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INTEGRAL OPERATOR TECHNIQUE OF LINE TRANSFER

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ABSTRACT

We present details of a new technique of obtaining a numerical solution of the radiative transfer equation in spherical symmetry in a polychromatic system. We employed the integral operators together with an interpolation formula of the specific intensity defined on radius-angle-frequency grid. We found that the method is quite stable and employs a large stepsize in all the variables on the grid. We employed complete redistribution in calculating the lines.

Key Words: Spectral lines, Radiative transfer equation in spherical symmetry.

1. Introduction

There are several methods of solving the line transfer equation (Kalkofen, 1984). Most of them are iterative in nature and therefore time consuming and not accurate to the desired degree. Although speed is necessary in many situations, the accuracy is more important particularly when we wish to calculate flux carried in a resonance line. Therefore we are forced to develop new methods which emphasize accuracy but we have still kept in mind the speed with which the computations have to be done. For such purposes, we need to have large stepsized algorithms. We have developed such an algorithm in this paper and applied to line formation problems.

2. Formulation of the Problem

The equation of line transfer can be written in spherical symmetry as, (Peraiah, 1984)

$$\mu \frac{\partial I'(r,\mu,x)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} \{I'(r,\mu,x)\}$$

$$= K_L(r,\mu,x) \; \{\beta+\phi(r,\mu,x)\}\{s(r,\mu,x)-I'(r,\mu,x)\}$$

$$\text{for} \quad 0 \le \mu \le 1 \tag{1}$$

and

$$-\mu \frac{\partial I(r,-\mu,x)}{\partial r} - \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} \{I'(r,-\mu,x)\}$$

$$= K_L(r,-\mu,x) \; \{\beta+\phi(r,-\mu,x)\} \; \{s(r,-\mu,x)-I'(r,-\mu,x)\} \tag{2}$$

Where $I'(r,\mu,x)$ is the specific intensity of the ray making an angle $\cos^{-1}\mu$ with the radius vector K_L is the line absorption coefficient, β is the ratio of continuum to line absorption coefficients $\phi(r,\mu,x)$ is the profile function and $s(r,\mu,x)$ is the source function The source function is given by,

$$s(x,\mu,r) = \frac{\phi(x,\mu,r)}{\beta+\phi(x,\mu,r)} S_L(r) + \frac{\beta}{\beta+\phi(x,\mu,r)} S_C(r) \tag{3}$$

Where S_L and S_C are the source functions in the line and continuum respectively

Further more

$$S_C(r) = \rho(r) B(\nu_0, T_e(r)) \tag{4}$$

Where $B(\nu_0, T_e(r))$ is the Planck function at frequency ν_0 and temperature T_e. We assume that the factors ρ and B are independent of r The mean optical depth at frequency ν_0 is defined in terms of geometrical depth r by,

$$d\tau = K_L(r)dr = \frac{h\nu_0}{4\pi\Delta} (N_1 B_{12} - N_2 B_{21})dr \tag{5}$$

Where B_{12} and B_{21} are the Einstein's coefficients and $N_1(r)$ and $N_2(r)$ are the population densities of the lower and upper states respectively The line source function for complete redistribution is given by,

$$S_L(r) = A_{21} N_2(r)/(B_{12} N_1(r) - B_{21} N_2(r)) \tag{6}$$

The statistical equilibrium equation is given by

$$N_1 [B_{12} \int_{-\infty}^{+\infty} \phi(x) J_x dx + C_{12}] = N_2 [A_{21} + C_{21} + B_{21} \int_{-\infty}^{+\infty} \phi(x) J_x dx] \tag{7}$$

When we combine equations (6) and (7) we obtain,

$$S_L(r) = (1-\varepsilon) \int_{-\infty}^{+\infty} \phi(x) J_x dx + \varepsilon B \tag{8}$$

Where J_x is the mean intensity given by

$$J_x = \frac{1}{2} \int_{-1}^{+1} I'_x \, d\mu \tag{9}$$

and

$$\epsilon = \frac{C_{21}}{C_{21}+A_{21}[1-\exp(-h\nu_o/KT)]^{-1}} \qquad (10)$$

Where ϵ is the probability per scatter that a photon will be destroyed by collisional de-excitation. For LTE, $\epsilon = 1$ and for non-LTE, $\epsilon \ll 1$. We shall specify the quantities $\beta, \rho, \epsilon, \phi$ and B in advance.

3. Method of Obtaining the Solution

By writing that

$$I(r,\mu,x) = 4\pi r^2 I'(r,\mu,x)$$

$$S(r,\mu,x) = 4\pi r^2 s(r,\mu,x) \qquad (11)$$

We can rewrite the equations (1) and (2) as

$$\mu \frac{\partial I(r,\mu,x)}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \mu} \{(1-\mu^2) I(r,\mu,x)\}$$

$$= K_L(r,\mu,x) \{\beta + \phi(r,\mu,x)\}\{S(r,\mu,x)-I(r,\mu,x)\} \qquad (12)$$

and

$$-\mu \frac{\partial I(r,-\mu,x)}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \mu} \{(1-\mu^2) I(r,-\mu,x)\}$$

$$= K_L(r,-\mu,x)\{\beta + \phi(r,-\mu,x)\}\{S(r,-\mu,x)-I(r,-\mu,x)\} \qquad (13)$$

We shall expand the specific intensity $I(r,\mu,x)$ by the following interpolation formula (Peraiah and Varghese,1985).

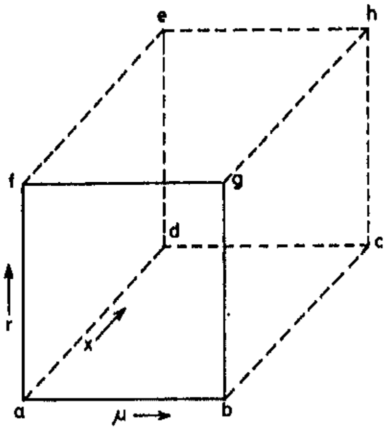
$$I(r,\mu,x) = I_o + I_r\alpha + I_\mu\beta + I_x\gamma + I_{r\mu}\alpha\beta$$

$$+ I_{\mu x}\beta\gamma + I_{xr}\gamma\alpha + I_{r\mu x}\alpha\beta\gamma \qquad (14)$$

