Phase diagram of a bosonic ladder with two coupled chains.

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We study a bosonic ladder with two coupled chains using the finite size density matrix renormalisation group method. We show that in a commensurate bosonic ladder the critical on-site interaction (U_C) for the superfluid to Mott insulator transition becomes larger as the inter-chain hopping ($t_\perp$)increases. We analyze this quantum phase transition and obtain the phase diagram in the $t_\perp - U$ plane.

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I. INTRODUCTION

Quantum phase transitions in ultracold atoms provide important insights into the behaviour of matter at very low temperatures [1, 2, 3, 4]. An important example of this class of transitions is the transition from a superfluid(SF) to a Mott insulator(MI) which has been observed in cold bosonic atoms in $3D$ optical lattices [5, 6] as predicted by Jaksch *et al* [7]. Subsequently, this transition has been observed in a $1D$ optical lattice [6]. Detailed theoretical studies of this transition have been carried out by Pai *et al* [8, 9]. An important question to address is how do the characteristics of this transition alter in going from one to two dimensions. In coupled bosonic chains, competition exists between the ratio of atomic interactions to the intra-chain hopping and the inter-chain hopping. A large value of the former favors a Mott insulator state overcoming the effect of the inter-chain hopping while if the latter dominates it would tend to delocalize the bosons and drive the system to a superfluid state. It is not practical to perform numerical studies for the above mentioned transitions in a very large number of coupled chains, so one must restrict to a finite number of such chains. The aim of the present work is to study the effect of the inter-chain hopping on the SF-MI transition for a bosonic ladder consisting of two coupled chains. Although a substantial amount of theoretical and numerical work has been done in this direction for the case of spin-less fermionic ladders and spin ladders [10, 11, 12],no work has been done to our knowledge

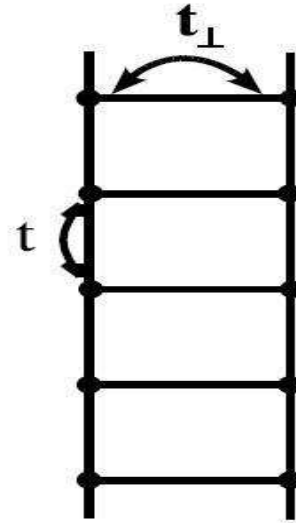for the bosonic ladders except a recent work using the Bosonization method. [13, 14]. The Hamiltonian of the

FIG. 1: Schematic picture of a two-leg bosonic ladder. t and $t_\perp$ are, respectively, inter-chain and intra-chain hopping amplitudes.

bosonic ladder(as shown in Fig.1)is given by

$$\mathcal{H} = -t\sum_{i,\alpha}(a^\dagger_{i,\alpha}a_{i+1,\alpha} + h.c)$$
$$+\frac{U}{2}\sum_{i,\alpha}n_{i,\alpha}(n_{i,\alpha} - 1)$$
$$-t_\perp\sum_i(a^\dagger_{i,1}a_{i,2} + h.c). \qquad (1)$$

In this model (1) $a^\dagger_{i,\alpha}$ ($a_{i,\alpha}$) represents bosonic creation (annihilation) operator for the site i of the chain with

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index $\alpha = 1, 2$. t and U are the intra-chain hopping amplitude between the nearest neighboring sites of chain α and the on-site interaction between the bosons respectively. The last term in this model (1) represents inter-chain hopping with an amplitude $t_\perp$ between corresponding sites on the two chains. We set our energy scale by taking $t = 1$.

This model has been studied using the Bosonization technique [13, 14] at or close to commensurate filling of one boson per site. This study predicts a transition from a Mott insulator to a superfluid phase when the inter-chain hopping is increased and it is in the Beresenskii-Kosterlitz-Thouless (BKT) universality class at commensurate filling. In the present work we verify these predictions and thus complement the earlier analytical results. For this purpose, we study the variation of critical on-site interaction U_C for the superfluid phase to the Mott insulator phase transition with the change in the inter-chain hopping amplitude $t_\perp$ using the finite size density matrix renormalization group method (FSDMRG) [9, 15, 16]and obtain the phase diagram in the $(t_\perp - U)$ plane. To the best of our knowledge, the present work is the first application of FSDMRG method to bosonic ladders.

