MEASUREMENT OF VECTOR MAGNETIC FIELDS

I. *Theoretical Approach to the Instrumental Polarisation of the Kodaikanal Solar Tower*

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Abstract. The observations of Stokes line profiles require an accurate knowledge of the instrumental polarisation caused by optical components in the path of the light beam. In this context we present a theoretical approach to the instrumental polarisation caused by the 3-mirror coelostat system of the Kodaikanal Solar Tower.

1. Introduction

Stokes profiles of Zeeman sensitive spectral lines are significantly distorted by reflecting mirrors. Physically this distortion can be understood in terms of the differential response of the electric vectors in and orthogonal to the plane of incidence respectively. Ideally, one must avoid several oblique reflections while measuring the polarisation of light. However, one cannot prevent this in existing installations not originally meant for polarisation measurements. For this reason, it is essential that the instrumental polarisation be determined precisely for all possible orientation of the telescope. This has been done by several groups (Moe, 1968; Makita and Nishi, 1970; Livingston and Harvey, 1971; Bachman and Pflug, 1983) for their respective installations. In the present paper we describe in detail a theoretical approach for estimating the instrumental response of the Kodaikanal Solar Tower to arbitrarily polarised light. In a forthcoming paper, we intend to supplement these results with experimentally determined values of the composite Mueller matrix of the telescope.

2. The Telescope

The Kodaikanal Tower (Bappu, 1967) consists of a 3-mirror coelostat system. The primary mirror is an equatorial mount (latitude 10° 14′ N) 61 cm in diameter. The sunlight from the primary mirror is reflected onto a secondary mirror that in turn reflects the light vertically down onto a tertiary mirror. The tertiary mirror reflects the light horizontally onto a 38 cm lens which images the Sun on to the slit of the spectrograph.

Let us now consider a cartesian coordinate system whose origin is the centre of the primary mirror (Figure 1), i.e. (0, 0, 0). The secondary mirror has coordinates

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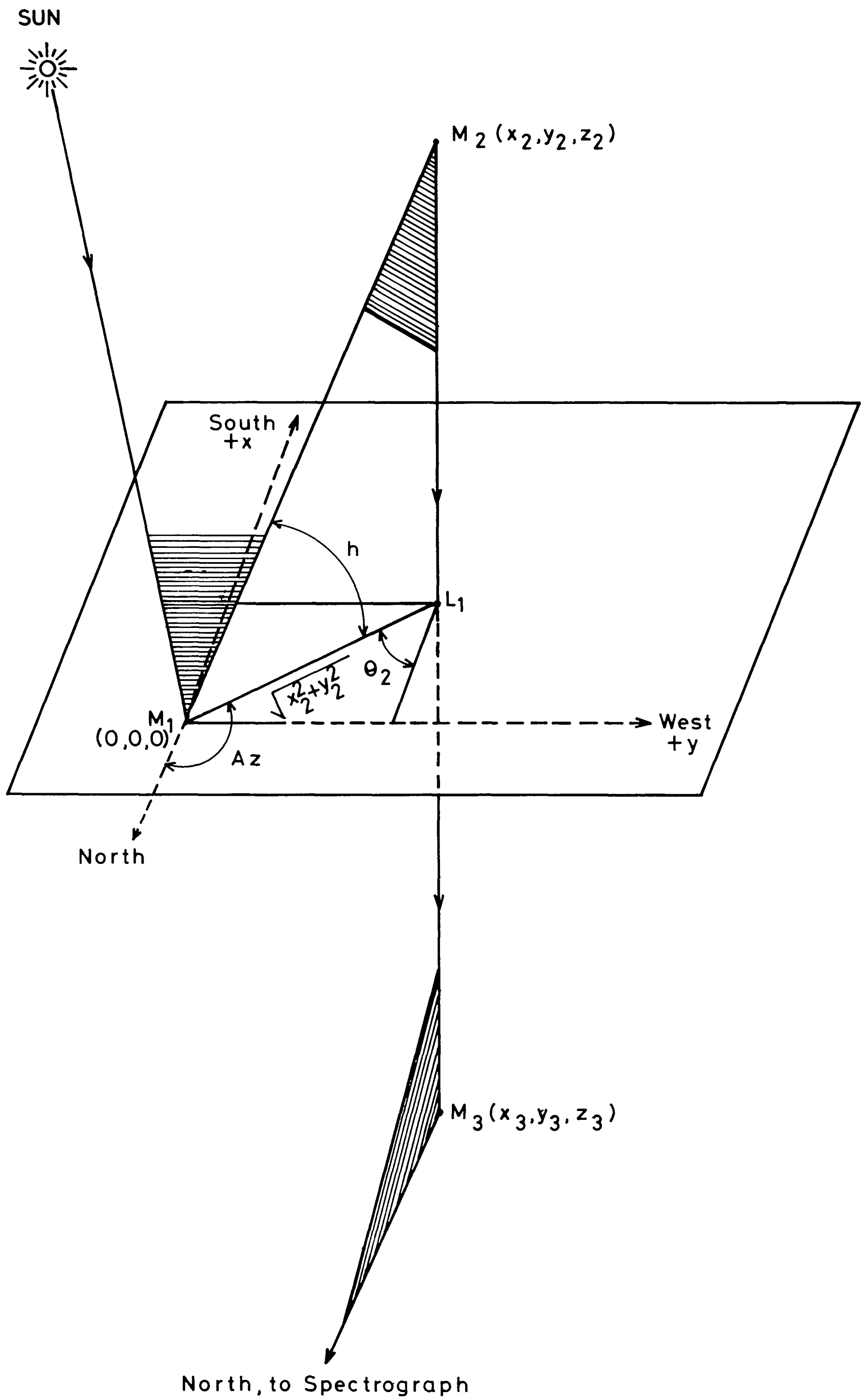

Fig. 1. Schematic representation of the 3-mirror system at Kodaikanal. M_1, M_2, and M_3 are the primary, secondary and tertiary mirrors respectively. In practice y_2 and z_2 have fixed values of 83 cm and 74 cm respectively and x_2 changes with declination.

