

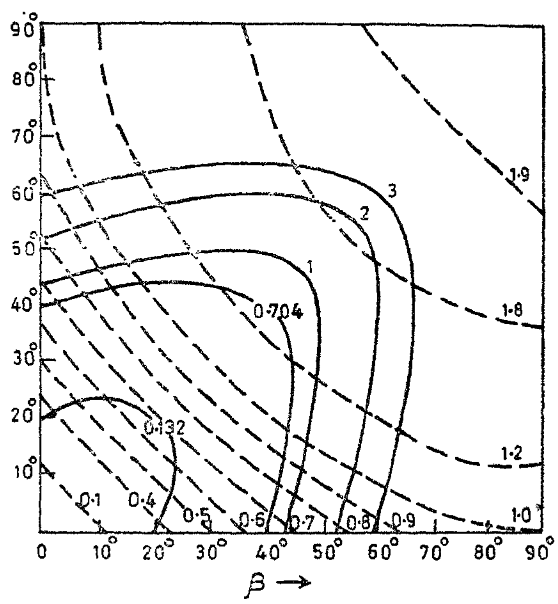

Fig. 4—Variation of D_3 (α, β) with wavelength (———) and λ (α, β) (– – –). Wavelength is in μm and σ_0 = 1 μm

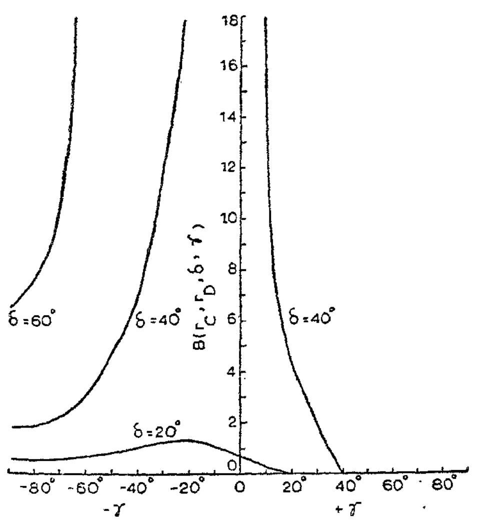

Fig. 5—Variation of B (r_C, r_D, δ, γ) with γ for different values of δ, with $r_C = 2\, r_D$

variation of B (r_C, r_D, δ, γ), with γ and for different values of δ for $r_C = 2\, r_D$. For maximum resolution at a particular wavelength one can find out from Fig. 4 the value of D_3 and from Fig. 5 at $D_3 = B$, one can find out the suitable values of δ and γ at $r_C = 2\, r_D$.

4.2 Type II Mounting

In a way similar to those given above one can find out the various relations for type II mounting

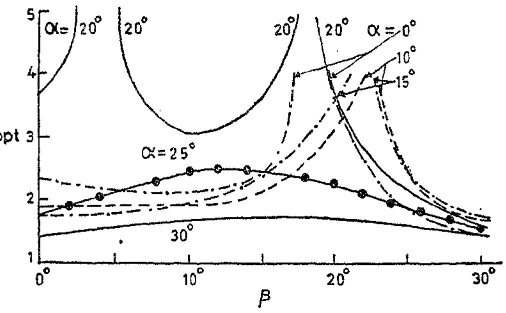

Fig. 6—Variation of W_{opt} with β for different values of α [Table 2]