The remaining part of the paper is organized in the following manner.The FSDMRG method in the context of bosonic ladders is briefly described in section II. Our results are described and discussed in Section III and our conclusions are stated in Section IV.

II. FSDMRG METHOD

We use the FSDMRG technique to obtain the energies and the correlation functions of the ground state. This method is very efficient and has proven to give accurate results for 1D quantum lattice systems and has been applied to low-dimensional strongly correlated fermionic and bosonic systems [9, 17]. We give below some pertinent details of this method adapted to the system of two coupled chains that we have considered.

We begin with a super-block configuration $B^l_{\frac{L}{2}-1} \bullet \bullet B^r_{\frac{L}{2}-1}$ of L rungs as shown in Fig.2. The left $B^l_{\frac{L}{2}-1}$ and the right block $B^r_{\frac{L}{2}-1}$ have $\frac{L}{2} - 1$ rungs each and the $\bullet$ represents one rung of two sites, one from each chain. Thus in every iteration, the new left and right blocks are $B^l_{\frac{L}{2}} = B^l_{\frac{L}{2}-1} \bullet$ and $B^r_{\frac{L}{2}} = \bullet B^r_{\frac{L}{2}-1}$ respectively. We increase the size of the system in every iteration by adding two rungs which increases the number of lattice sites by 4. To keep the density $\rho = 1$ fixed, we also increase the number of bosons in the system by 4. The truncation of states of left (right) block in each iterations corresponds to choosing M highest weighted states out of $2 \times n_{\max} \times M$ of the left (right) density matrix. Here $n_{\max}$ is the number of states kept at each site, which is in general infinity, but we truncate it for a feasible numerical calculation. We keep $n_{\max} = 4$ in this calculation which is found to be

sufficient for the values of U considered here [9]. The value of M is chosen such that the truncation error in our calculation is always less than 10^{-5}.
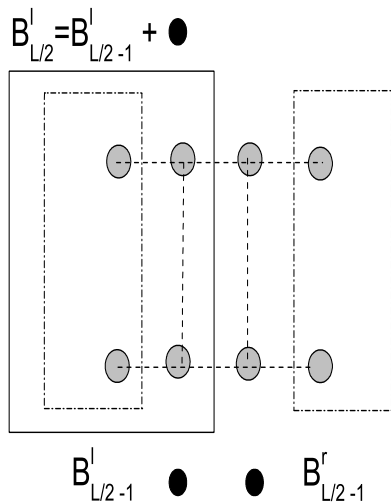

FIG. 2: A scheme of superblock configuration for the FSDMRG algorithm

III. RESULTS AND DISCUSSION

The Mott insulator phase has a finite gap in its energy spectrum. The single particle gap is defined as

$$G_L = E_L(N+1) - E_L(N) - (E_L(N) - E_L(N-1)) \quad (2)$$

where $E_L(N)$ is the ground-state energy of two leg bosonic ladder with length L having N bosons. The MI phase is signaled by the opening up of the gap $G_{L\to\infty}$. However, G_L is finite for finite systems and we must extrapolate to the $L \to \infty$ limit, which is best done by using finite-size scaling [9]. In the critical region, i.e., SF region, the gap

$$G_L \approx L^{-1} f(L/\xi), \quad (3)$$

where the scaling function $f(x) \sim x$, $x \to 0$ and ξ is the correlation length. $\xi \to \infty$ in the SF region. Thus plots of LG_L versus U, for different system sizes L, consist of curves that intersect at the critical point at which the correlation length for $L = \infty$ diverges and gap G_∞ vanishes. The phase diagram, as discussed below, is obtained from such plots.

It is now well known that the single chain Bose-Hubbard model with density $\rho = 1$ shows a SF-MI transition with the critical on-site interaction $U_C \sim 3.4$[9]. In order to understand the effect of the inter-chain hopping on this transition, we varied $t_\perp$ from 0 to 20 and