(x_2, y_2, z_2) and the tertiary mirror has coordinates (x_3, y_3, z_3). The coordinates are such that the positive x-direction is towards the south, the positive y-direction is towards the west, and the positive z-direction is towards the zenith, all distances being reckoned from the centre of the primary mirror. The primary mirror can be moved in the north-south direction in order to direct the sunlight on to the secondary for different declinations of the Sun. This means that x_2 will change with a change in declination. The primary mirror can also be secured on two symmetrical positions east and west of the second mirror to avoid the shadow of the secondary mirror falling on the primary mirror at certain times of the year. In this case y_2 becomes $-y_2$. Thus y_2 and z_2 are fixed quantities. The centre of the secondary mirror is fixed, but the mirror itself can be rotated about two axes to facilitate the diversion of the light beam vertically down to the tertiary mirror.

3. The Method

An incident Stokes vector $[\mathbf{I}] = (I, Q, U, V)^T$ referred to coordinate system defined by the electric vector in and orthogonal to the plane of incidence on the primary transforms to the observed Stokes vector

$$[\mathbf{I}'] = (I', Q', U', V')^T$$

as

$$[\mathbf{I}'] = [\mathbf{M}_3]\,[\mathbf{R}_2]\,[\mathbf{M}_2]\,[\mathbf{R}_1]\,[\mathbf{M}_1]\,[\mathbf{I}],$$

where $[\mathbf{M}_1]$, $[\mathbf{M}_2]$, and $[\mathbf{M}_3]$ are the Mueller matrices of the primary, secondary, and tertiary mirrors, $[\mathbf{R}_1]$ and $[\mathbf{R}_2]$ are the matrices describing the rotation of the planes of incidence between successive reflections. The output Stokes vector $[\mathbf{I}']$ is analysed in a coordinate system where the plane of incidence of the third mirror coincides with the slit direction of the spectrograph. The Mueller matrix on reflection from a metallic surface is defined by (Kawakami, 1983)

$$[\mathbf{M}] = \tfrac{1}{2}\begin{pmatrix} 1+\mathbf{X}^2 & 1-\mathbf{X}^2 & 0 & 0 \\ 1-\mathbf{X}^2 & 1+\mathbf{X}^2 & 0 & 0 \\ 0 & 0 & 2X\cos\tau & 2X\sin\tau \\ 0 & 0 & -2X\sin\tau & 2X\cos\tau \end{pmatrix},$$

where X is ratio of the reflection coefficients of linearly polarised light perpendicular and parallel to the plane of incidence, and τ is the phase difference between reflected light polarised perpendicular and parallel to the plane of incidence (Makita and Nishi, 1970). Here

$$X^2 = \frac{a^2 + b^2 - 2a\sin i\tan i}{a^2 + b^2 + 2a\sin i\tan i}$$

and

$$\tan\tau = \frac{2b\sin i\tan i}{\sin^2 i - (a^2 + b^2)},$$

where

$$a^2 = \tfrac{1}{2}[n^2 - k^2 - \sin^2 i + \sqrt{(n^2 - k^2 - \sin^2 i) + 4n^2k^2}]$$

and

$$b^2 = \tfrac{1}{2}[-n^2 + k^2 + \sin^2 i + \sqrt{(n^2 - k^2 - \sin^2 i) + 4n^2k^2}]\,,$$

i being the angle of incidence; n and k are the real and imaginary parts of the complex refractive index, respectively. The rotation matrix is given by

$$[\mathbf{R}(\theta)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where θ is the angle of rotation of the planes of incidence.

3.1. Determination of the angles of incidence

The first step in determining the Mueller matrix of each reflection is to determine the angles of incidence. This is done in the following way:

In spherical polar coordinates, let the origin again be at the centre of the primary mirror. The secondary mirror will now have coordinates (Figure 1)

$$\begin{aligned} x_2 &= -d_1 \cos h \cos A_z\,, \\ y_2 &= d_1 \cos h \sin A_z\,, \\ z_2 &= d_1 \sin A_z\,, \end{aligned} \tag{1}$$

where h is the altitude, A_z is the azimuth of the secondary mirror as viewed from the origin and $d_1 = \sqrt{x_2^2 + y_2^2 + z_2^2}$ is the distance from the centre of the primary to the secondary. The azimuth is positive when recorded from north towards the west.

Consider now the celestial sphere with origin at the centre of the primary mirror. Let the Sun have the coordinates $h_\odot$ and $A_{z\odot}$ as its altitude and azimuth, respectively. Let t and δ be the hour angle and declination of the Sun. The primary mirror is mounted equatorially and the normal to the primary mirror always points at the equator. Let the hour angle of the normal to the primary be T. If ϕ is the latitude of the place then from the spherical triangle PzM_2 (Figure 2) we can write

$$-\sin\delta = \sin\phi \sin h + \cos\phi \cos h \cos A_z\,. \tag{2}$$

Substituting Equation (2) in Equation (1) we get

$$\sin\delta = \frac{x_2}{d_1}\cos\phi - \frac{z_2}{d_1}\sin\phi\,. \tag{3}$$

For a given declination of the Sun the value of x_2 is unique so that the light from the primary is always reflected into the secondary. One can solve for x_2 from Equation (3) as

