THE PLANCK LENGTH AS A COSMOLOGICAL CONSTRAINT

C. SIVARAM

Indian Institute of Astrophysics, Bangalore, India

(Received 27 May, 1986)

Abstract. Any attempt at unification of gravity with quantum physics inevitably leads to the Planck length, usually interpreted as defining the distance scale at which quantum corrections to general relativity are expected to become important. Here we arrive at a scalelength of the same magnitude from the cosmological requirement that gravitating vacuum or zero-point energy does not overdominate the dynamics of the Universe. Other cosmological considerations are again seen to imply such a constraining lower scalelength.

The Planck length given by $L_p = (\hbar G/c^3)^{1/2} \simeq 10^{-33}$ cm, provides a natural length unit involving gravity and quantum physics; G being the Newtonian gravitational constant; $\hbar$ and c being Planck's constant and the velocity of light. It defines the distance scale at which quantum corrections to general relativity are expected to be significant. It is also commonly thought to provide the ultimate ultraviolet momentum cut-off rendering finite, divergent particle self-energies and enabling gravity to play the role of a universal regulator to other fundamental interactions such as electromagnetism. For instance, in quantum electrodynamics the electron self-energy arises from the electromagnetic interaction of the 'bare' electron with the virtual photons of all momenta from zero to arbitrarily high values and is expressed as a divergent integral. In practice one usually applies a cut-off to the momenta at some arbitrarily high value and recovers a finite value for the electron mass. This led to the old suggestion (cf. Landau, 1955; Klein, 1956) that gravitation might provide a natural cut-off to the virtual photon energies at wavelengths of the order of L_p. This can be pictured physically as follows: A photon of wavelength $\lambda \simeq (\hbar G/c^3)^{1/2} \simeq 10^{-33}$ cm has a Schwarzschild radius equal to its wavelength! Thus, by a well-known result in general relativity, such a photon will be trapped inside its own gravitational field and will not be able to propagate as the region of space-time inside the Schwarzschild radius is inaccessible to the outside world and the ordering of space-time events will no longer be possible. The same arguments hold for a material particle with a finite rest mass. The wave packet for the particle or photon cannot be localized to a distance of less than L_p, which gives an upper limit to the energy that can be carried by a material particle or photon, i.e., $E_p \simeq (\hbar c^5/2G)^{1/2} \simeq 10^{19}$ GeV, corresponding to the so-called Planck mass of $M_p \simeq (\hbar c/G)^{1/2} \simeq 10^{-5}$ g. We see that at energies above this the particles are always inside their Schwarzschild radii. We can arrive at these results by noting that in both general relativity and quantum mechanics we have restrictions on the localizability of a particle or wave packet. In general relativity (GR) a particle of the total mass m cannot be localized with an accuracy greater than $r_s = 2Gm/c^2$ whereas in quantum mechanics the particle cannot be specified to a distance less than $r_c = \hbar/mc$. For particles at ordinary energies $r_s \ll r_c$ showing that

Astrophysics and Space Science **127** (1986) 133–137.

This is the contribution of a curved space to zero-point energy density. The renormalised constant is the flat space value which is usually normalised to zero. Consider the first term. k is the wave number. Assume the integral cut-off at K_{max} assumed to be Nk_0 where $k_0 \simeq 1/R_H$ cm^{-1}; R_H being the Hubble radius; and k_0 is the smallest wave number. Then the vacuum energy density is given by

$$\rho_{vac} \simeq \tfrac{1}{2}\hbar c R \times N^2 k_0^2 . \tag{2}$$

Given a Hubble constant H_0, the critical density of the Universe is

$$\rho_c \simeq 3H_0^2/8\pi G . \tag{3}$$

Thus in order that the vacuum energy density does not predominate (vacuum energy also contributes to gravitation!) we have

$$\rho_{vac} \lesssim \rho_c . \tag{4}$$

Using for the average curvature R as $R \approx R_H^{-2}$ and $H_0 \simeq 100$ km s^{-1} Mpc^{-1}, corresponding to $\rho_c \simeq 10^{-8}$ ergs cc^{-1}, we get $\hbar c \times N^2 \times R_H^{-4} \simeq 10^{-8}$ and with $R_H \simeq 10^{28}$ cm, we have $N \simeq 10^{61}$ or $K_{max} \simeq NK_0 \simeq 10^{33}$ cm^{-1} or the short wavelength cut-off $\lambda_{min} \simeq 1/K_{max} \simeq 10^{-33}$ cm which is the same as the Planck length! It can be shown that K_{max} is independent of the epoch and λ_{min} is a universal value for the wavelength cut-off (cf. Sivaram, 1986a). The higher-order terms do not significantly contribute to ρ_{vac}. To see this consider the R^n term. Its contribution to ρ_{vac} is: $\rho_{vac} \sim \hbar c/(R_H)^{n+3}$ which becomes very small for large n. The Planck length arising as a cosmological constraint on vacuum energy can also be heuristically seen by using the expression $(T_{00})_{vac} = \Sigma \tfrac{1}{2}\hbar\omega$. Expanding in multiples of the lowest ω given by $\omega_{min} = C/R_H$, we find that the summation can be expressed as $\tfrac{1}{2}\hbar\omega_{min}N_{max}(N_{max} + 1)$, where the cut-off frequency is $\omega_{max} = N_{max}\omega_{min}$. If we now insist that the total vacuum energy be less than the mass of the Universe (M_U) – i.e.,

$$\tfrac{1}{2}\hbar\omega_{min}N_{max}^2 \leq M_U c^2 , \tag{5}$$

$$N_{max} \simeq \left(\frac{M_U c R_H}{\hbar}\right)^{1/2} \simeq 10^{61} , \tag{6}$$