obtained the corresponding critical on-site interaction $U_C(t_\perp)$ for the SF-MI transition. We found that U_C increases with $t_\perp$ and saturates in the limit $t_\perp \to \infty$. These results are highlighted by the plots of scaling of gap LG_L versus U for different values of $t_\perp$ and lengths L. For example, in the Fig.3 we plot LG_L versus U for $t_\perp = 0.4$. The coalescence of LG_L curves for different values of L below $U < 6.6$ demonstrates the SF-MI transition with $U_C(t_\perp = 0.4) \sim 6.6$ which is much larger than the corresponding value for the single chain $U_C(t_\perp = 0) \sim 3.4$. Fig.4 represents similar plots for $t_\perp = 1$. For this case the critical on-site interaction increases further to $U_C((t_\perp = 1) \sim 7.9$.
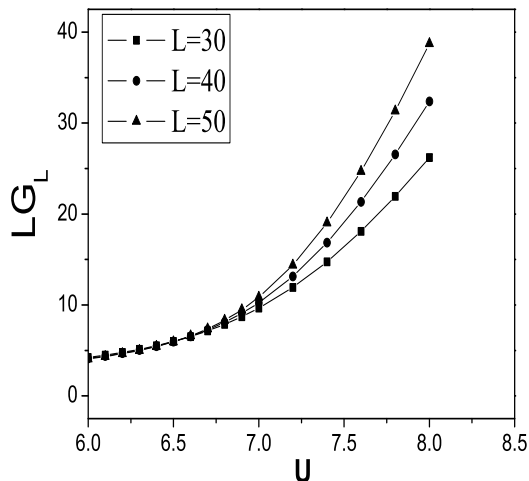

FIG. 4: Scaling of gap LG_L as a function of U for $t_\perp = t = 1.0$ and different lengths. The coalescence of curves for different lengths for $U < U_C \sim 7.9$ shows a superfluid phase and a Mott insulator with finite gap for $U > U_C$. Comparing this figure with Fig.3, we observe that the critical U_C increases with $t_\perp$.

FIG. 3: Scaling of gap LG_L as a function of U for $t_\perp = 0.4$ and different lengths. The coalescence of curves for different lengths for $U < U_C \sim 6.6$ shows a superfluid phase and a Mott insulator with finite gap for $U > U_C$.

From similar plots of LG_L versus U, we obtain the phase diagram for model (1) in the $t_\perp - U$ plane and it can be seen in Fig.5. U_C for the SF-MI transition initially increases sharply as the inter-chain hopping $t_\perp$ increases. This phase diagram verifies the prediction of MI-SF transition with respect to increase in the inter-chain hopping $t_\perp$[10]. For higher values of $t_\perp$, U_C tends to saturate. For $t_\perp \gg t, U$, each rung has two one particle states: corresponding to bonding or anti-bonding. As predicted in the Bosonization [10] study, this problem then maps onto a single chain Bose Hubbard model with commensurate density $\rho = 2$ and on-site interaction $U/2$. To confirm this prediction we plot the variation of U_C with respect to $t_\perp$ in Fig.6. Critical on-site interaction U_C for large $t_\perp$ converges to a value equal to 12.5 ± 0.3. Plotting LG_L versus U for single chain Bose-Hubbard model for $\rho = 2$ in Fig.7 we find that $U_C \sim 6.3$, which is one half the converged value of $U_C(t_\perp = \infty) \sim 12.5$ for the bosonic ladder confirming the prediction made in
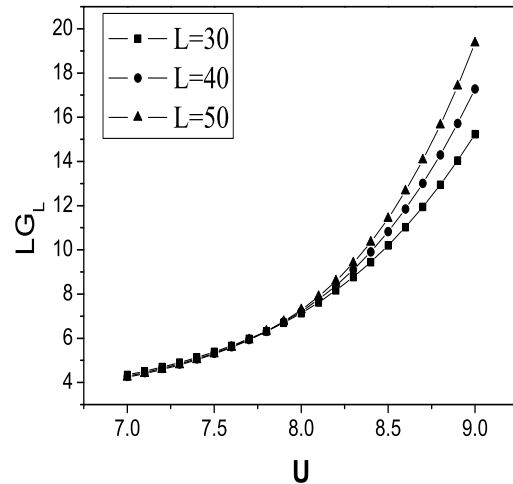
Bosonization study[10].

The Mott insulator to superfluid transition is found to be Beresinskii-Kosterlitz-Thouless (KT) universality class at commensurate filling[10]. The correlation function that characterizes the superfluid phase is given by $\Gamma_\alpha(r) = \langle a_{i,\alpha}^\dagger a_{i+r,\alpha} \rangle$ which decays as a power law in the limit $r \to \infty$. Here the expectation value is taken with respect to the ground state. However, in the Mott insulator phase it has an exponential decay due to the finite gap in the energy spectrum. The power law decay of this correlation function has been obtained using the Bosonization method [10] and it is predicted to go as

$$\Gamma_\alpha(r) \propto \frac{1}{r^{1/4K_s}} \qquad (4)$$

with the Luttinger Liquid parameter K_s stated to be 1 at the superfluid to Mott insulator transition point.

