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ASTRONOMY AND ASTROPHYSICS

Formation of the Hydrogen Lyman α Line in Expanding Spherical Nebulae with Dust

A. Peraiah* and R. Wehrse

Institut für Theoretische Astrophysik der Universität Heidelberg, Im Neuenheimer Feld 294, D-6900 Heidelberg, Federal Republic of Germany

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Summary. The radiation field in the hydrogen Ly α line is calculated for a dust filled spherical gaseous nebula. It is assumed that the nebula with a ratio of inner to outer radius of 3.3 has an optical depth at the line center of about 1000 and is either static or expands uniformly with 26 km s^{-1}. For the solution of the radiative transfer equation the discrete space theory with partial redistribution is used.

It is found that the dust gives only rise to small changes if it purely scatters. However, if it absorbs it is very effective in reducing the mean intensities J in the whole nebula, e.g. for an optical depth in the dust of $\tau=1$ decreases J by about a factor of 7. The results also show that for dust filled nebulae the effects of velocities are of minor importance in contrast to dust-free configurations.

Key words: gaseous nebulae — dust — H Ly α line — spherical radiative transport — expansion

I. Introduction

Infrared observations show that most (or even all) planetary nebulae contain dust (see e.g. Miller, 1974) which is mixed with the ionized gas (Osterbrock, 1974). Although the dust-to-gas ratio is smaller in planetary nebulae than in the interstellar gas (Köppen, 1977) it has been demonstrated that the dust influences the structure and dynamics of planetary nebulae in various respects, e.g. it causes a reduction of the size of the Strömgren sphere (Mathis 1971, Petrosian et al. 1972) and affects the Balmer lines (Cox and Mathews, 1969, Mathis, 1970, Balick, 1975). In the evolution of a planetary nebula dust also plays an important role (Mathews, 1969; Ferch and Salpeter, 1975).

In this paper we want to contribute to these investigations and consider the influence of dust on the strength and the profile of the hydrogen Lyman α line. We also want to discuss the importance of this line for the heating of the dust on the basis of these calculations (see Ferch and Salpeter, 1975, comp. also Wynn-Williams and Becklin, 1974).

Strong changes in the radiation field due to dust are to be expected since competing absorbers are hardly present in the nebula and since—as a consequence of the large number of scatterings—photons have to move very long distances in the plasma before they can escape so that the probability they are absorbed by dust is rather high. It is also evident that the different character of the redistribution in the core and the wing (see e.g. Jefferies, 1968) results in distortions of the profiles if dust is present. Detailed calculations however have not yet been published to our knowledge.

It seems to us that the main reason for this is the great complexity of the problem (compare Wehrse and Peraiah, 1977). It is necessary to take into account simultaneously the sphericity of the configuration, the expansion of the nebula and the partial redistribution of the photons after being absorbed by neutral hydrogen atoms. It will be shown in the discussion that the radiation field is significantly influenced by all of them. In addition we have to consider the fact that the dust both scatters and absorbs.

In Section II we describe the solution radiative transfer equation with dust. The numerical results are presented and discussed in Chapter III.

II. The Radiative Transfer in the Presence of Dust

The transfer equation for the hydrogen Lyman α line for a dusty moving medium is written as

$$\mu\frac{\partial I(x,\mu,r)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I(x,\mu,r)}{\partial\mu}+(\chi(x,\mu,r)+\chi_d)I(x,\mu,r)$$