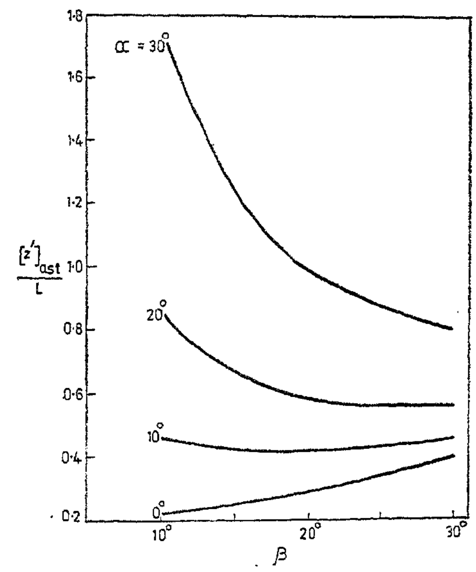

Fig. 7 –Variation of $[z']_{ast}/L$ with β for different values of α

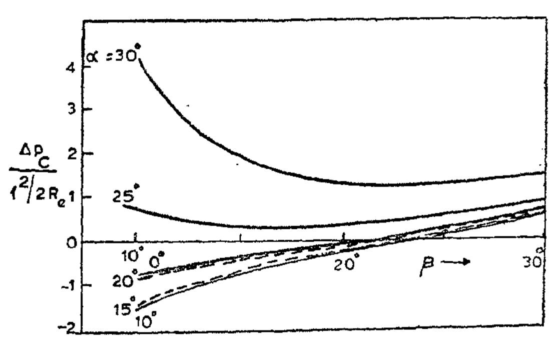

Fig. 8—Variation of $\Delta p_C/(l^2/2R_e)$ with β for different values of α

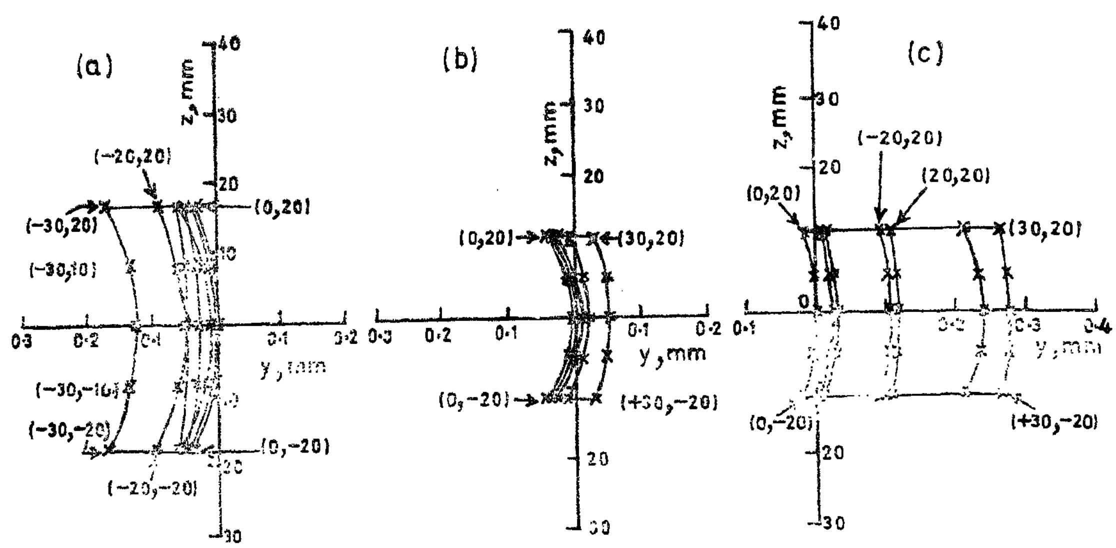

Fig. 9—Images obtained by ray tracing for the example studied (Table 2) [λ_{image} in µm: (a) 0·65264; (b) 0·86574; and (c) 1·06568]

and hence the required design parameters for a particular problem. From a plot of secondary focal curves, we see that these curves cut the corresponding primary focal curves at two points. That is in this case we can get stigmatic images at two wavelengths.

The optimum grating width is given by

$$W_{opt} = \left(\frac{\sigma_0}{2m}\right)^{1/3}\left[\frac{R_e^2}{\tan\beta\,(\sin\alpha + \sin\beta)\,\sec\beta - B}\right] \quad ...(26)$$

It is obvious from this expression that by proper choice of R_e and B, i.e. recording parameters, the resolution of HRPDG can be improved.

5. Design of Gratings

Now we shall apply the above treatment for designing HRPDG, and give one suitable example for type I mounting. The parameters for this example are given in Table 2. Figs. 6, 7 and 8 represent the variation of W_{opt}, $[z']_{ast}/L$ and Δp_C with β for different values of α. It is found that $\alpha = 20°$ is the most suitable value for this design covering the spectrum from $\beta = 0$-$30°$. The coma is zero at one wavelength and reduced at other nearby wavelengths. The astigmatism is very high in this case. In general, the W_{opt} is 1·5 cm for $R_e = 100$ cm. At $\alpha = 20°$, W_{opt} is infinite at two wavelengths. These results shown in Figs. 6, 7 and 8 are quite general, that is, for example, if W_{opt} at other values of R_e are required, the values given in Fig. 6 should be multiplied by $R_e^{2/3}$. In these design parameters one can choose the suitable value of r_D to make the grating as fast as other commercially available gratings.

6. Ray-Tracing and Spot Diagrams

Following Noda *et al.*,[9] the spot diagrams for this example at $\lambda_{image} = 0·65264$ µm, 0·86574 µm and 1·06568 µm are presented in Figs. 9 (a), (b) and (c), respectively. These diagrams are only for $\lambda_0 = 0·6328$ µm and the corresponding parameters as shown in Table 2. It is evident from these diagrams that the results predicted earlier in this paper are correct.

7. Conclusion

It is clear from this study that HRPDG can be a useful optical element, comparable to a concave diffraction grating. The self-focussing property of HRPDG can be usefully exploited for designing new types of spectrographs and monochromators. Details of design etc. for the mountings suitable for such spectrographs and monochromators based on these investigations will be communicated elsewhere.

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