ON THE STABILITY OF HELICAL VELOCITY AND MAGNETIC FIELDS

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(Received 4 November, 1977)

Abstract. In this paper we study the stability of an infinitely conducting, incompressible, inviscid infinite cylinder with non-parallel helical velocity and magnetic field. It is shown that the system is stable if the energy in the φ-component of the velocity field is larger than that in the φ-component of the magnetic field.

1. Introduction

Trehan (1959) has considered the stability of an inviscid, incompressible, infinitely conducting cylinder in which the streamlines of fluid particles and of the magnetic lines of force are helices and parallel to each other:

$$\vec{U} = U\left(0, \frac{\varpi}{p}, \perp\right) \tag{1}$$

and

$$\bar{B} = B\left(0, \frac{\varpi}{p}, \perp\right), \tag{2}$$

where (ϖ, φ, z) denotes a system of cylindrical coordinates, $2\pi p$ is the pitch of the helices, and U and B are constants with

$$U = \alpha B, \tag{3}$$

where α is a constant. It is found that the system is stable if $\alpha \geqslant 1$. The value $\alpha = 1$ corresponds to the equipartition of energy; the stability of such a system was discussed by Trehan (1958). The system is unstable in the case of a pure magnetic field, $\alpha = 0$ – a result first obtained by Roberts (1956).

In recent times there has been a renewed interest in the stability of systems with twisted magnetic fields in the context of studies of magnetic fields in the solar atmosphere (see, e.g., Parker, 1974). In the present paper we consider the stability of an idealized infinite cylinder in which the magnetic and velocity fields are helices but are not parallel to each other.

Astrophysics and Space Science **55** (1978) 395–400. *All Rights Reserved*

2. Formulation of the Problem

The equations governing the hydromagnetic behaviour of an incompressible inviscid fluid of infinite electrical conductivity are

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_k}(u_j u_k - h_j h_k) = -\frac{\partial \pi}{\partial x_j} \tag{4}$$

and

$$\frac{\partial h_j}{\partial t} + \frac{\partial}{\partial x_k}(h_j u_k - u_j h_k) = 0, \tag{5}$$

where

$$h_j = B_j(4\pi\rho)^{-1/2}, \tag{6}$$

$$\pi = \frac{p}{\varrho} + V + \tfrac{1}{2}h^2. \tag{7}$$

In the above equations, V is the gravitational potential, $\bar{B}$ is the magnetic field, and the remaining symbols have their usual meaning. These equations admit the stationary solution

$$U = A\left(0, \alpha\frac{\varpi}{p}, \beta\right), \tag{8}$$

$$H = A\left(0, \frac{\varpi}{p}, \perp\right) \tag{9}$$

and

$$\pi = \text{constant} + (\alpha^2 - 1)\frac{\varpi^2 A^2}{2p^2}. \tag{10}$$

In the above, α and β are arbitrary constants. At the surface of the cylinder ($\varpi = R$) there is a current sheet, and the total pressure at the surface will be assumed constant.

Let the perturbed state be characterized by

$$U_j + u_j, \qquad H_j + h_j \quad \text{and} \quad \pi + P, \tag{11}$$

where $\vec{u}$, $\vec{h}$ and P are small quantities, the squares and higher powers of which will be neglected. The equations governing the perturbed state are obtained by linearizing Equations (4) and (5). In terms of the variables

$$\xi_j = h_j - u_j \quad \text{and} \quad \eta_j = h_j + u_j \tag{12}$$

the linearized equations are of the form

$$\frac{\partial \xi_j}{\partial t} + \frac{\partial}{\partial x_k}\{(H_j - U_j)\eta_k + (H_k + U_k)\xi_j\} = \frac{\partial P}{\partial x_j} \tag{13}$$

and

$$\frac{\partial \eta_j}{\partial t} - \frac{\partial}{\partial x_k}\{(H_j + U_j)\xi_k + (H_k - U_k)\eta_j\} = -\frac{\partial P}{\partial x_j}. \tag{14}$$