Send offprint requests to: R. Wehrse

* Present address: Indian Institute of Astrophysics, Bangalore 560034, India

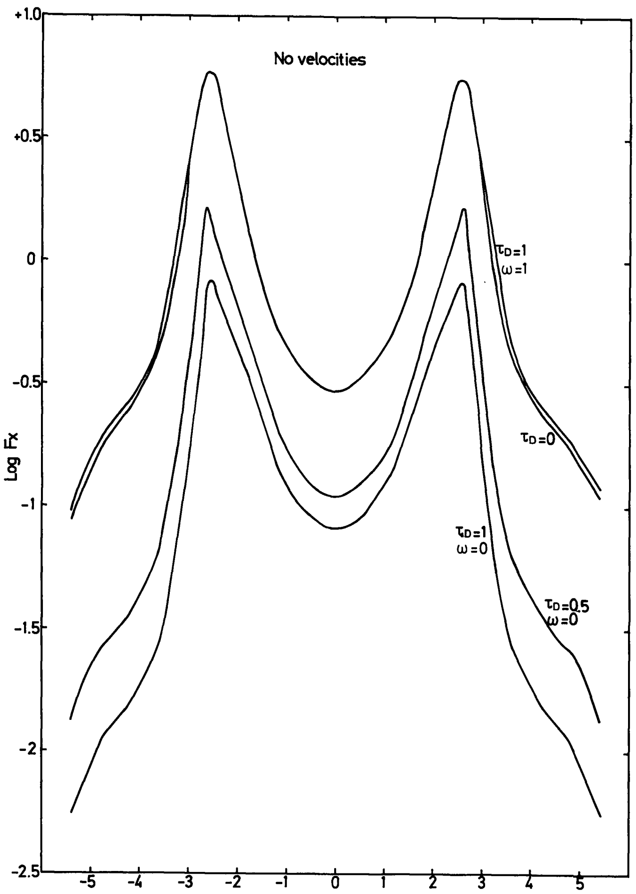

Fig. 1. Profiles of the emergent hydrogen Ly α line for static nebulae. All data are given in the same arbitrary scale

$$=\chi(0,\mu,r)\left\{\frac{1}{2}\int_{-\infty}^{\infty} dx' \int_{-1}^{+1} R(x,x')I(x',\mu',r)d\mu' + \phi(x,\mu)S(r)\right\} + \chi_d\left\{\frac{1}{2}\tilde{\omega}\int_{-1}^{+1} P(\mu,\mu')I(x,\mu',r)d\mu' + (1-\tilde{\omega})B(r)\right\} \quad (1)$$

A similar equation for the oppositely directed beam can be given by replacing μ with $-\mu$ and changing the sign of the second term on the left hand side of Equation (1). The symbols in the above equation are defined as follows:

$I(x,\mu,r)$ = specific intensity for frequency $x=(\nu-\nu_0)/\Delta\nu_D$ ($\Delta\nu_D$ is the Doppler width) making an angle $\cos^{-1}\mu$ with the radius vector r.

$\chi(x,\mu,r)$ = absorption coefficient of the hydrogen Lyman α line at frequency point x for the angle $\cos^{-1}\mu$. χ and the emission profile function ϕ (assumed to be a Voigt function) depend on μ, since the medium is moving.

$R(x,x')$ = the angle averaged redistribution function.

$S(r)$ = source term representing the generation of photons by recombination.

χ_d = absorption coefficient of the dust (A constant dust density is assumed).

$\tilde{\omega}$ = albedo for single scattering.

$P(\mu,\mu')$ = phase function and $B(r)$ = Planck function.

The last three quantities refer also to the dust. Equation (1) is integrated following the lines of Grant and Peraiah (1972) and Peraiah and Grant (1973) (GP and PG respectively). Since the inclusion of dust leads to a number of additional terms in the discrete form, the equations actually used are reproduced in the appendix. The details of the numerical evaluation are the same as in Wehrse and Peraiah (1978). All results presented in this paper are obtained also with the same ionization model that was used in that investigation. It has an inner radius of $r_{in}=3.6\ 10^{16}$ cm and an outer radius of $r_{out}=1.2\ 10^{17}$ cm so that the ratio r_{out}/r_{in} equals 3.3. The electron density and temperature are assumed to be 10^3 cm^{-3} and 10^4 K. At the line center the total optical depth is about 1000. The inner boundary is assumed to be perfectly reflecting.