again giving ω_{max} or $\lambda_{min} \simeq 10^{-33}$ cm. In earlier papers (Sivaram, 1982a, b, 1984a, b) it was pointed out that the characteristic length

$$l_0 = e^2/2m_e c^2 = g^2/2m_p c^2 = \hbar/m_\pi c \simeq 1.4 \times 10^{-13}\ \text{cm} \tag{7}$$

was a very pertinent scale-length for nuclear and atomic fundamental processes (m_π is the pion mass; e^2 and g^2 relate to electromagnetic and nuclear couplings; $e^2/\hbar c = \frac{1}{137}$, $g^2/\hbar c \simeq 14$; m_p and m_e are the photon and electron masses). If it be argued that the initial state of the Universe was a configuration with scale l_0, then its gravitational self-energy would have been $GM_U^2/l_0 \sim 10^{119}$ ergs, some 10^{42} times the total rest energy of the constituents, i.e., the gravity would have been too strong for the Universe to have emerged out of the initial epoch, it would have simply recollapsed. The only way to avoid it would have been if a large cosmological repulsive (Λ-term) had overcome this gravitational self-energy and set the system into an initial inflationary phase. The magnitude of the initial curvature (Λ_0) required can be estimated by equating the curvature energy $\Lambda_0 c^2 M_U l_0^2$ to GM_U^2/l_0, giving

$$\Lambda_0 \simeq GM_U/c^2 l_0^3 \simeq GM_U c^4 m_e^3/e^6 \simeq 10^{66}\, cm^{-2}\,; \tag{8}$$

which agrees with the Planck curvature ($\Delta_{Pl} \sim c^3/\hbar G) \sim 10^{66}$ cm^{-2}).

Such a large Λ-term might have been induced at the earliest epoch when the scale invariance of the high-energy gravitational action $L \sim C^2$ (C is the Weyl curvature) was broken at Planck energies. We see the necessity of a post-Planckian inflationary epoch to get the Universe going so it could overcome its self-gravity. We have shown elsewhere (cf. Sivaram, 1986a) that the present value of the cosmological constant is constrained by

$$\Lambda \simeq 6\hbar H_0 m_e^3 c^2 G/e^6 \leq 10^{-57}\ \mathrm{cm}^{-2}\,. \tag{9}$$

As a final example of how a scale of a Planck length might arise from cosmological considerations, we take a particle at ultra-relativistic energy for which we can write $v/c = \sqrt{1 - m_0^2/m^2} \simeq 1 - \frac{1}{2}(m_0^2/m^2)$ (for $m \gg m_0$). The velocity difference from that of light can be expressed as $(C - V) \simeq C/2(m_0/m)^2$ and this can be made arbitrarily small.

The length given by Equation (7) along with the Hubble time $1/H_0$ can be combined to give the smallest measurable $(C - V)$ using atomic or nuclear processes as

$$(C - V) \simeq H_0 \hbar/m_\pi c\,; \tag{10}$$

imposing a cosmological limit on the local applicability of special relativity. This gives the maximum energy to which the particle can be meaningfully said to be accelerated as $(m/m_0) \simeq (m_\pi c^2/\hbar H_0)^{1/2}$ or with $m_0 = m_\pi$ (a typical particle rest mass) the maximum kinetic mass is $m \simeq (m_\pi^3 c^2/\hbar H_0)^{1/2}$; the corresponding Lorentz contracted length is $(\hbar/m_\pi c)\,(\hbar H_0/m_\pi c^2)^{1/2}$ and this would be the *smallest definable* length, as length and mass, scale inversely. Identifying this with the Planck length $(\hbar G/c^3)^{1/2}$ we would get

$$(\hbar/m_\pi c)\,(\hbar H_0/m_\pi c^2)^{1/2} \simeq (\hbar G/c^3)^{1/2} \tag{11}$$

or

$$m_\pi = (\hbar^2 H_0/Gc)^{1/3}\,; \tag{12}$$

which is the intriguing relation pointed out by Weinberg (1972) and which was sought to be explained as a cosmological constraint on particle masses (Sivaram, 1983).

References

Klein, O.: 1956, *Helv. Phys. Acf. Suppl.* **4**, 93.
Landau, L. D.: 1955, in *Niels Bohr and Development of Physics*, McGraw-Hill, New York.
Motz, L.: 1962, *Nuovo Cimento* **26**, 1.
Motz, L.: 1972, *Nuovo Cimento* **12B**, 239.

Pauli, W.: 1956, *Helv. Phys. Act. Suppl.* **4**, 136.
Sakharov, A.: 1968, *Soviet Phys. Dokl.* **12**, 1040.
Sivaram, C.: 1974, *Nuovo Cimento Letters* **10**, 227.
Sivaram, C.: 1982a, *Astrophys. Space Sci.* **88**, 507.
Sivaram, C.: 1982b, *Am. J. Phys.* **50**, 279.
Sivaram, C.: 1983, *Am. J. Phys.* **51**, 277.
Sivaram, C.: 1984a, in De Sabbata (ed.), *Relativistic Astrophysics and Cosmology*, World Scientific, Singapore, pp. 228–247.
Sivaram, C.: 1984b, *Astrophys. Space Sci.* **100**, 467.
Sivaram, C.: 1986a, *Int. J. Theor. Phys.* **25**, 825.
Sivaram, C.: 1986b, to be published.
Weinberg, S.: 1972, *Gravitation and Cosmology*, Wiley and Sons, New York.