$$x_2 = r_2 \tan\beta\,, \tag{4}$$

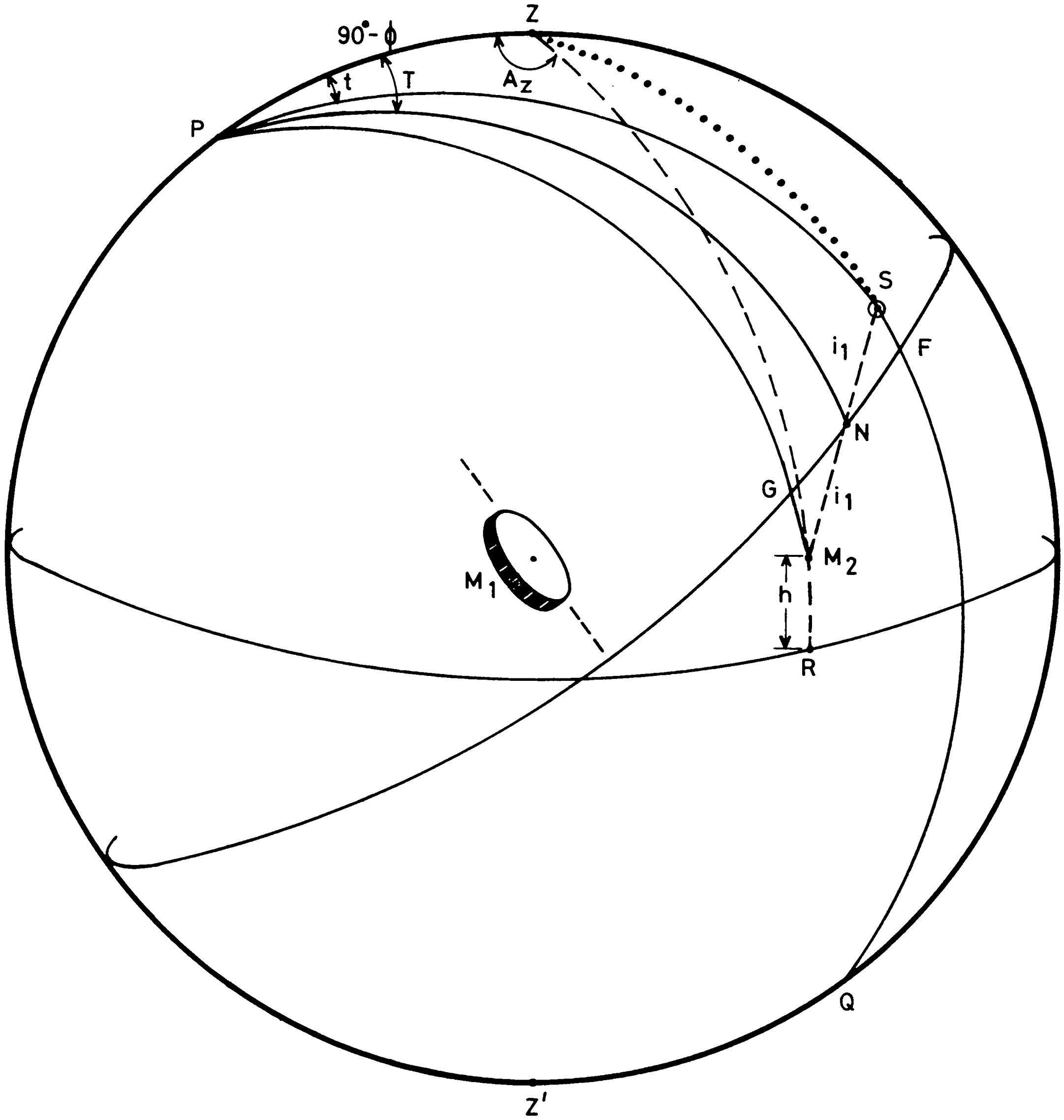

Fig. 2. The celestial sphere representation of the first two reflections. SM_1 is the incoming ray reflected along SM_2. This ray is finally reflected towards the nadir at M_2 and hence will be parallel to ZZ'.

where

$$\beta = A + \sin^{-1}\left(\frac{\delta}{B}\right),$$

$$A = \tan^{-1}\left(\frac{z_2}{r^2}\tan\phi\right),$$

$$B = \left(\cos^2\phi + \sin^2\phi\left(\frac{z_2}{r_2}\right)^2\right),$$

$$r_2 = (y_2^2 + z_2^2)^{1/2}.$$

Hence knowing the value of δ for the Sun at any given moment, we can adjust the position of the primary mirror. The maximum rate of change of declination is 24 arc sec day^{-1} at $\delta = 0$ and the minimum rate of change of declination is zero at $\delta = \pm 23\frac{1}{2}°$. As a result the maximum positional error of x_2 in terms of declination, if the mirror is adjusted every one hour of observation will be approximately one arc sec.

To determine the angles of incidence of the first mirror, consider the spherical triangle PZM_2 (Figure 2). Knowing that

$$P\hat{Z}M_2 = A_z ,$$

$$PM_2 = PG + GM_2 = 90 + \delta ,$$

$$Z\hat{P}M_2 = (2T - t) ,$$

$$ZM_2 = (90 - h) ,$$

we can apply the sine formula to obtain

$$\frac{\sin P\hat{Z}M_2}{\sin PM_2} = \frac{\sin Z\hat{P}M_2}{\sin ZM_2}$$

or

$$\cos h \, \sin A_z = \cos\delta \, \sin(2T - t) . \tag{5}$$

Now consider the spherical triangle PSN. Here $PN = 90°$ hence from the cosine formula (Smart, 1979)

$$\cos i_1 = \cos\delta \cos(T - t) . \tag{6}$$

Equations (5) and (6) can be used to determine i_1 using the known values of h and A_z. To determine the angle of incidence of the second mirror, (Figure 1) consider the triangle defined by the centre of the primary mirror (M_1) the centre of the secondary mirror (M_2) and the point of intersection of the (x_2, y_2) plane and the line joining the secondary and tertiary mirrors, i.e., L_1. The required angle of incidence i_2 is half the angle $M_1M_2L_1$. Thus

$$i_2 = \tfrac{1}{2} \tan^{-1}\left(\frac{M_1L_1}{M_2L_1}\right) ,$$

$$i_2 = \tfrac{1}{2} \tan^{-1}\left(\frac{\sqrt{x_2^2 + y_2^2}}{z_2}\right) .$$

Since the third mirror merely reflects the beam from the second mirror into the horizontal direction towards its north, the angle of incidence i_3 is always 45°.

3.2. Determination of the Angles of Rotation

Since the rotation matrix $\mathbf{R}(\theta)$ involves the terms $\cos 2\theta$ and $\sin 2\theta$ it suffices to determine the acute angle between the planes of incidence. However, the sense of rotation, namely clockwise or anticlockwise, must be determined in each case with the convention that an anticlockwise rotation is positive when the light beam common to the planes of incidence is towards the observer. We will now proceed to determine the magnitudes and sign of the angles θ_1 and θ_2.

The Magnitude of θ_1

The magnitude of the angle θ_1 is the angle made by the plane containing the Sun (S), primary mirror (M_1), and secondary mirror (M_2) with the plane containing the primary mirror (M_1), secondary mirror (M_2), and the zenith (Z) (also refer Figure 2). We are interested in the angle ZM_2S.

Here

$$ZM_2 = (90 - h), \qquad SM_2 = 2i_1 \quad \text{and} \quad ZS = (90 - h_\odot).$$

Thus from the cosine formula we have:

$$\cos(90 - h_\odot) = \cos 2i_1 \cos(90 - h) + \sin 2i_1 \sin(90 - h) \cos \theta_1$$

and hence

$$\cos \theta_1 = \frac{\sin h_\odot - \cos 2i_1 \sin h}{\sin 2i_1 \cos h},$$

where the altitude of the Sun is given by (Smart, 1979)

$$\sin h_\odot = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t.$$

Table I shows the sign of θ_1 obtained from geometrical considerations (Figure 1).