We now assume that all the perturbations are of the form

$$(\text{Function of } \varpi) \times \exp i(\sigma t + kz + m\varphi), \tag{15}$$

where σ is the frequency of oscillation, k is the wave number of the perturbation along the z-axis, and m is a positive or negative integer. Then Equations (13) and (14) lead to the following equations for the various components of $\bar{\xi}$ and $\bar{\eta}$

$$i\omega_1\xi_\varpi - \frac{A}{p}(1 + \alpha)\xi_\varphi - \frac{A}{p}(1 - \alpha)\eta_\varphi = \mathrm{D}P, \tag{16}$$

$$i\omega_1\xi_\varphi + \frac{A}{p}(1 + \alpha)\xi_\varpi + \frac{A}{p}(1 - \alpha)\eta_\varpi = i\frac{mP}{\varpi}, \tag{17}$$

$$i\omega_1\xi_z = ikP, \tag{18}$$

$$i\omega_2\eta_\varpi + \frac{A}{p}(1 + \alpha)\xi_\varphi + \frac{A}{p}(1 - \alpha)\eta_\varphi = -\mathrm{D}P, \tag{19}$$

$$i\omega_2\eta_\varphi - \frac{A}{p}(1 + \alpha)\xi_\varpi - \frac{A}{p}(1 - \alpha)\eta_\varpi = -i\frac{mP}{\varpi} \tag{20}$$

and

$$i\omega_2\eta_z = -ikP, \tag{21}$$

where

$$\mathrm{D} \equiv \mathrm{d}/\mathrm{d}\varpi, \tag{22}$$

$$\omega_1 = \sigma + A\left(\frac{m}{p} + k\right) + A\left(\alpha\frac{m}{p} + \beta k\right) \tag{23}$$

and

$$\omega_2 = \sigma - A\left(\frac{m}{p} + k\right) + A\left(\alpha\frac{m}{p} + \beta k\right). \tag{24}$$

From Equations (16)–(21) we obtain, between the components of $\bar{\xi}$ and $\bar{\eta}$, the relation

$$\omega_1\xi_\varpi + \omega_2\eta_\varpi = 0, \tag{25}$$

and two similar relations for the φ- and z-components. Eliminating $\bar{\eta}$ from Equations (16) and (17) with the aid of Equation (25) we obtain

$$\left[\omega_1^2\omega_2^2 - \frac{A^2}{p^2}\{(1 - \alpha)\omega_1 - (1 + \alpha)\omega_2\}^2\right]\xi_\varpi = $$
$$= i\omega_2\left[-\omega_1\omega_2\,\mathrm{D}P + \frac{A}{p}\{(1 - \alpha)\omega_1 - (1 + \alpha)\omega_2\}\frac{mP}{\varpi}\right] \tag{26}$$

and

$$\left[\omega_1^2\omega_2^2 - \frac{A^2}{p^2}\{(1-\alpha)\omega_1 - (1+\alpha)\omega_2\}^2\right]\xi_\varphi =$$

$$= -\omega^2\left[-\omega_1\omega_2\frac{mP}{\varpi} + \frac{A}{p}\{(1-\alpha)\omega_1 - (1+\alpha)\omega_2\}\,\mathrm{D}P\right]. \tag{27}$$

Equation (18) implies that

$$\xi_z = kP/\omega_1. \tag{28}$$

We now substitute for the three components of $\bar{\xi}$ from Equations (26)–(28) in

$$\mathrm{D}_*\xi_\varpi + \frac{im}{\varpi}\xi_\varphi + ik\xi_z = 0, \tag{29}$$

where

$$\mathrm{D}_* = \mathrm{D} + 1/\varpi. \tag{30}$$

Equation (29) represents the solenoidal character of $\bar{\xi}$ and leads to the differential equation for P of the form

$$\mathrm{D}_*\mathrm{D}P + \left(\kappa^2 - \frac{m^2}{\varpi^2}\right)P = 0, \tag{31}$$

where

$$\kappa^2 = k^2\left[\frac{A^2}{p^2}\frac{\{(1-\alpha)\omega_1 - (1+\alpha)\omega_2\}^2}{\omega_1^2\omega_2^2} - 1\right]. \tag{32}$$