When dust is present this model is no longer consistent because the effect of the dust grains on the degree of ionization is not taken into account. But since we want to study the influence of dust on the formation of the Ly α line by transfer effects it seemed to us preferable to restrict the discussion mainly to one ionization model and to avoid the additional complications that result from changes in the model structure. However, we will return to this problem at the end of the next section.

III. Results and Discussion

We take the optical depth of the dust at 1215 Å $\tau_D=1.0$, 0.5, and 0 and the albedo for single scattering $\omega=0$ and 1 corresponding to pure absorption and pure scattering. We assume that the dust scatters isotropically so that the phase function is $P(\mu,\mu')=1$. The Planck function B of the dust is taken to be zero because the reemission is extremely small for the wavelengths considered. In the case of the expanding medium we use a uniform velocity distribution of $v=2$ Doppler velocities which corresponds to hydrogen velocities of about 26 km s^{-1}.

In Figure 1 the monochromatic emergent fluxes are plotted for static nebulae with dust and for comparison without dust (this curve is taken from Wehrse and Peraiah, 1978). It is immediately seen that for absorbing grains the fluxes are considerably reduced. The effect is much stronger in the line wings than in the cores, for example when $\tau_d=1$ and $\omega=0$ (i.e. pure absorption) the flux in the wing is reduced by more than 20 times whereas in the core the reduction factor is only 3.6. This is because of the fact that the wing photons have great mean free path lengths (e.g. for $x=2.5$ the total optical depth is approximately unity) and consequently the probability of these photons to be absorbed is large. In the core the optical depth of the line is so high that the chance for a photon to diffuse to the line wing before being absorbed is quite large and therefore the dust is not as efficient in absorbing these photons as in wings. Because of the fact that for $x \gtrsim 3$ there is approximately coherent scattering it is difficult in the line wing to replace a photon which is absorbed by the dust, whereas in the core, since there is strong diffusion, absorbed photons can easily be replenished. This behaviour is also evident from the point of view that the photons are mainly generated near the core and that in the process of diffusion to the wing their age increases implying a greater chance to be absorbed by dust.

The effectiveness of dust absorbing Ly α photons can also be seen from the fact that even for $\tau_D = 0.5$ the flux in the wings is about 9 times smaller than in the dust-free case, whereas in the core the reduction factor still amounts to 2.7. In total for $\tau_d = 1$ only 14% of the generated Ly α photons can leave the nebula (for $\tau_d = 0.5$ 24%). The rest is absorbed by the the dust and used for heating the grains.

These data also show that the emergent flux has a stronger dependence on τ_D than $\exp(-\tau_D)$ so that no simple approximation seems to be possible.

When it is assumed that the dust scatters isotropically without absorption the effects are minimal, in particular in the region of the line core. In the wings we find increases of the fluxes up to a factor of 1.07. The explanation for this is that the relative importance of scatterings by dust grains is much higher in the wings than in the core and that for $x \gtrsim 3$ we have practically coherent scattering both by the dust and the gas.

In Figure 2 the corresponding emergent fluxes are given for a nebula expanding uniformally with two thermal velocity units. The results are given for optical depths in the dust of 0 and 1, the latter for pure absorption and pure scattering. For the non dust-case the data are taken from Wehrse and Peraiah (1978). Again we find strong differences in the profiles for the lines formed by a gas with and without dust. For most frequencies however they are not as large as in the case of a static nebula, because in a moving medium the photons need not undergo so many scatterings to leave the system through the surface (comp. Kunasz and Hummer, 1974). For the case of purely scattering dust grains only minor differences occur.

In general all profiles are asymmetric with a similar shape. However, the profiles formed by the gas with scattering dust are more asymmetric than those formed by the gas mixed with absorbing dust. This can be seen simply by comparing the heights of the emission peaks on either side of the lines. This relative decrease of the red emission peak is again due to the increase in the mean free path length as explained earlier for the static nebula.