Where I_o, I_r, I_μ, I_x, $I_{r\mu}$, $I_{\mu x}$, I_{xr}, $I_{r\mu x}$ are the interpolation coefficients and the quantities α, β, γ are given by

$$\alpha = \frac{r - \bar{r}}{\Delta r/2}$$

$$\beta = \frac{\mu - \bar{\mu}}{\Delta\mu/2}$$



$a \longrightarrow (r_{i-1}, \mu_{j-1}, x_{k-1})$

$b \longrightarrow (r_{i-1}, \mu_{j}, x_{k-1})$

$c \longrightarrow (r_{i-1}, \mu_{j}, x_{k})$

$d \longrightarrow (r_{i-1}, \mu_{j-1}, x_{k})$

$e \longrightarrow (r_{i}, \mu_{j-1}, x_{k})$

$f \longrightarrow (r_{i}, \mu_{j-1}, x_{k-1})$

$g \longrightarrow (r_{i}, \mu_{j}, x_{k-1})$

$h \longrightarrow (r_{i}, \mu_{j}, x_{k})$

Fig.1 Schematic diagram of the radius - angle - frequency mesh

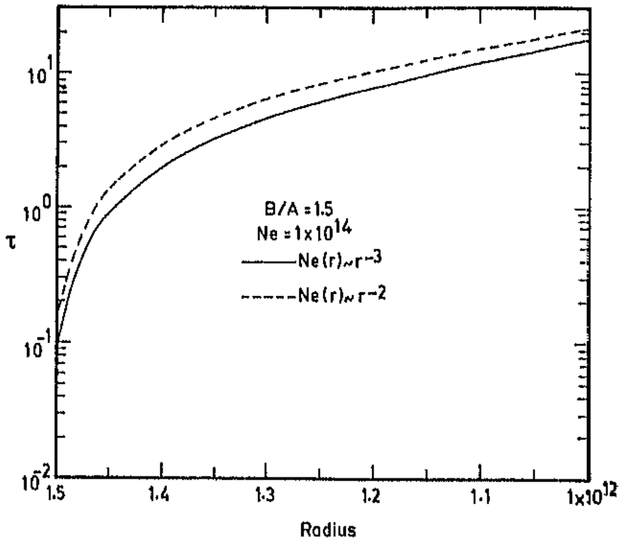


Fig.2 Optical depth is plotted against radius for case 1 and case 2 for B/A = 1.5

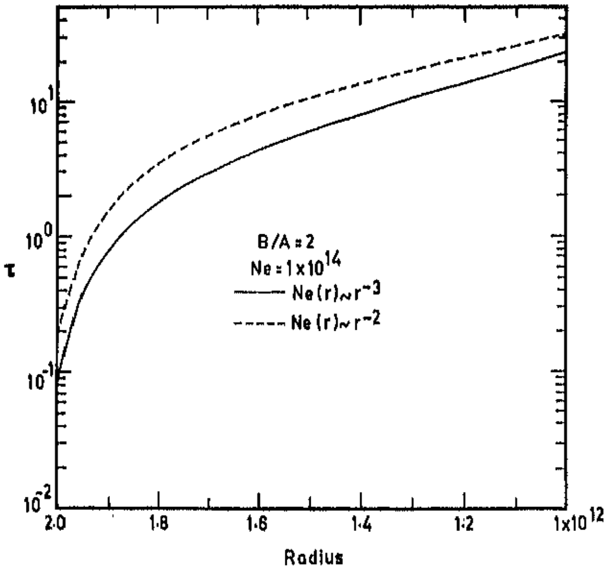


Fig.3 Same as those in Figure 1 with B/A = 2

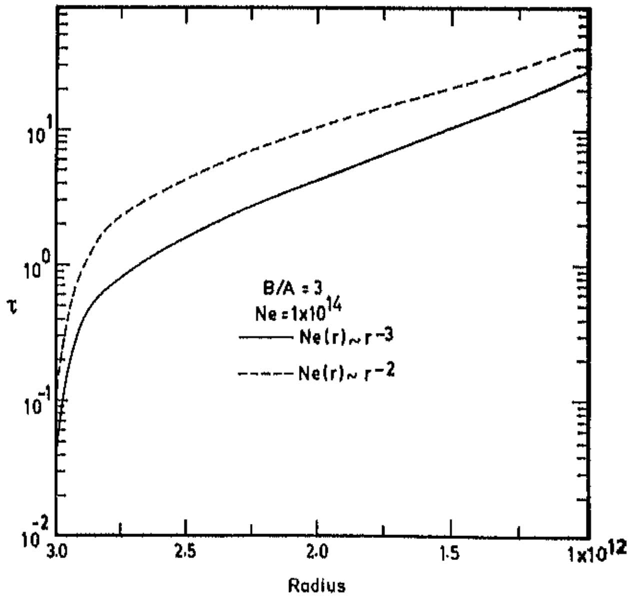


Fig.4 Same as those in Figure 1 but with B/A = 3

$$\gamma = \frac{x - \bar{x}}{\Delta x/2} \tag{15}$$

Where

$$\bar{r} = \frac{1}{2}\,(r_{i+1} + r_i), \quad \Delta r = (r_{i+1} - r_i)$$

$$\bar{\mu} = \frac{1}{2}\,(\mu_{j+1} + \mu_j), \quad \Delta\mu = (\mu_{j+1} - \mu_j)$$

$$\bar{x} = \frac{1}{2}\,(x_{k+1} + x_k), \quad \Delta x = (x_{k+1} - x_k)$$

Similarly we can express the source function as,

$$\begin{aligned} S(r,\mu,x) = {} & S_o + S_r\alpha + S_\mu\beta + S_x\gamma + S_{r\mu}\alpha\beta \\ & + S_{\mu x}\beta\gamma + S_{xr}\gamma\alpha + S_{r\mu x}\alpha\beta\gamma \end{aligned} \tag{16}$$