TABLE I

Sign convention for angle θ_1

	$-x_2$	$+x_2$
$-Y_2$	anticlockwise (positive)	clockwise (negative)
$+Y_2$	clockwise (negative)	anticlockwise (positive)

TABLE II

Sign convention for angle θ_2

	$-x_2$	$+x_2$
$-Y_2$	clockwise (negative)	anticlockwise (positive)
$+Y_2$	anticlockwise (positive)	clockwise (negative)

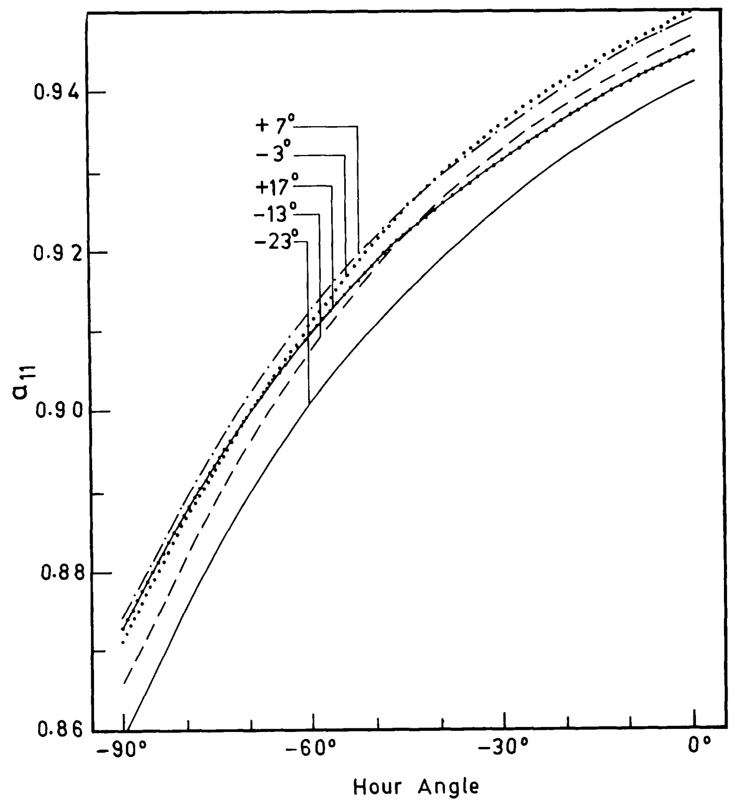

Fig. 3a. Depicts a_{11} as a function of hour angle for various values of the declination (marked in the figure). Only one half of the day is shown since the curves are symmetrical about 0°.

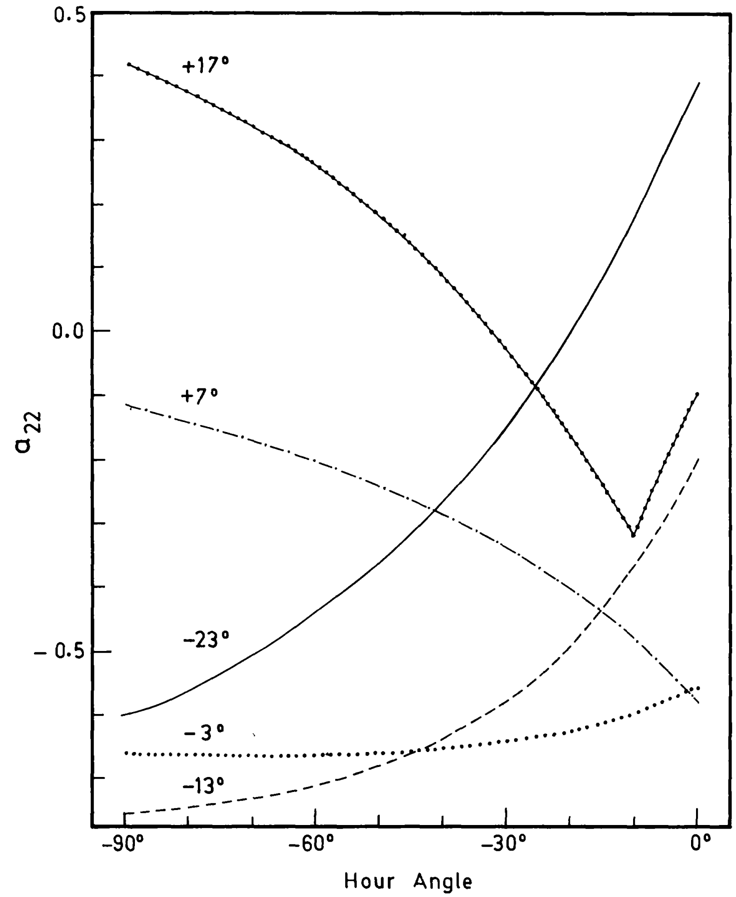

Fig. 3b. Shows a_{22} as a function of hour angle. These curves are also symmetrical about 0°.

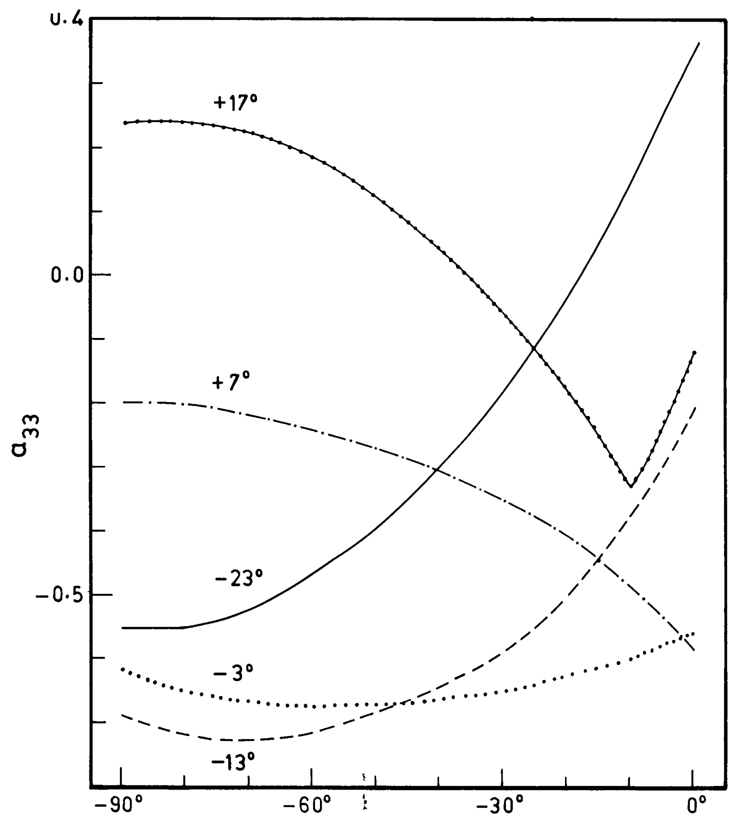

Fig. 3c. Shows a_{33} with similar symmetry.