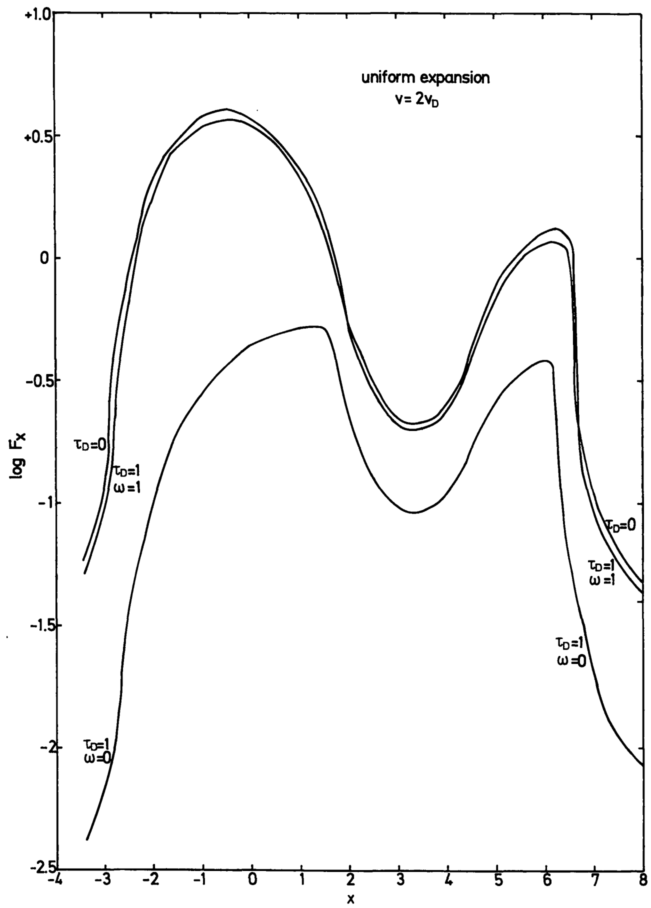

Fig. 2. Profiles of the emergent hydroge Ly α line for a nebula expanding uniformly with twice the thermal velocity. The data are given in the observer's frame

The influence of dust on the radiation field of the line at intermediate radial points is shown in Figure 3, where the total mean intensities $\int_{\text{line}} J(r,x)dx$ are plotted as functions of the shell numbers (radii). The graph shows the great effectiveness of dust absorption for all parts of the nebula in preventing the intensities to build up too high. So a uniform expansion of 2 thermal units decreases J only by a factor of about 2 whereas dust with $\tau_d = 1$ results in a factor of about 7. Even $\tau_d = 0.5$ reduces the mean intensity about five times. The predominance of dust absorption can also be seen from the fact that for dust-filled nebulae there is hardly a difference in J between the static and the expanding case for most shells.

As was already evident from the emergent fluxes the influence of scattering dust on the radiation field is only comparatively small. For a static configuration it simply increases the intensity proportional to τ_d because the total number of scatterings is increased correspondingly. For an expanding nebula the situation is more complicated: In the inner parts J is significantly larger, whereas for the outer shell it becomes smaller.