In order to obtain the Luttinger Liquid parameter K_s we fix $U = 6$ and vary the inter-chain hopping $t_\perp$ and obtain the Mott insulator to superfluid transition. The scaling of gap LG_L as a function of $t_\perp$ is given in Fig.8. The critical inter-chain hopping $t_\perp^C = 0.24 \pm 0.05$ for the MI to SF transition. The correlation functions $\Gamma_\alpha(r)$ for $U = 6$ different values of $t_\perp$ are given in Fig.9. The Luttinger Liquid parameter K_s which is obtained by fitting $\Gamma_\alpha(r)$ with the expression given in Eq.4 is plotted as a function of $t_\perp$ in Fig.10. From these values the critical $t_\perp^C$ for which $K_s = 1$ is given by 0.3 ± 0.03 which is consistent with values obtained from the scaling of the gap.
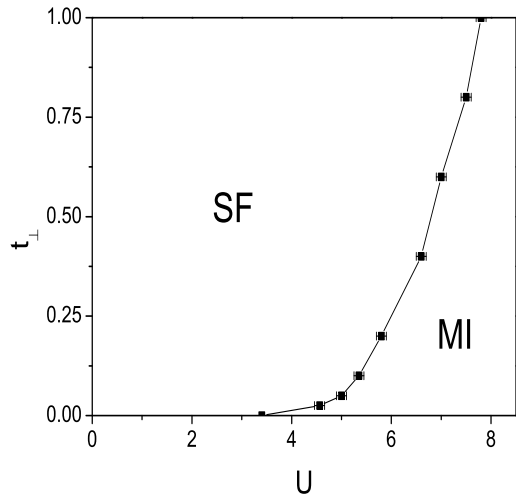
The results we have obtained could have experimen-

FIG. 5: Phase diagram of model (1) as a function of inter-chain hopping $t_\perp$ and on-site interaction U for density $\rho = 1$. Note that we have set intra-chain hopping $t = 1$.
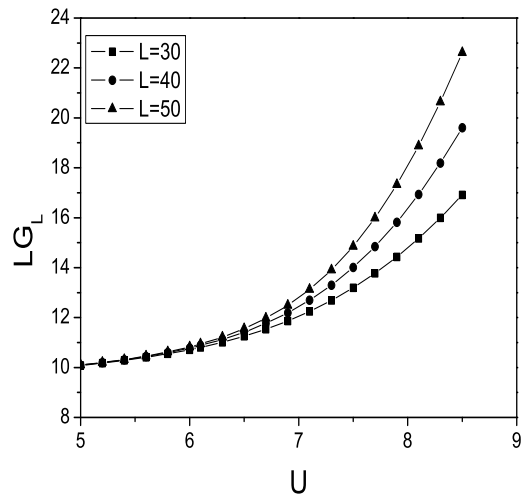

FIG. 7: Scaling of gap LG_L as a function of U for the single chain Bose-Hubbard model with density $\rho = 2$. The coalescence of curves for different lengths for $U < U_C \sim 6.3$ shows a superfluid to a Mott insulator transition.
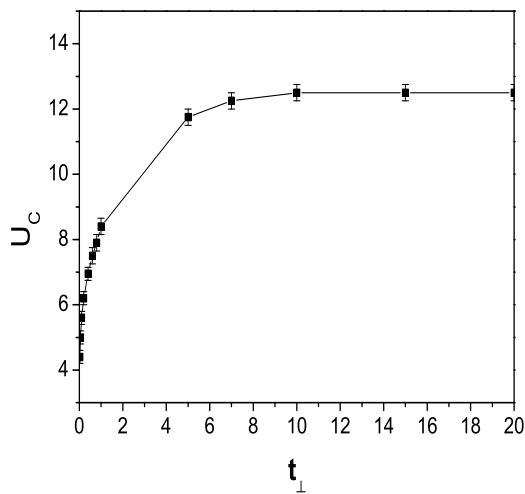