We substitute equations (14) and (16) in equations (12) and (13) and obtain,

$$\begin{aligned} & \frac{2\mu}{\Delta r}\,\{I_r + I_{r\mu}\beta + I_{xr}\gamma + I_{r\mu x}\beta\gamma\} \\ & - \frac{2\mu}{r}\,\{I_o + I_r\alpha + I_\mu\beta + I_x\gamma + I_{r\mu}\alpha\beta \\ & \quad + I_{\mu x}\beta\gamma + I_{xr}\gamma\alpha + I_{r\mu x}\alpha\beta\gamma\} \\ & + \frac{1-\mu^2}{r} \cdot \frac{2}{\Delta\mu}\,\{I_\mu + I_{r\mu}\alpha + I_{\mu x}\gamma + I_{r\mu x}\alpha\gamma\} \\ & = K_L(r,\mu,x)\,[(S_o + S_r\alpha + S_\mu\beta + S_x\gamma + S_{r\mu}\alpha\beta \\ & + S_{\mu x}\beta\gamma + S_{xr}\gamma\alpha + S_{r\mu x}\alpha\beta\gamma) - (I_o + I_r\alpha + I_\mu\beta \\ & + I_x\gamma + I_{r\mu}\alpha\beta + I_{\mu x}\beta\gamma + I_{xr}\gamma\alpha + I_{r\mu x}\alpha\beta\gamma)] \end{aligned} \tag{17}$$

and

$$\begin{aligned} & - \frac{2\mu}{r}\,\{I_r + I_{r\mu}\beta + I_{xr}\gamma + I_{r\mu x}\beta\gamma\} + \frac{2\mu}{r}\,\{I_o + I_r\alpha \\ & + I_\mu\beta + I_x\gamma + I_{r\mu}\alpha\beta + I_{\mu x}\beta\gamma + I_{xr}\gamma\alpha + I_{r\mu x}\alpha\beta\gamma\} \end{aligned}$$

$$- \frac{1-\mu^2}{r} \quad \frac{2}{\Delta\mu} \{ I_\mu + I_{r\mu}\alpha + I_{\mu x}\gamma + I_{r\mu x}\alpha\gamma \}$$

$$= K(r,-\mu,x) \; [(S_o + S_r\alpha + S_\mu\beta + S_x\gamma + S_{r\mu}\alpha\beta + S_{\mu x}\beta\gamma + S_{xr}\gamma\alpha$$

$$+ S_{r\mu x}\alpha\beta\gamma) - (I_o + I_r\gamma + I_x\gamma + I_{r\mu}\alpha\beta + I_{\mu x}\beta\gamma$$

$$+ I_{xr}\gamma\alpha + I_{r\mu x}\alpha\beta\gamma)] \tag{18}$$

We apply the following integral operators on the radius-angle-frequency mesh (see Fig.1)

$$M_\mu = \frac{1}{\Delta\mu} \int_{\Delta\mu} \ldots\ldots \, d\mu$$

$$R_r = \frac{1}{V} \int_{\Delta r} \ldots\ldots \, 4\pi r^2 dr$$

$$\text{and} \quad F_x = \frac{1}{\Delta x} \int_{\Delta x} \ldots\ldots \, dx \tag{19}$$

We shall apply the operator M_μ on equations (17) and (18) and obtain,

$$\frac{2}{\Delta r} [\bar{\mu}(I_r + I_{xr}\gamma) + \frac{1}{6} \Delta\mu (I_{r\mu} + I_{r\mu x}\gamma)]$$

$$+ \frac{2}{r} \frac{1}{\Delta\mu} (1-\bar{\mu}^2) [I_\mu + I_{r\mu}\alpha + I_{\mu x}\gamma + I_{r\mu x}\alpha\gamma]$$

$$- \frac{2\bar{\mu}}{r} \{ I_o + I_r\alpha + I_x\gamma + I_{xr}\gamma\alpha \} - \frac{1}{3} \frac{\Delta\mu}{r}$$

$$\{ I_\mu + I_{r\mu}\alpha + I_{\mu x}\gamma + I_{r\mu x}\alpha\gamma \}$$

$$= K[(S_o + S_r\alpha + S_x\gamma + S_{xr}\gamma\alpha) - (I_o + I_r\alpha + I_x\gamma + I_{xr}\gamma\alpha)] \tag{20}$$

and

$$- \frac{2}{\Delta r} [\bar{\mu}(I_r + I_{xr}\gamma) + \frac{\Delta\mu}{6} (I_{r\mu} + I_{r\mu x}\gamma)]$$

$$- \frac{2}{r} \quad \frac{1}{\Delta\mu} (1-\mu^2) [I_\mu + I_{r\mu}\alpha + I_{\mu x}\gamma + I_{r\mu x}\alpha\gamma]$$

$$+ \frac{2}{r} \bar{\mu}[I_o + I_r\alpha + I_x\gamma + I_{xr}\gamma\alpha]$$

$$+ \frac{1}{3} \frac{\Delta\mu}{r} [I_{\mu} + I_{r\mu}\alpha + I_{\mu x}\gamma + I_{r\mu x}\alpha\gamma]$$

$$= K [S_{o} + S_{r}\alpha + S_{x}\gamma + S_{xr}\gamma\alpha - (I_{o} + I_{r}\alpha + I_{x}\gamma + I_{xr}\alpha\gamma)] \quad (21)$$

We shall apply the operator R_r on equations (20) and (2i) and obtain,

$$\frac{2}{\Delta r} [\bar{\mu}(I_{r} + I_{xr}\gamma) + \frac{1}{6}\Delta\mu(I_{r\mu} + I_{r\mu x}\gamma)]$$

$$+ \frac{2}{\Delta\mu} (1-\bar{\mu}^{2}) [(I_{\mu} + I_{\mu x}\gamma) - \frac{1}{2} \frac{\Delta A}{V} + (I_{r\mu} + I_{r\mu x}\gamma) . \frac{1}{\Delta r}$$

$$(2-\bar{r} . \frac{\Delta A}{V})] - 2 \bar{\mu} [(I_{o} + I_{x}\gamma) \frac{1}{2} \frac{\Delta A}{V}$$

$$+ (I_{r} + I_{xr}\gamma) \frac{1}{\Delta r} (2-\bar{r} . \frac{\Delta A}{V})]$$

$$- \frac{\Delta\mu}{3} [(I_{\mu} + I_{\mu x}\gamma) . \frac{1}{2} \frac{\Delta A}{V} + (I_{r\mu} + I_{r\mu x}\gamma) . \frac{1}{\Delta r}$$

$$(2-\bar{r} \frac{\Delta A}{V})] = K [\{(S_{o} + S_{x}\gamma)+(S_{r} + S_{xr}\gamma) . \frac{1}{6} \frac{\Delta A}{\bar{A}}$$

$$- \{(I_{o} + I_{x}\gamma)+(I_{r} + I_{xr}\gamma) . \frac{1}{6} \frac{\Delta A}{\bar{A}} \}] \quad (22)$$

and

$$- \frac{2}{\Delta r} [\bar{\mu}(I_{r} + I_{xr}\gamma)+ \frac{\Delta\mu}{6} (I_{r\mu} + I_{r\mu x}\gamma)]$$

$$- \frac{2}{\Delta\mu} (1-\bar{\mu}^{2}) [(I_{\mu} + I_{\mu x}\gamma) . \frac{1}{2} \frac{\Delta A}{V} + (I_{r\mu} + I_{r\mu x}\gamma) . \frac{1}{\Delta r}(2-\bar{r} \frac{\Delta A}{V})]$$