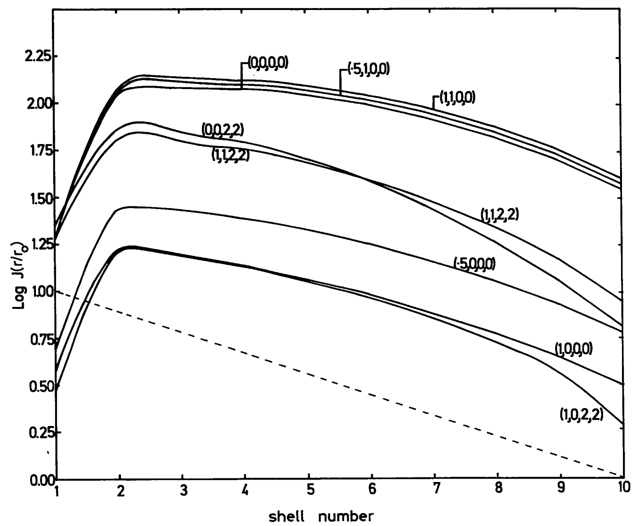

Fig. 3. Run of the total mean intensities as functions of the radial positions indicated by the shell number (No. 1 and No. 10 refer to the outer- and innermost shell; compare text). The different cases are described by the quadrupels (τ_d, $\tilde{\omega}$, v_1, v_{10}) which give the total optical depth of the dust, the albedo for single scattering and the velocity of the outer- and innermost shell respectively. The straight line gives the run of the geometry factor $(r/r_0)^2$. (Note: The ordinate is Log $J(r/r_0)^2$)

The reason for this behaviour is that the anisotropy in the scattering of the gas induced by the velocity is damped by the presence of dust scattering coherently and isotropically. For the dust-filled expanding nebula we have therefore relatively more photons being scattered to the inner parts than towards the outer surface.

It was already noted earlier that dust also changes the ionization structure of the nebula. In order to study the resulting modifications we constructed a model, in which the (absorbing) dust was consistently taken into account (comp. Baschek and Wehrse, 1975). The parameters were again those given in the last section. The main change was a slightly lower ionization degree, which leads to an increase of the Ly α optical depths by a factor of 2 approximately. The line profiles calculated with this model are somewhat broader and less high than those shown here. The differences in the radiation fields are negligible for the innermost parts of the nebula, since there the ionization is hardly changed. The greatest differences are found in the outermost shell for the static case, where the (far-) wing fluxes approximately double, while in the core the fluxes are reduced by about 30%. These calculations thus show that the profile of Ly α is noticeably influenced by changes in the model, but that the main modifications are due to direct dust absorption in the line.

In the discussion on the applicability of these results it has to be kept in mind that most observed nebulae have much higher optical depths than our model (see e.g. Osterbrock, 1974) so that there the mean number of scatterings is higher and consequently even more photons are absorbed by the dust (assuming that it is not totally scattering). We avoided larger optical depths because they would have lead to an appreciable increase in the computing time (due to the necessary increase in number of angles and doublings), in particular for the expanding cases. For the profiles presented here the CPU time on the IBM 370/168 of the university of Heidelberg was already 20 min. However, since $\tau_d=1$ at λ 1215 Å under the assumption of $\tau_d \alpha \lambda^{-2}$ corresponds to $\tau_d=0.06$ at the wavelength of Hβ, it is evident from our calculations that in a large number of planetary nebulae the radiation field in the hydrogen Ly α line is strongly influenced by the dust.

Although most Ly α photons are absorbed by the dust, the graph shows that most of them are not near the place of their generation (in that case the curves should be straight lines as given by the on-the-spot-approximation).

The maximum number of absorbed photons per cm^{-3} is reached at about shell number 6, where J has its highest value. We therefore should also find a temperature maximum in that region provided the dust is optically thin in the infrared and that heating by direct radiation from the central star is negligible. But since it is very difficult to fulfil the last condition it seems that this effect has no observable consequences. However, the decrease of the mean intensities near the surface causes lower dust temperatures than calculated under the assumption of the on-the-spot-approximation. Since the relative importance of Ly α dust heating compared to heating by direct stellar radiation increases proportional to

$$r^2 J \int_0^\infty S_\nu \frac{\chi_\nu^d}{\chi_{\mathrm{Ly}\,\alpha}^d} h\nu e^{-\tau_\nu(r)} d\nu$$

(S_ν number of photons the star emits at frequencies ν ... $\nu+1$, $\chi_\nu^d/\chi_{\mathrm{Ly}\,\alpha}^d$ = relative run of the dust absorption with frequency ν and radius r) this reduction of the temperatures may show up in center-to-limb observations in the near-infrared, where the dependence of the intensities on T is largest (Wien part of the spectrum).