FIG. 6: Variation of critical on-site interaction U_C with respect to inter-chain hopping $t_\perp$. U_C increases sharply for small values of $t_\perp$ and saturate to 12.5 ± 0.3 as $t_\perp \to \infty$.
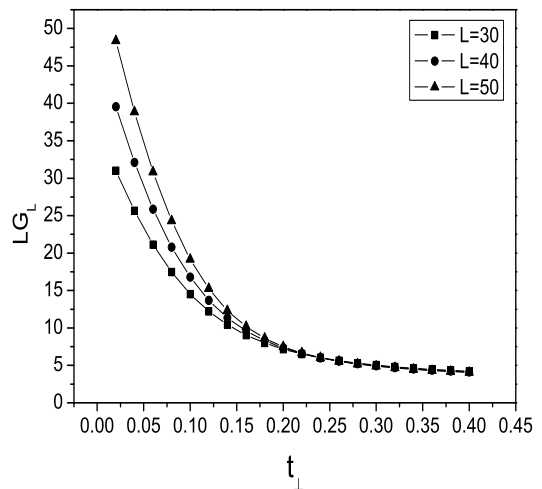

FIG. 8: Scaling of LG_L as a function of $t_\perp$ for $U = 6$.

IV. CONCLUSIONS

tal implications. It is now possible to prepare bosonic ladders by growing optical superlattices in the form of double well potential along one direction[18]. The tunneling between the double well potential will control the inter-chain hopping. By changing dynamically the optical lattice parameters one can control all the interaction and hopping parameters of the model (1).

We have studied the ground state properties of a two chain bosonic ladder with commensurate filling of one boson per site using the finite size density matrix renormalization group method. The critical on-site interaction for SF-MI phase transition increases sharply for small values of inter-chain hoping amplitude $t_\perp$. However, it saturates in the limit $t_\perp \to \infty$. Thus in the presence of
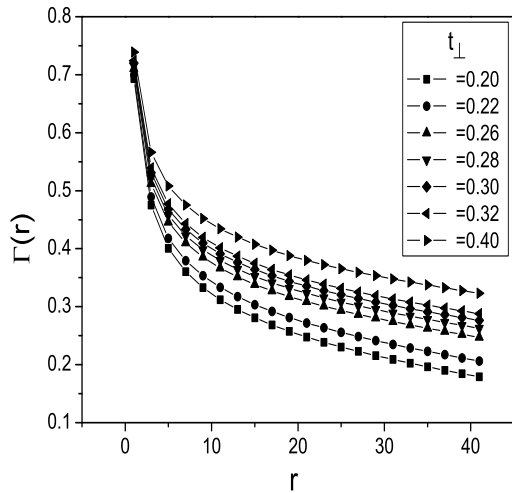
large inter-chain hopping, the system continues to be in the superfluid state even though the single chain is a Mott insulator. Thus we confirm the prediction of Mott insulator to superfluid transition as a function of $t_\perp$. We have obtained the Luttinger Liquid parameter and compared it with the analytical results. In addition to verifying and complementing the predictions made by the Bosonization technique, we have pointed out the possible experimental verification of our results. We hope that our analysis of the SF-MI transition in bosonic ladders will stimulate experimental studies in this direction.

FIG. 9: Power law decay of $\Gamma_\alpha(r)$ for various values of $t_\perp$ for $U = 6$.
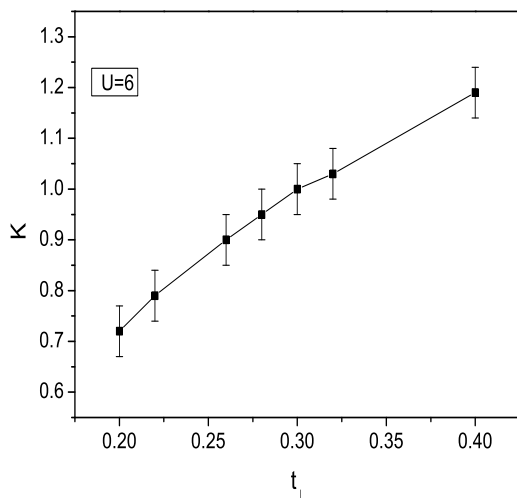

FIG. 10: Variation of Luttinger liquid parameter K_s as a function of $t_\perp$ near MI to SF transition for $U = 6$.

V.   ACKNOWLEDGMENTS

VI.   REFERENCES

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