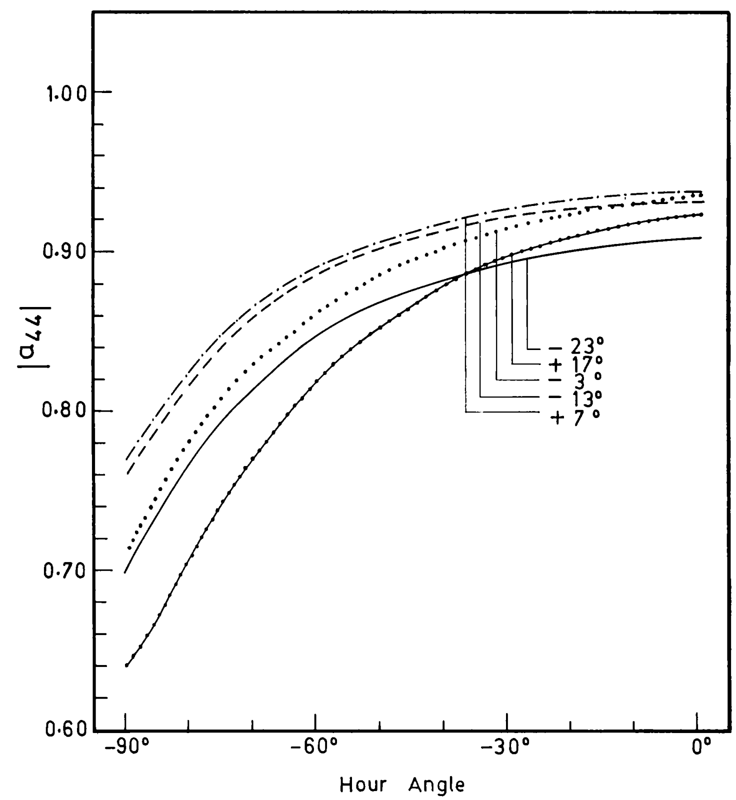

Fig. 3d. a_{44} also possesses same symmetry as a_{11}.

Fig. 3a-d. Diagonal elements of the composite Mueller matrix.

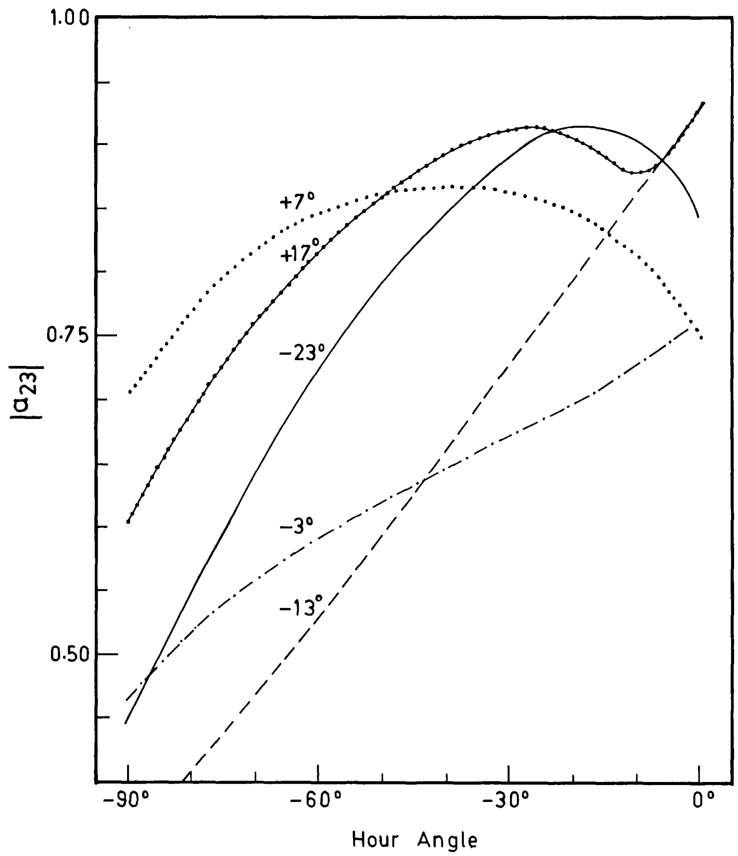

Fig. 4a. a_{23} is antisymmetric about 0° and is positive for negative hour angles for declinations of − 23° and − 13°. It assumes opposite sign for declinations of − 3°, + 7°, and + 17°.

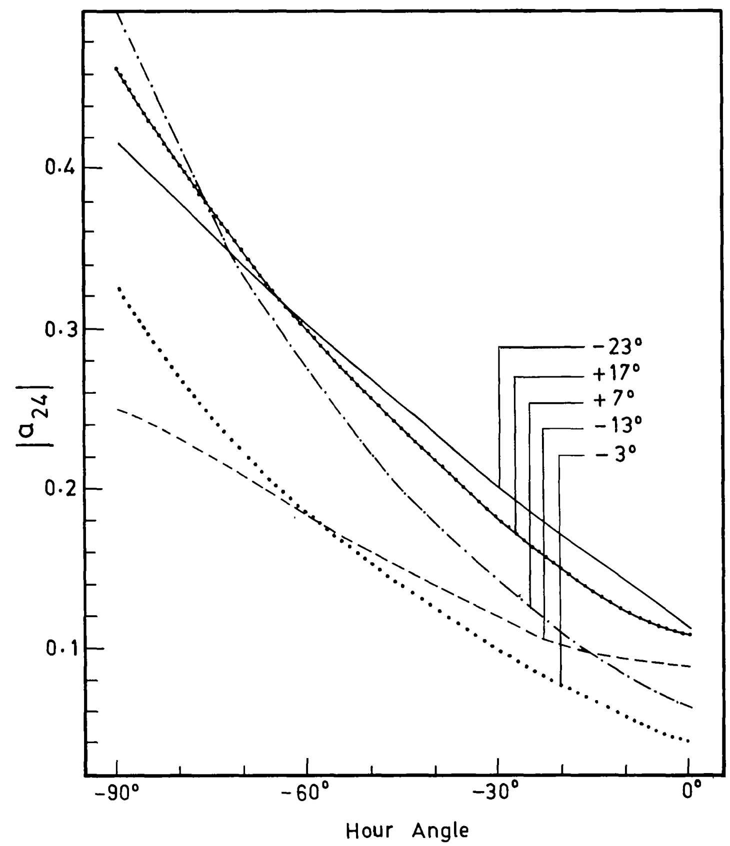

Fig. 4b. a_{24} is also antisymmetric about 0° hour angle and takes on negative values for declinations of − 23° and − 13°. It assumes positive values for all other cases.