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Appendix

Analogously to the treatment of GP and PG we obtain for the two oppositely directed beams

$$\begin{aligned}
&\boldsymbol{M}[\boldsymbol{u}^+_{n+1}-\boldsymbol{u}^+_n]+\rho_c[\boldsymbol{\Lambda}^+\boldsymbol{u}^+_{n+1/2}+\boldsymbol{\Lambda}^-\boldsymbol{u}^-_{n+1/2}]\\
&\quad+\tau_{n+1/2}\boldsymbol{\phi}^+_{n+1/2}\,\boldsymbol{u}^+_{n+1/2}+\tau^d_{n+1/2}\boldsymbol{E}\boldsymbol{u}^+_{n+1/2}\\
&\quad=\tau_{n+1/2}\boldsymbol{S}^+_{n+1/2}\\
&\quad+\tfrac{1}{2}\tau_{n+1/2}[\boldsymbol{R}^{++}\boldsymbol{w}^{++}\boldsymbol{u}^+ +\boldsymbol{R}^{+-}\boldsymbol{w}^{+-}\boldsymbol{u}^-]_{n+1/2}\\
&\quad+\tfrac{1}{2}\tau^d_{n+1/2}\tilde{\omega}[\boldsymbol{P}^{++}\boldsymbol{c}\boldsymbol{u}^+ +\boldsymbol{P}^{+-}\boldsymbol{c}\boldsymbol{u}^-]\\
&\quad+\tau^d_{n+1/2}(1-\tilde{\omega})\boldsymbol{B}_{n+1/2}
\end{aligned}\tag{A1}$$

and

$$\begin{aligned}
&\boldsymbol{M}[\boldsymbol{u}^-_n-\boldsymbol{u}^-_{n+1}]-\rho_c[\boldsymbol{\Lambda}\boldsymbol{u}_{n+1/2}+\boldsymbol{\Lambda}^-\boldsymbol{u}^+_{n+1/2}]\\
&\quad+\tau_{n+1/2}\boldsymbol{\phi}^-_{n+1/2}\,\boldsymbol{u}^+_{n+1/2}+\tau^d_{n+1/2}\boldsymbol{E}\boldsymbol{u}^-_{n+1/2}\\
&\quad=\tau_{n+1/2}\boldsymbol{S}^-_{n+1/2}\\
&\quad+\tfrac{1}{2}\tau_{n+1/2}[\boldsymbol{R}^{-+}\boldsymbol{w}^{+-}\boldsymbol{u}^+ +\boldsymbol{R}^{--}\boldsymbol{w}^{--}\boldsymbol{u}^-]_{n+1/2}\\
&\quad+\tfrac{1}{2}\tau^d_{n+1/2}\tilde{\omega}[\boldsymbol{P}^{-+}\boldsymbol{c}\boldsymbol{u}^+ +\boldsymbol{P}^{--}\boldsymbol{c}\boldsymbol{u}^-]_{n+1/2}\\
&\quad+\tau^d_{n+1/2}(1-\tilde{\omega})\boldsymbol{B}_{n+1/2}
\end{aligned}\tag{A2}$$

where, for a shell bounded by the radii r_n and r_{n+1}

$$\boldsymbol{u}^\pm_n=4\pi r_n^2\begin{bmatrix} I(x_1,\ \pm\mu_1,r_n)\\ I(x_1,\ \pm\mu_2,r_n)\\ \cdots\cdots\cdots\\ I(x_1,\ \pm\mu_J,r_n)\\ \cdots\cdots\cdots\\ I(x_K,\ \pm\mu_1,r_n)\\ \cdots\cdots\cdots\\ I(x_K,\ \pm\mu_J,r_n)\end{bmatrix}$$