The solution of Equation (31), which is free of singularity at the origin, is

$$P = CJ_m(\kappa\varpi), \tag{33}$$

where C is a constant and J_m is Bessel's function of order m. The boundary condition on P is that P vanishes at $\varpi = R$. Then

$$\kappa R = x_{m,j}, \tag{34}$$

where $x_{m,j}$ is the jth zero of $J_m(x)$. Equation (32) then becomes

$$\omega_1\omega_2 = q\frac{A}{p}\{(1-\alpha)\omega_1 - (1+\alpha)\omega_2\}, \tag{35}$$

where

$$q = \pm\sqrt{\frac{x^2}{x^2 + x_{m,j}^2}} \quad \text{and} \quad x = kR. \tag{36}$$

We note that $|q|$ is a monotonically increasing function of x, $|q| \leqslant 1$, the value 1 being attained only for $x \to \infty$. Now substituting in Equation (35) for ω_1 and ω_2 from Equations (23) and (24), we obtain, on simplification,

$$\sigma^2 + 2A\left\{\alpha\left(\frac{m}{p} + \frac{q}{p}\right) + \beta k\right\}\sigma -$$
$$- A^2\left\{\left(\frac{m}{p} + \frac{q}{p} + k\right)^2 - \left[\alpha\left(\frac{m}{p} + \frac{q}{p}\right) + \beta k\right]^2 - (1 - \alpha^2)\frac{q^2}{p^2}\right\} = 0. \tag{37}$$

The roots of this equation are given by

$$\sigma\frac{R}{A} = -\{\alpha l(m + q) + \beta x\} \pm \{(lm + lq + x)^2 + (\alpha^2 - 1)l^2q^2\}^{1/2}, \tag{38}$$

where

$$l = R/p. \tag{39}$$

Since for any given value of x and m, there are two values of q ($\pm|q|$), there are four modes of oscillation corresponding to the four roots given by Equation (38).

3. Criterion for Stability

The system would be stable to small perturbations if σ is real. From Equation (38), we see that if $\alpha \geqslant 1$ the system is stable. For $\alpha < 1$ there exist modes which would make the system unstable. Let us translate the stability condition in terms of the energies of the velocity and the magnetic fields, which are, respectively,

$$T = \tfrac{1}{2}\pi\rho A^2R^2\left(\frac{\alpha^2 l^2}{2} + \beta^2\right) \tag{40}$$

and

$$M = \tfrac{1}{2}\pi\rho A^2R^2\left(\frac{l^2}{2} + 1\right). \tag{41}$$

Thus

$$T - M = \tfrac{1}{2}\pi\rho A^2R^2[\tfrac{1}{2}l^2(\alpha^2 - 1) + (\beta^2 - 1)]. \tag{42}$$

We see from Equations (40) and (41) that the system would be stable if the energy in the φ-component of the velocity field is greater than or equal to that in the φ-component of the magnetic field. In the special cases when the helical velocity and magnetic fields are parallel ($\alpha = \beta =$ constant) and when there is equipartition of energy ($\alpha = \beta = 1$), we note from Equation (42) that the stability criterion implies that the energy in the velocity field be greater than or equal to that in the magnetic field (Trehan, 1958, 1959).

An inspection of the radical term in Equation (38) reveals how the presence of fluid motions affects the stability of the system with a helical magnetic field. In the case of a pure magnetic field, $\alpha \doteq \beta = 0$, the presence of fluid motions in the form given by Equation (8) results in an additional positive term under the radical sign. Thus all modes which were stable in the case of a pure magnetic field continue to be stable when fluid motions (8) are present. A sufficiently large α ($\alpha^2 \geqslant 1$) ensures that even modes that are unstable in the absence of fluid motions are now stabilized. It is interesting to note that the stability criterion does not depend on β, which is a measure of the z-component of the velocity field.

References

Parker, E. N.: 1974, *Astrophys. J.* **191**, 245.
Roberts, P. H.: 1956, *Astrophys. J.* **124**, 430.
Trehan, S. K.: 1958, *Astrophys. J.* **127**, 446.
Trehan, S. K.: 1959, *Astrophys. J.* **129**, 475.