$$+ 2\mu[(I_{o} + I_{x}\gamma) . \frac{1}{2} \frac{\Delta A}{V} + (I_{r} + I_{xr}\gamma) . \frac{1}{\Delta r} (2-\bar{r} \frac{\Delta A}{V})]$$

$$+ \frac{\Delta\mu}{3} [(I_{\mu} + I_{\mu x}\gamma) . \frac{1}{2} \frac{\Delta A}{V} + (I_{r\mu} + I_{r\mu x}\gamma) \frac{1}{\Delta r} (2-\bar{r} \frac{\Delta A}{V})]$$

$$= K [\{(S_{o} + S_{x}\gamma)+(S_{r} + S_{xr}\gamma) \frac{1}{6} \frac{\Delta A}{\bar{A}}\}$$

$$- \{(I_{o} + I_{x}\gamma)+(I_{r} + I_{xr}\gamma) \frac{1}{6} \frac{\Delta A}{\bar{A}} \}] \quad (23)$$

Where

$$\bar{\mu}^{2} = (\bar{\mu})^{2} + \frac{1}{12} (\Delta\mu)^{2} \quad (24)$$

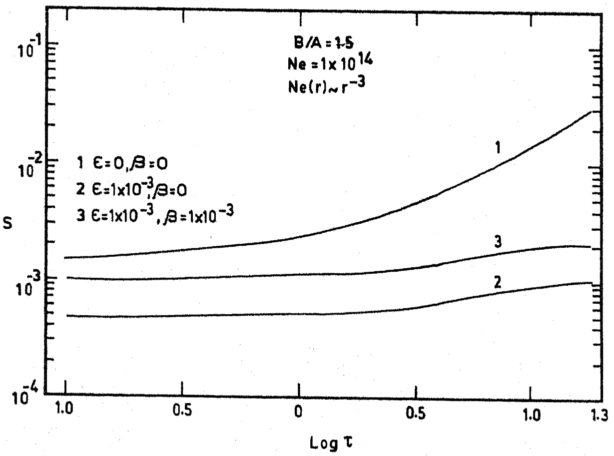


Fig.5 The source function is plotted against the optical depth. For the parameter given in the Figure with B/A = 1.5

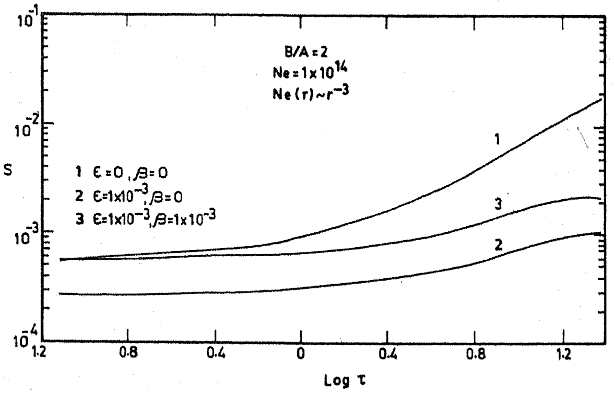


Fig.6 Same as those in Figure 4, with B/A = 2

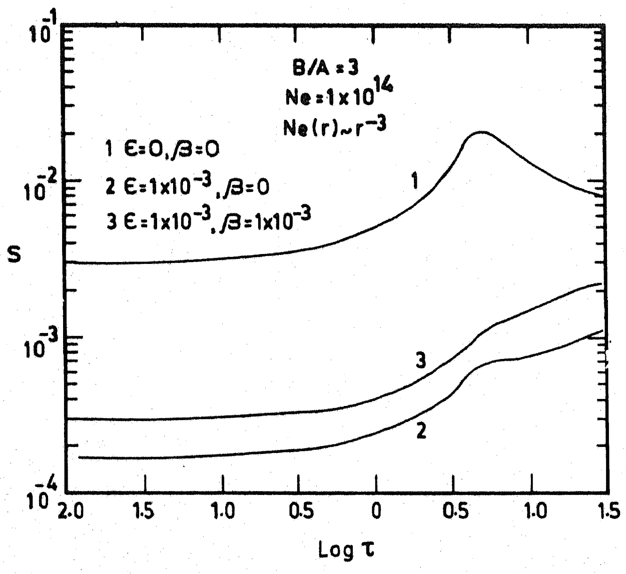


Fig.7 Same as those in Figure 4 with B/A = 3

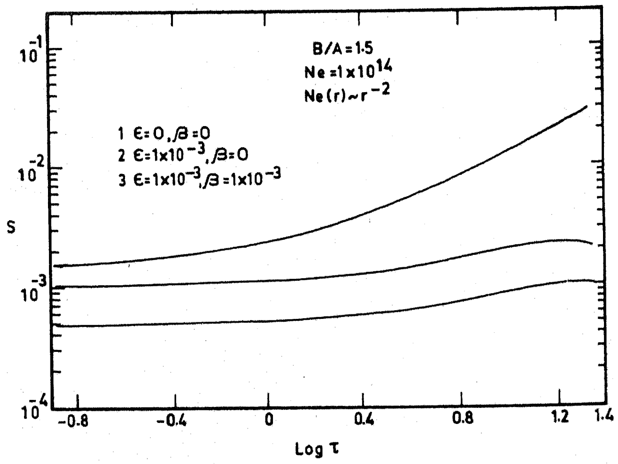


Fig.8 Source function are given with respect to the optical depth for case 2 with B/A = 1.5

$$V = \frac{4}{3} \pi (r_{i+1}^{3} - r_{i}^{3}) \tag{25}$$

$$\bar{A} = V/\Delta r \tag{26}$$

$$\Delta A = 4\pi (r_{i+1}^{2} - r_{i}^{2}) \tag{27}$$

We apply the operator F_x on equations (22) and (23) and obtain,

$$\begin{aligned} &\frac{2}{\Delta r} [\bar{\mu} I_r + \frac{1}{6}\Delta\mu \; I_{r\mu}] + \frac{2}{\Delta\mu} (1-\bar{\mu}^2)[\frac{1}{2}. \; \frac{\Delta A}{V} \; I_\mu \\ &+ \frac{1}{\Delta r} (2-\bar{r}. \; \frac{\Delta A}{V}) I_{r\mu}]-2\bar{\mu}[\; \frac{1}{2} \; \frac{\Delta A}{V} \; I_o \\ &+ \frac{1}{\Delta r} (2-\bar{r} \; \frac{\Delta A}{V}) I_r] - \frac{1}{3} \Delta\mu[\; \frac{1}{2} \frac{\Delta A}{V} \; I_\mu \\ &+ \frac{1}{\Delta r} (2-\bar{r} \; \frac{\Delta A}{V}) \; I_{r\mu}] = K \; [\{S_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} \; S_r\} \\ &- \{I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} \; I_r\}] \end{aligned} \tag{28}$$