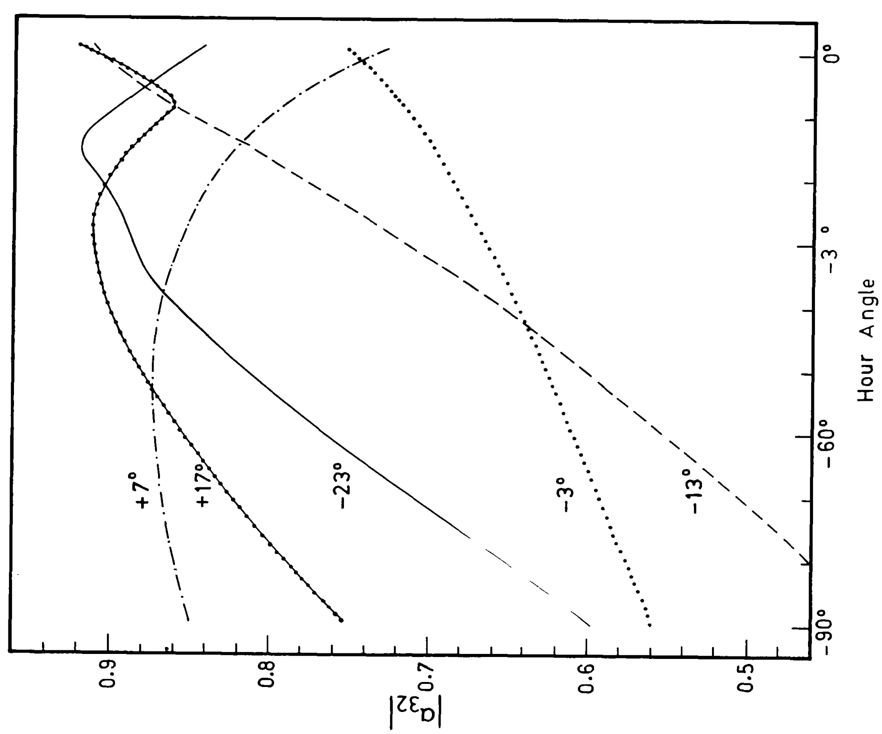

Fig. 4c. a_{32} has the same symmetry properties as a_{24}.

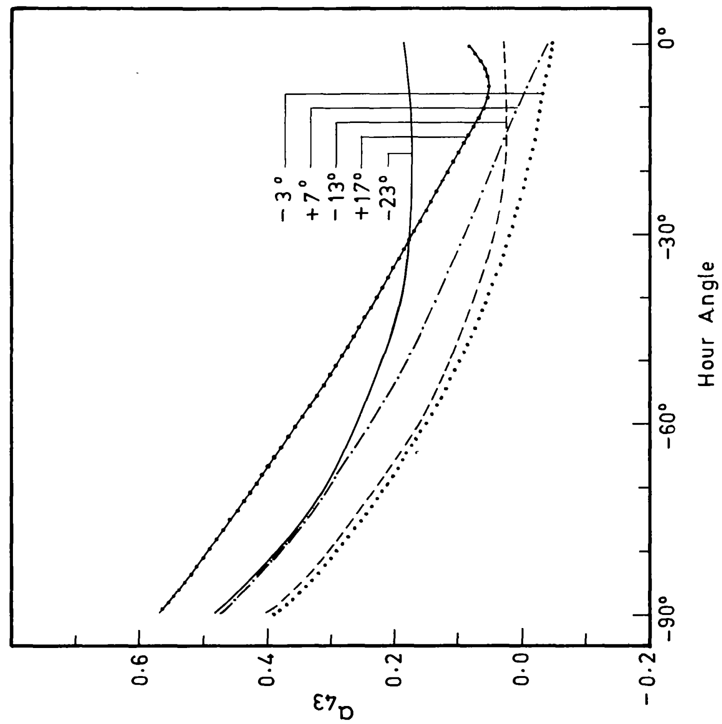

Fig. 4d. a_{43} is symmetric about 0° hour angle.

Fig. 4a-d. A few significantly large off-diagonal elements are shown.

The Magnitude of θ_2

To determine the magnitude of θ_2 consider Figure 1. It is immediately seen that $\tan \theta_2 = (|y_2|)/(|x_2|)$. The sign of θ_2 can be seen in Table II. The angle θ_1 varies on a shorter time scale which is a function of the hour angle of the Sun whereas θ_2 varies on a longer time scale which is a function of the declination.

4. Results

In order to estimate the total effects of instrumental polarisation of the coelostat mirrors we simulated the actual conditions for five declinations $-23°$, $-13°$, $-3°$, $+7°$, and $+17°$, for each fixed declination, running the hour angles from $-90°$ to $+90°$. The Mueller matrices due to each reflection was calculated assuming $n = 1.13$ and $k = 6.39$ valid at 6300 Å. This is a good approximation according to Makita and Nishi (1970). We, however, plan to experimentally check the validity of this approximation in the forthcoming paper. The Mueller matrices and rotation matrices were combined to form a single Mueller matrix.

The Mueller matrix elements are shown as a function of hour angle for the various declinations (Figures 3 and 4). The data for the Stokes parameters representing the following five forms of polarization was the input in our calculations: (a) unpolarised, (b) linearly polarised along the plane of incidence at the first mirror, (c) linearly polarised orthogonal to the plane of incidence at the first mirror, (d) right circularly polarised, and (e) left circularly polarised.