K and J are the number of frequency angle points respectively. ρ_c is the curvature factor defined as $\rho_c=\Delta r/r_n$ and $\Lambda^\pm$ are the curvature matrices (see PG for their derivation). $\phi^\pm$ represent the absorption profile and is defined by

$$\phi^\pm=\phi(x,\pm\mu)\,\delta_{lm}\qquad l=m=j+(K-1)J$$

ϕ being the profile function. The quantities with subscript $n+1/2$ refer averages over the shell.

$$\tau_{n+1/2}=\chi(0,0,\tau_{n+1/2})\,(r_n-r_{n+1})$$

and

$$\tau^d_{n+1/2}=\chi_d(r_n-r_{n+1})$$

$\boldsymbol{E}$ is the identity matrix and

$$\boldsymbol{P}^{++}=\begin{bmatrix}\boldsymbol{P}(+\mu,+\mu') & 0\\ 0 & \boldsymbol{P}(+\mu,+\mu')\end{bmatrix}$$

$\boldsymbol{P}(+\mu,+\mu')$ are identically equal to one for all elements of the matrix and for isotropic scattering similarly $\boldsymbol{P}^{+-}$ etc. are defined. The matrices $\boldsymbol{M}$ and $\boldsymbol{c}$ are defined as

$$\boldsymbol{M}=\begin{bmatrix}\boldsymbol{M}_J & & \\ & \ddots & \\ & & \ddots\end{bmatrix}\qquad \begin{bmatrix}\boldsymbol{c}_j & & \\ & \ddots & \\ & & \ddots\end{bmatrix}$$

where

$$\boldsymbol{M}_J=[\mu_j\delta_{jl}]\quad\text{and}\quad \boldsymbol{c}_j=[c_j\delta_{jl}]$$

μ_j and c_j being the roots and weights of the angle quadrature

$$\boldsymbol{R}^{++}=R(x+\mu v,\ x'+\mu' v)$$

where v is the velocity of the gas in units of Doppler velocities

$$\boldsymbol{S}^\pm_{n+1/2}=\boldsymbol{\phi}^\pm S_{n+1/2}$$

$\boldsymbol{w}^{++}$ contains the normalizing weights for the redistribution function.

By replacing the average intensities $u^\pm_{n+1/2}$ by the diamond scheme and comparing the resulting equations in (2) and (3) with the canonical form [see GP, Eqs. (2.23) and (2.24) respectively] we obtain

$$\begin{bmatrix}
\boldsymbol{M}+\rho_c\boldsymbol{\Lambda}^+/2+\{\tau(\boldsymbol{\phi}^+-\boldsymbol{R}^{++}\boldsymbol{w}^{++}/2)/2\} +\tau_d(\boldsymbol{E}-\tilde{\omega}\boldsymbol{P}^{++}\boldsymbol{c}/2)/2 & \rho_c\boldsymbol{\Lambda}^- -\tau\boldsymbol{R}^{+-}\boldsymbol{w}^{+-}/4-\tau_d\tilde{\omega}\boldsymbol{P}^{++}\boldsymbol{c}^-/4\\
-\rho_c\boldsymbol{\Lambda}^-/2-\tau\boldsymbol{R}^{-+}\boldsymbol{w}^{-+}/4-\tau_d\tilde{\omega}\boldsymbol{P}^{-+}\boldsymbol{c}/4 & \boldsymbol{M}-\rho_c\boldsymbol{\Lambda}^+/2+\{\tau(\boldsymbol{\phi}^- -\boldsymbol{R}^{--}\boldsymbol{w}^{--})/2\} +\tau_d(\boldsymbol{E}-\tilde{\omega}\boldsymbol{P}^{--}\boldsymbol{c}^-/2)/2
\end{bmatrix}\begin{bmatrix}\boldsymbol{u}^+_{n+1}\\ \boldsymbol{u}^-_n\end{bmatrix}=$$