and

$$\begin{aligned} &- \frac{2}{\Delta r} [\bar{\mu} I_r + \frac{1}{6}\Delta\mu I_{r\mu}] \\ &- [\; \frac{1}{2} \frac{\Delta A}{V} \; I_\mu + \frac{1}{\Delta r} (2-\bar{r} \; \frac{\Delta A}{\bar{A}}) \; I_{r\mu}] \; \frac{2(1-\bar{\mu}^2)}{\Delta\mu} + 2\bar{\mu} \\ &[\; \frac{1}{2} \frac{\Delta A}{V} \; I_o + \frac{1}{\Delta r} (2-\bar{r} \; \frac{\Delta A}{V}) I_r] + \frac{1}{3}\Delta\mu \; [\; \frac{1}{2} \frac{\Delta A}{V} \; I_\mu \\ &+ \frac{1}{\Delta r} (2-\bar{r} \; \frac{\Delta A}{V}) I_{r\mu}] = K \; [\{S_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} \; S_r\} - \{I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} \; I_r\}] \end{aligned} \tag{29}$$

We shall replace the quantities I_o, I_r, I_μ, I_x, $I_{r\mu}$, $I_{\mu x}$, I_{xr}, and $I_{r\mu x}$ by the nodal values I_A, I_B, I_C, I_D, I_E, I_F, I_G and I_H. This gives us,

$$\begin{aligned} I_o &= \frac{1}{8} [+I_A + I_B + I_C + I_D + I_E + I_F + I_G + I_H] \\ I_r &= \frac{1}{8} [-I_A - I_B - I_C - I_D + I_E + I_F + I_G + I_H] \\ I_\mu &= \frac{1}{8} [-I_A + I_B + I_C - I_D - I_E - I_F + I_G + I_H] \end{aligned}$$

$$I_x = \frac{1}{8}[-I_A - I_B + I_C + I_D + I_E - I_F - I_G + I_H]$$

$$I_{r\mu} = \frac{1}{8}[+I_A - I_B - I_C + I_D - I_E - I_F + I_G + I_H]$$

$$I_{\mu x} = \frac{1}{8}[+I_A - I_B + I_C - I_D - I_E + I_F - I_G + I_H]$$

$$I_{xr} = \frac{1}{8}[+I_A + I_B - I_C - I_D + I_E - I_F - I_G + I_H]$$

$$I_{\mu r x} = \frac{1}{8}[-I_A + I_B - I_C + I_D - I_E + I_F - I_G + I_H] \qquad (30)$$

Similarly, we can write the quantities S_o, S_r, S_μ etc. in terms of S_A, S_B etc. Substituting equations (30) into equations (28) and (29) we obtain,

$$A_a I_A^+ + A_b I_B^+ + A_c I_C^+ + A_d I_D^+ + A_e I_E^+ + A_f I_F^+$$

$$+ A_g I_G^+ + A_h I_H^+ = \tau^-(S_A^+ + S_B^+ + S_C^+ + S_D^+] + \tau^-(S_E^+ + S_F^+ + S_G^+ + S_H^+) \qquad (31)$$

and

$$A'_a I_A^- + A'_b I_B^- + A'_c I_C^- + A'_d I_D^- + A'_e I_E^- + A'_f I_F^-$$

$$+ A'_g I_G^- + A'_h I_H^- = \tau^-(S_A^- + S_B^- + S_C^- + S_D^-) + \tau^+(S_E^- + S_F^- + S_G^- + S_H^-) \qquad (32)$$

Where

$$A_a = -l - m + n + p$$
$$A_b = -l + m - n + p$$
$$A_c = -l + m - n + p$$
$$A_d = -l - m + n + p$$
$$A_e = l - m - n + p$$
$$A_f = l - m - n + p$$
$$A_g = l + m + n + p$$
$$A_h = l + m + n + p \qquad (33)$$

$$A'_a = -l' - m' + n' + p'$$
$$A'_b = -l' + m' - n' + p'$$

$$\begin{aligned}
A'_c &= -l' + m' + n' + p' \\
A'_d &= -l' - m' + n' + p' \\
A'_e &= l' - m' - n' + p' \\
A'_f &= l' - m' - n' + p' \\
A'_g &= l' + m' + n' + p' \\
A'_h &= l' + m' + n' + p'
\end{aligned} \tag{34}$$

$$\begin{aligned}
l &= \frac{1}{6}\tau_x \frac{\Delta A}{\bar{A}} - 2\bar{\mu}\,(1 - \frac{\bar{r}}{\Delta r} \,.\, \frac{\Delta A}{\bar{A}}) \\
m &= \frac{\Delta A}{\bar{A}}\,[\,\frac{1-\bar{\mu}^2}{\Delta\mu} - \frac{1}{6}\,\Delta\mu] \\
n &= \frac{1}{3}\,\Delta\mu(\,\frac{\bar{r}}{\Delta r} \,.\, \frac{\Delta A}{\bar{A}} - 1) + \frac{2}{\Delta\mu}\,(1-\bar{\mu}^2)(2 - \frac{\bar{r}}{\Delta r}\,\frac{\Delta A}{\bar{A}}) \\
p &= \tau_x - \bar{\mu}\,\frac{\Delta A}{\bar{A}}
\end{aligned} \tag{35}$$

$$\begin{aligned}
l' &= \frac{1}{6}\tau_x \frac{\Delta A}{\bar{A}} + 2\bar{\mu}(1 - \frac{\bar{r}}{\Delta r} . \frac{\Delta A}{\bar{A}}) \\
m' &= \frac{\Delta A}{\bar{A}}\,\{\frac{1}{6}\Delta\mu - \frac{1-\bar{\mu}^2}{\Delta\mu}\} \\
n' &= \frac{1}{3}\Delta\mu(1 - \frac{\bar{r}}{\Delta r} . \frac{\Delta A}{\bar{A}}) + \frac{2}{\Delta\mu}\,(1-\bar{\mu}^2)(\frac{\bar{r}}{\Delta r} . \frac{\Delta A}{\bar{A}} - 2) \\
p' &= \tau_x + \bar{\mu}\,\frac{\Delta A}{\bar{A}}
\end{aligned} \tag{36}$$

$$\begin{aligned}
\tau_x &= \tau\phi\,(x) = K.\Delta r\phi\,(x) \\
\tau^{\pm} &= \tau\{1 \pm \frac{1}{6}\,\frac{\Delta A}{\bar{A}}\}
\end{aligned} \tag{37}$$