The output Stokes parameters

$$[\mathbf{I}'] = [\mathbf{M}]\,[\mathbf{I}]$$

for each set of input parameters was determined again as a function of hour angle for each of the five declinations. The following quantities were determined in each case from the output Stokes parameters (Clark and Grainger, 1971):

(i) percentage polarisation

$$p = \frac{\sqrt{Q'^2 + \mathbf{U}'^2 + \mathbf{V}'^2}}{I'}\ ;$$

(ii) azimuth of the ellipse

$$\zeta = \tfrac{1}{2}\tan^{-1}(\mathbf{U}'/Q')\ ;$$

(iii) ratio of the semi-major axis to the semi-minor axis, $b/a = \eta$, where

$$\frac{2\eta}{1 + \eta^2} = \frac{|\mathbf{V}'|}{I}\ .$$

The percentage of polarisation and η produced from unpolarised light is shown in Figure 5 while Figure 6 shows η corresponding to the four polarised forms of the input Stokes vectors.

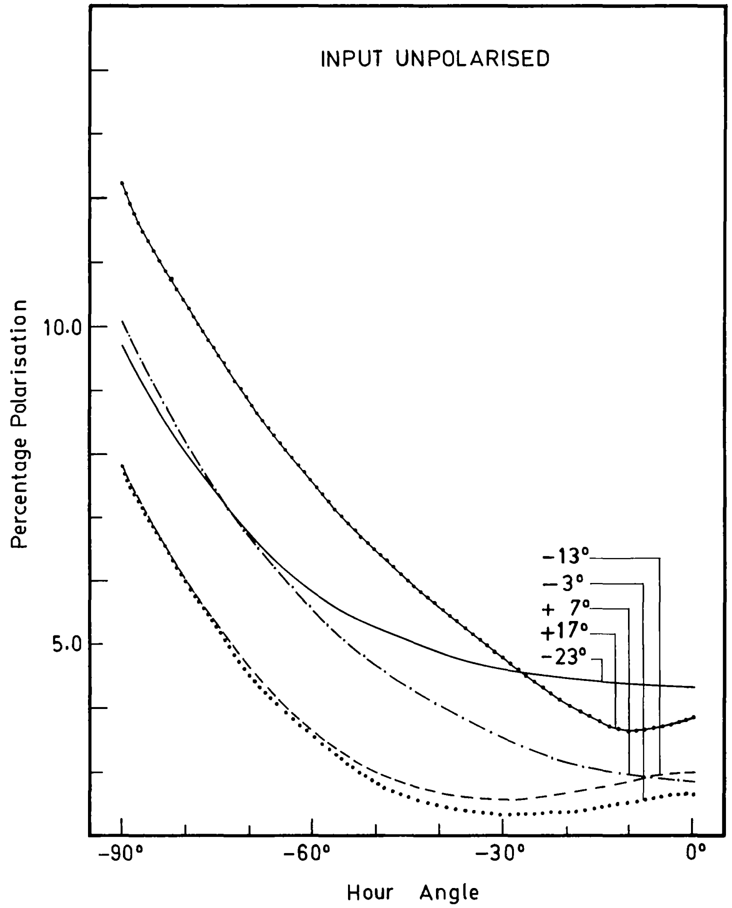

Fig. 5a.

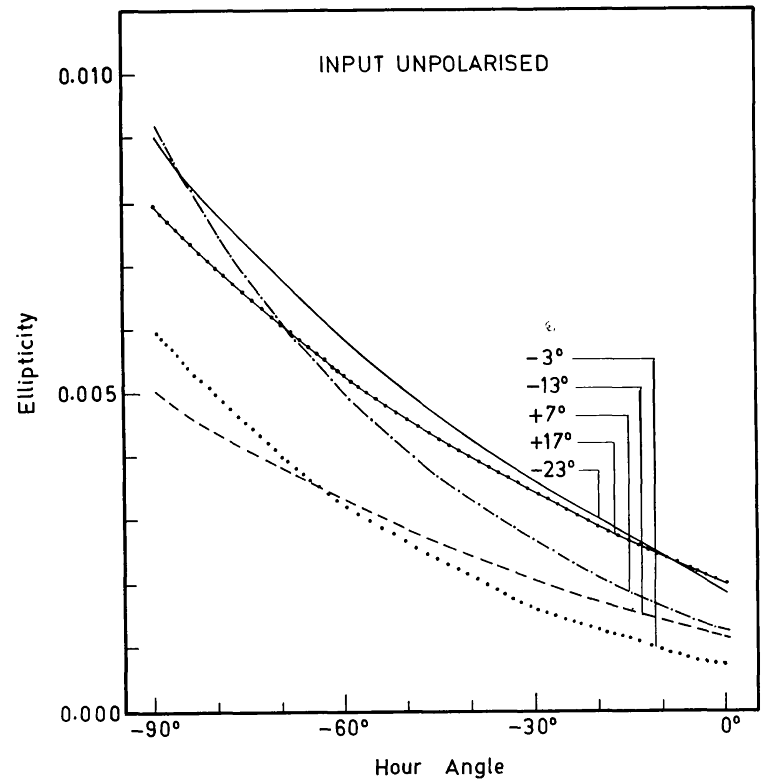

Fig. 5b.

Fig. 5a-b. (a) Percentage of total polarisation produced by the mirrors as a function of hour angle for various declinations of the Sun. The input beam is unpolarised. (b) The ellipticity of the polarised component of light produced by reflection of unpolarised light in the three-mirror system.

5. Discussion

Let us first look at the results for the variation of the elements of the composite Mueller matrix. The element a_{11} represents the scattering of I into I and thus is a measure of the partially polarizing property of the system. It is seen that for all declinations, a_{11} is near unity, thereby showing that not much partial polarisation is produced. The elements a_{12}, a_{13}, and a_{14} represent the scattering of Q, U, and V into I and thus are a measure of the depolarisation produced. These elements have very small magnitude ($\approx 1\%$). Moreover, in the measurement of Stokes profiles of Zeeman sensitive lines, the Q, U, and V signals would be in general quite small thus reducing the contamination of I still further. The only context where these might assume some significance is in the comparison of the bisectors of Zeeman sensitive and insensitive line profiles.

The elements a_{21}, a_{31}, and a_{41} are more serious since they represent the contamination of Q, U, and V respectively from the I profile. However, these elements also remain very small ($\approx 1\%$) and will pose no numerical problems while inverting the composite Mueller matrix for off-line compensation.