$$\begin{bmatrix}
\boldsymbol{M}-\rho_c\boldsymbol{\Lambda}^+/2-\{\tau(\boldsymbol{\phi}^+-\boldsymbol{R}^{++}\boldsymbol{w}^{++}/2)/2\} -\tau_d(\boldsymbol{E}-\tilde{\omega}\boldsymbol{P}^{++}\boldsymbol{c}/2)/2 & -\rho_c\boldsymbol{\Lambda}^-/2+\tau\boldsymbol{R}^{+-}\boldsymbol{w}^{+-}/4+\tau_d\tilde{\omega}\boldsymbol{P}^{+-}\boldsymbol{c}/4\\
\rho_c\boldsymbol{\Lambda}^-/2+\tau\boldsymbol{R}^{-+}\boldsymbol{w}^{-+}/4+\tau_d\tilde{\omega}\boldsymbol{P}^{-+}\boldsymbol{c}/4 & \boldsymbol{M}+\rho_c\boldsymbol{\Lambda}^+/2-\{\tau(\boldsymbol{\phi}^- -\boldsymbol{R}^{--}\boldsymbol{w}^{--}/2)/2\} -\tau_d(\boldsymbol{E}-\tilde{\omega}\boldsymbol{P}^{--}\boldsymbol{c}/2)/2
\end{bmatrix}\begin{bmatrix}\boldsymbol{u}^+_n\\ \boldsymbol{u}^-_{n+1}\end{bmatrix}$$

$$+\tau\begin{bmatrix}S^+\\S^-\end{bmatrix}+\tau_d(1-\tilde{\omega})B\begin{bmatrix}1\\1\end{bmatrix} \tag{A3}$$

From Equation (A3) we obtain the two pairs of transmission and reflection matrices, which are given by

$$t(n+1,n)=G^{+-}[\Delta^+A+g^{+-}g^{-+}]$$
$$t(n,n+1)=G^{-+}[\Delta^-D+g^{-+}g^{+-}]$$
$$r(n+1,n)=G^{-+}g^{-+}[E+\Delta^+A]$$
$$r(n,n+1)=G^{+-}g^{+-}[E+\Delta^+D]$$

The source terms are

$$\Sigma^+_{n+1/2}=G^{+-}[\Delta^+\{\tau S^++\tau_d(1-\tilde{\omega})B\}+g^{+-}\Delta^-\{\tau S^-+\tau_d(1-\tilde{\omega})B\}]$$

and

$$\Sigma^-_{n+1/2}=G^{-+}[\Delta^-\{\tau S^-+\tau_d(1-\tilde{\omega})B\}+g^{-+}\Delta^-\{\tau S^++\tau_d(1-\tilde{\omega})B\}$$

where

$$G^{+-}=[E-g^{+-}g^{-+}]^{-1}$$
$$g^{+-}=\tau\Delta^+Y_-/2$$

G^{-+} and g^{-+} are obtained by interchanging the signs. The other matrices used are defined by

$$A=M-\tau Z_+/2$$
$$D=M-\tau Z_-/2$$
$$\Delta^+=[M+\tau Z_+/2]^{-1}$$
$$\Delta^-=[M+\tau Z_-/2]^{-1}$$
$$Y_+=\rho_c\Lambda^-/\tau+R^{-+}W^{-+}/2+\tilde{\omega}P^{-+}c/2$$
$$Y_-=-\rho_c\Lambda^-/\tau+R^{+-}W^{+-}/2+\tilde{\omega}P^{+-}c/2$$
$$Z_+=\phi^+-R^{++}w^{++}/2+\rho_c\Lambda^+/\tau+E-\tilde{\omega}P^{++}c/2$$
$$Z_-=\phi^--R^{--}w^{--}-\rho_c\Lambda^+/\tau+E-\tilde{\omega}P^{--}c/2$$

By using these matrices together with the procedure given in Equations (5.1) to (5.6) of PG we obtain the diffuse radiation field.