From Figure 1 we have, further more

$$\begin{aligned}
&I^+_A = I^{i,+}_{j,k},\ I^+_B = I^{i,+}_{j+1,k},\ I^+_C = I^{i,+}_{j+1,k+1},\ I^+_D = I^{i,+}_{j,k+1} \\
&I^+_E = I^{i+1,+}_{j,k+1},\ I^+_F = I^{i+1,-}_{j,k},\ I^+_G = I^{i+1,+}_{j+1,k},\ I^+_H = I^{i+1,+}_{j+1,k+1} \\
&I^-_A = I^{i,-}_{j,k},\ I^-_B = I^{i,-}_{j+1,k},\ I^-_C = I^{i,-}_{j+1,k+1},\ I^-_D = I^{i,-}_{j,k+1}
\end{aligned} \tag{38}$$

$$I_E^- = I_{j,k+1}^{i+1,-},\quad I_F^- = I_{j,k}^{i+1,-},\quad I_G^- = I_{j+1,k}^{i+1,-},\quad I_H^- = I_{j+1,k+1}^{i+1,-}$$

$$S_A^+ = S_{j,k}^{i,+},\quad S_B^+ = S_{j+1,k}^{i,+},\quad S_C^+ = S_{j+1,k+1}^{i,+},\quad S_D^+ = S_{j,k+1}^{i,+}$$

$$S_E^+ = S_{j,k+1}^{i+1,+},\quad S_F^+ = S_{j,k}^{i+1,+},\quad S_G^+ = S_{j+1,k}^{i+1,+},\quad S_H^+ = S_{j+1,k+1}^{i+1,+}$$

$$S_A^- = S_{j,k}^{i,-},\quad S_B^- = S_{j+1,k}^{i,-},\quad S_C^- = S_{j+1,k+1}^{i,-},\quad S_D^- = S_{j,k+1}^{i,-}$$

$$S_E^- = S_{j,k+1}^{i+1,-},\quad S_F^- = S_{j,k}^{i+1,-},\quad S_G^- = S_{j+1,k}^{i+1,-},\quad S_D^- = S_{j+1,k+1}^{i+1,-} \tag{39}$$

Where S's are the source functions at all the nodal points. They represent the scattering and the non-scattering parts of the source function. We replace the integrals by the quadrature sums. For example,

$$S_A^+ = S_{j,k}^{i,+} = \frac{1}{2}(1-\varepsilon)\ \phi_{j,k}^{i,+} \sum_{K'=-K}^{K} a_{k'}\ \phi_{k'}^{i} \sum_{j'=1}^{J} C_{j'}$$
$$(I_{j+1,k}^{i,+} + I_{j'+1,k}^{i,-})$$
$$+ \ (\varepsilon\phi_{j,k}^{i,+} + \rho\beta)B_{j,k}^{i,+} \tag{40}$$

Where the term with $B_{j,k}^{i,+}$ represent the interval source. With the above definition, we can write equation (31) for angles μ_j and μ_{j+1}. This is given by,

$$A_j^a\ I_{j,k}^{i,+} + A_{j+1}^b\ I_{j+1,k}^{i,+} + A_{j+1}^c\ I_{j+1,k+1}^{i,+} + A_j^d\ I_{j,k+1}^{i,+}$$
$$A_j^e\ I_{j,k+1}^{i+1,+} + A_j^f\ I_{j,k}^{i+1,+} + A_{j+1}^g\ I_{j+1,k}^{i+1,+} + A_{j+1}^h\ I_{j+1,k+1}^{i+1,+} =$$
$$\tau^-\{[\sigma\phi_{j,k}^{i,+} \sum_{k'=-k}^{K} a_{k'}\ \phi_{k'}^{i} \sum_{j'=1}^{J} C_{j'}(I_{k,j'}^{i,+} + I_{j',k}^{i,-})] + (\varepsilon\phi_{j,k}^{i,+} + \rho\beta)B_{j,k}^{i,+}$$
$$+ \ [\sigma\phi_{j+1,k}^{i,+} \sum_{k'=-k}^{K} a_{k'}\ \phi_{k'}^{i} \sum_{j'=1}^{J} C_{j'}(I_{j',k}^{i,+} + I_{j',k}^{i,-})]$$
$$+ \ (\varepsilon\phi_{j+1,k}^{i,+} + \rho\beta)B_{j+1,k}^{i,+}$$
$$+ \ [\sigma\phi_{j,k+1}^{i,+} \sum_{k'=-k}^{K} a_{k'}\ \phi_{k'}^{i} \sum_{j'=1}^{J} C_{j'}(I_{j',k+1}^{i,+} + I_{j',k+1}^{i,-})]$$

$$+ (\epsilon\phi^{i,+}_{j,k+1} + \rho\beta)B^{i,+}_{j,k+1}$$

$$+ [\sigma\phi^{i,+}_{j+1,k+1} \sum_{k'=-k}^{K} a_{k'} \phi^{i}_{k'} \sum_{j'=1}^{J} C_{j'}(I^{i,+}_{j',k+1} + I^{i,-}_{j', k+1})]$$

$$+ (\epsilon\phi^{i,+}_{j+1,k+1} + \rho\beta)B^{i,+}_{j+1,k+1}\}$$

$$+ \tau^{+}\{[[\sigma\phi^{i+1,+}_{j,k+1} \sum_{k'=-k}^{K} a_{k'} \phi^{i+1}_{k'} \sum_{j'=1}^{J} C_{j'}(I^{i+1,+}_{j',k+1} + I^{i+1,-}_{j',k+1})]$$

$$+ (\epsilon\phi^{i+1,+}_{j,k+1} + \rho\beta)B^{i+1,+}_{j,k+1}$$

$$+ [\sigma\phi^{i+1,+}_{j+1,k} \sum_{k'=-k}^{K} a_{k'} \phi^{i+1}_{k'} \sum_{j'=1}^{J} C_{j'}(I^{i+1,+}_{j',k} + I^{i+1,-}_{j',k})]$$

$$+ (\epsilon\phi^{i+1,+}_{j+1,k} + \rho\beta) B^{i+1,+}_{j+1,k}$$

$$+ [\sigma\phi^{i+1,+}_{j+1,k} \sum_{k'=-k}^{K} a_{k'} \phi^{i+1}_{k'} \sum_{j'=1}^{J} C_{j'}(I^{i+1,+}_{j',k} + I^{i+1,-}_{j',k})]$$

$$+ (\epsilon\phi^{i+1,+}_{j+1,k+1} + \rho\beta) B^{i+1,+}_{j+1,k+1}$$

$$+ [\sigma\phi^{i+1,+}_{j+1,k+1} \sum_{k'=-k}^{K} a_{k'} \phi^{i+1}_{k'} \sum_{j'=1}^{J} C_{j'}(I^{i+1,+}_{j',k+1} + I^{i+1,-}_{j',k+1})]$$

$$+ (\epsilon\phi^{i+1,+}_{j+1,k+1} + \rho\beta) B^{i+1,+}_{j+1,k+1}\} \qquad (41)$$