The diagonal elements a_{22}, a_{33}, and a_{44} represent the sensitivity of the system to pick up the Q, U, and V signals, respectively. Unfortunately, a_{22} and a_{33} are not very large and consequently require some care during the matrix inversion for off-line compensation. The element a_{44} is $\gtrsim 80\%$ and thus the V measurements would be less prone to error.

The element a_{24} is particularly important for the measurement of the transverse component of a field which is inclined at a small angle to the line of sight. In this case V would be much larger than Q and a_{24} would further contaminate the Q signal. Other off-diagonal elements that are significantly larger than zero are a_{23}, a_{32}, and a_{43}. All the coefficients describing cross talk between Q, U, and V are very serious and have to be very carefully considered for a successful compensation.

Next let us consider the results for the percentage polarisation produced from natural light. This is large ($\approx 12\%$) when the Sun is just rising but quickly drops to about 2% at hour angles of $\sim -30°$ in spring and autumn. This value of about 2% is not a great impediment for the measurement of moderate field strengths. For weaker fields it would be necessary to introduce some on-line compensation. For purely polarised light, it was seen that the percentage polarisation remained at 100% thereby indicating that no depolarisation is produced in the system.

However, linearly polarised light is converted to slightly elliptically polarized light as seen by the value of η plotted in Figure 6(a) ($\lesssim 0.1$). Circularly polarised light is affected in the same way whether it is right or left circularly polarised (Figure 6b) and thus shows that no compensation is necessary if the system is used to feed a longitudinal magnetograph. There is a very small asymmetry, however, which could become important for very sensitive velocity measurements using the magnetograph.

In conclusion, it is seen that most favourable conditions for the measurement of polarisation of sunlight using the Kodaikanal Solar Tower/Telescope are available between hour angles of $-60°$ to $60°$ and during the seasons of small declination.

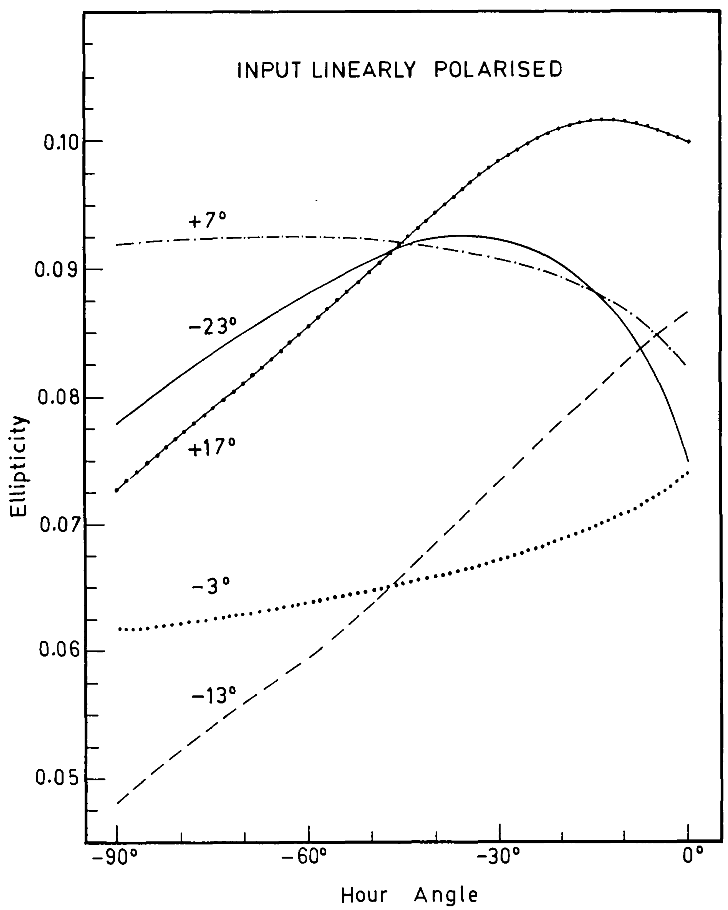

Fig. 6a. Ellipticity of the polarised light produced by reflection of linearly polarised light. The results for its orthogonally polarised component are very nearly the same and are not shown separately.

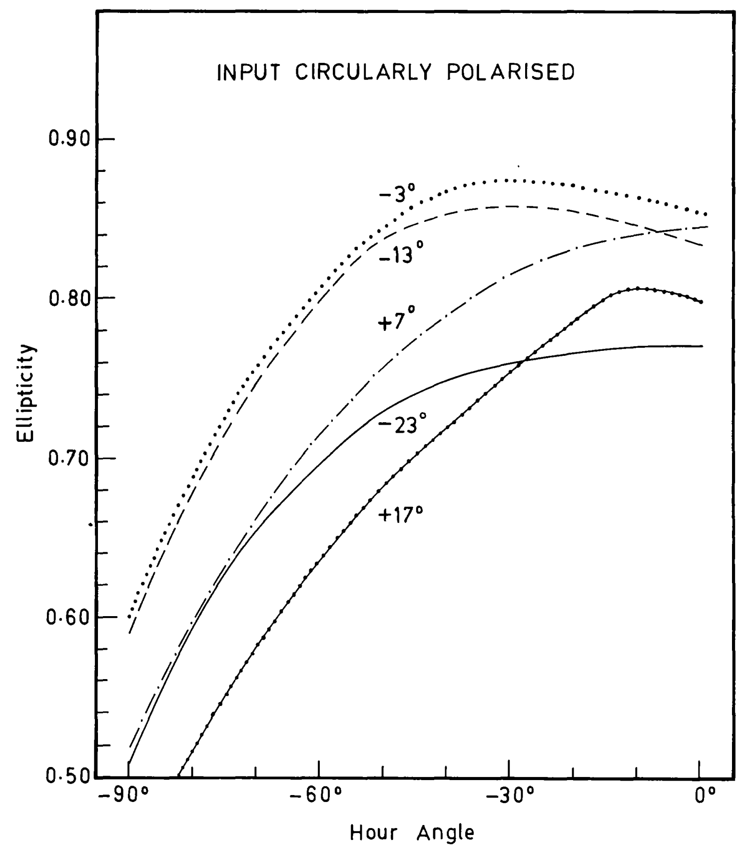

Fig. 6b. Ellipticity produced by reflection of circularly polarised light. The two orthogonal components show similar behaviour and therefore are not shown separately.

However, for any observing period, the off-line compensation requires careful consideration of the possible errors, in particular for the determination of the Stokes Q and U parameters.

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