In a similar way equation (32) can be rewritten. This is given by,

$$A'^{a}_{j} I^{i,-}_{j;k} + A'^{b}_{j+1} I^{i,-}_{j+1,k} + A'^{c}_{j+1} I^{i,-}_{j+1,k+1} + A'^{d}_{j,k+1} I^{i,-}_{j,k+1}$$

$$+ A'^{e}_{j} I^{i+1,-}_{j,k+1} + A'^{f}_{j} I^{i+1,-}_{j,k} + A'^{g}_{j+1} I^{i+1,-}_{j+1,k} + A'^{h}_{j+1} I^{i+1,-}_{j+1,k+1}$$

$$= \tau^{-}\{[\sigma\phi^{i,-}_{j,k}(Y^{i,+}_{k} + Y^{i,+}_{k})] + [\epsilon\phi^{i,-}_{j,k} + \rho\beta] B^{i,-}_{j,k}$$

$$+ [\sigma\phi^{i,-}_{j+1,k}(Y^{i,+}_{k} + Y^{i,-}_{k})] + [\epsilon\phi^{i,-}_{j,k+1} + \rho\beta] B^{i,-}_{j+1,k}$$

$$+ [\sigma\phi^{i,-}_{j,k+1} (Y^{i,+}_{k+1} + Y^{i,-}_{k+1})] + [\epsilon\phi^{i,-}_{j,k+1} + \rho\beta] B^{i,-}_{j,k+1}$$

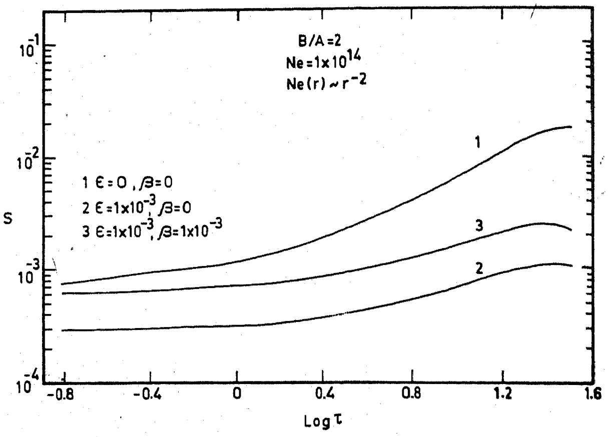


Fig.9 Same as those in Figure 7 with B/A = 2

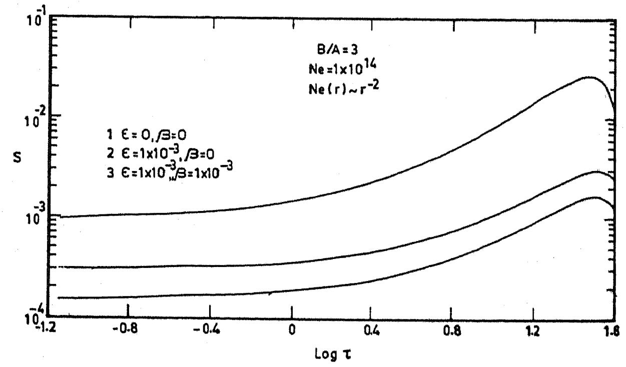


Fig.10 Same as those in Figure 7 with B/A = 3

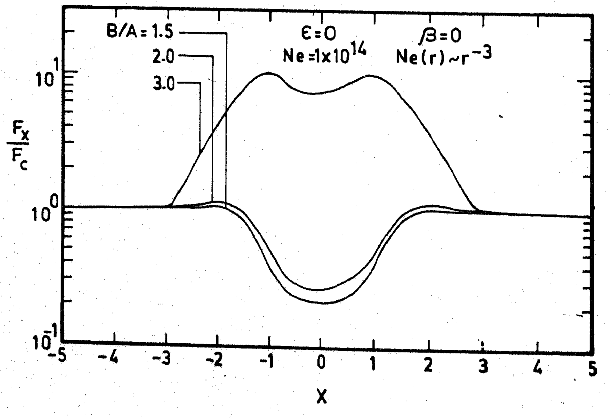


Fig.11 Flux profiles for case 1 in a scattering medium

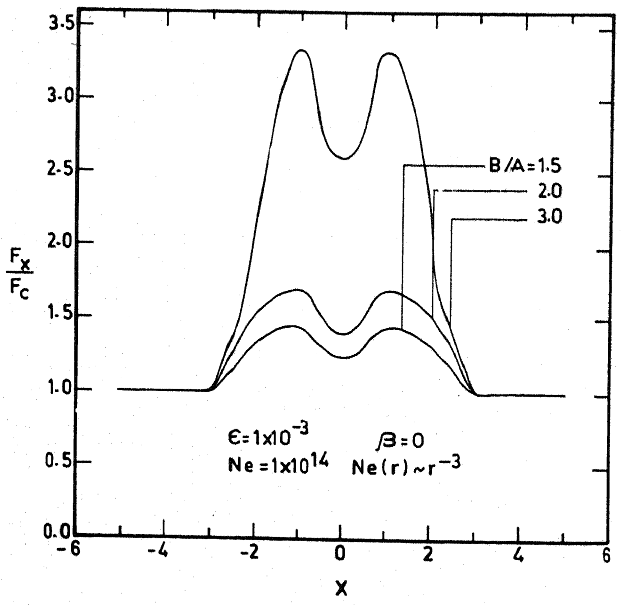


Fig.12 Flux profiles in line emitting medium for case 1

$$+ [\sigma\phi^{i,-}_{j+1,k+1} (Y^{i,+}_{k+1} + Y^{i,-}_{k})] + [\sigma\phi^{i,-}_{j+1,k+1} + \rho\beta] B^{i,-}_{j+1,k+1}$$

$$+ \tau^{+} \{[\sigma\phi^{i+1,-}_{j,k} (Y^{i+1,+}_{k} + Y^{i+1,-}_{k})] + [\epsilon\phi^{i+1,-}_{j,k} + \rho\beta] B^{i+1,-}_{j,k}$$

$$+ [\sigma\phi^{i+1,-}_{j+1,k} (Y^{i+1,+}_{k} + Y^{i+1,-}_{k})] + [\epsilon\phi^{i+1,-}_{j+1,k} + \rho\beta] B^{i+1,-}_{j+1,k}$$

$$+[\sigma\phi^{i+1,-}_{j,k+1} (Y^{i+1,-}_{k+1} + Y^{i+1,-}_{k+1})] + [\epsilon\phi^{i+1,-}_{j,k+1} + \rho\beta] B^{i+1,-}_{j,k+1} \tag{42}$$

$$+ [\sigma\phi^{i+1,-}_{j+1,k+1} (Y^{i+1,+}_{k+1} + Y^{i+1,-}_{k+1})] + [\epsilon\phi^{i+1,-}_{j+1,k+1} + \rho\beta] B^{i+1,-}_{j+1,k+1}\}$$

Where

$$Y^{i,\pm}_{k} = \sum_{k'=-k}^{K} a_{k'} \phi^{i}_{k'} \sum_{j'=1}^{J} C_{j'} I^{i,\pm}_{k',j'} \tag{43}$$

We can rewrite equations (40) and (41) as, ($\sigma = \frac{1}{2}(1-\epsilon)$)

$$[\underset{\sim}{A}^{ab} \underset{\sim}{I}^{i,+}_{k} - \tau^{-}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{i,+}_{k} \underset{\sim}{Y}^{+}_{k}] + [\underset{\sim}{A}^{dc} \underset{\sim}{I}^{i,+}_{k+1} - \tau^{-}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{k,+}_{k+1} \underset{\sim}{Y}^{+}_{k+1}]$$

$$+ [\underset{\sim}{A}^{fg} \underset{\sim}{I}^{i+1,+}_{k} - \tau^{+}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{i+1,+}_{k} \underset{\sim}{Y}^{+}_{k}] + [\underset{\sim}{A}^{eh} \underset{\sim}{I}^{i+1,+}_{k+1} - \tau^{+}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{i+1,+}_{k+1} \underset{\sim}{Y}^{+}_{k+1}]$$

$$=\tau^{-}\sigma\underset{\sim}{Q} [\underset{\sim}{\phi}^{i,+}_{k} \underset{\sim}{Y}^{-}_{k} + \underset{\sim}{\phi}^{i,+}_{k+1} \underset{\sim}{Y}^{-}_{k+1}]$$

$$+ \tau^{+}\sigma\underset{\sim}{Q} [\underset{\sim}{\phi}^{i+1,+}_{k} \underset{\sim}{Y}^{-}_{k} + \underset{\sim}{\phi}^{i+1,+}_{k+1} \underset{\sim}{Y}^{-}_{k+1}]$$

$$+ \tau^{-}\sigma\underset{\sim}{Q}[\rho\beta+\epsilon\underset{\sim}{\phi}^{i,+}_{k}]\underset{\sim}{B}^{i,k}_{k} + \tau^{-}\sigma\underset{\sim}{Q} [\rho\beta+\epsilon\underset{\sim}{\phi}^{i,+}_{k+1}] \underset{\sim}{B}^{i,+}_{k+1}$$

$$+ \tau^{+}\sigma\underset{\sim}{Q} [\rho\beta+\epsilon\underset{\sim}{\phi}^{i+1,+}_{k}]\underset{\sim}{B}^{i+1,+}_{k} + \tau^{+}\sigma\underset{\sim}{Q} [\rho\beta+\epsilon\underset{\sim}{\phi}^{i+1,+}_{k+1}] \underset{\sim}{B}^{i+1,+}_{k+1} \tag{44}$$

and

$$[\underset{\sim}{A}'^{ab} \underset{\sim}{I}^{i,-}_{k} - \tau^{-}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{i,-}_{k} \underset{\sim}{Y}^{-}_{k}] + [\underset{\sim}{A}'^{dc} \underset{\sim}{I}^{i,-}_{k+1} - \tau^{-}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{i,-}_{k+1} \underset{\sim}{Y}^{-}_{k+1}]$$

$$+ [\underset{\sim}{A}'^{fg} \underset{\sim}{I}^{i+1,-}_{k} - \tau^{+}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{i+1,-}_{k} \underset{\sim}{Y}^{-}_{k}] + [\underset{\sim}{A}'^{eh} \underset{\sim}{I}^{i+1,-}_{k+1} - \tau^{+}\sigma\underset{\sim}{Q}\underset{\sim}{\phi}^{i+1,-}_{k+1} \underset{\sim}{Y}^{-}_{k+1}]$$

$$= \tau^{-}\sigma\underset{\sim}{Q}[\underset{\sim}{\phi}^{i,-}_{k} \underset{\sim}{Y}^{+}_{k} + \underset{\sim}{\phi}^{i,-}_{k+1} \underset{\sim}{Y}^{+}_{k+1}] + \tau^{+}\sigma\underset{\sim}{Q}[\underset{\sim}{\phi}^{i+1,-}_{k} \underset{\sim}{Y}^{+}_{k} + \underset{\sim}{\phi}^{i+1,-}_{k+1} \underset{\sim}{Y}^{+}_{k+1}]$$

$$+ \tau^{-}\sigma\underset{\sim}{Q}[\rho\beta+\epsilon\underset{\sim}{\phi}^{i,-}_{k}] \underset{\sim}{B}^{i,-}_{k} + \tau^{-}\sigma\underset{\sim}{Q}[\rho\beta+\epsilon\underset{\sim}{\phi}^{i,-}_{k+1}] \underset{\sim}{B}^{i,-}_{k+1}$$

Where

$$\underset{\sim}{Q} = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \tag{46}$$

$$\underset{\sim}{A}^{ab} = \begin{bmatrix} A_j^a & A_{j+1}^b & & & \\ & A_{j+1}^a & A_{j+2}^b & & \\ & & \ddots & \ddots & \\ & & & A_{J-1}^a & A_J^b \\ & & & & A_J^a \end{bmatrix} \tag{47}$$

$$\underset{\sim}{A}^{dc} = \begin{bmatrix} A_j^d & A_{j+1}^c & & & \\ & A_{j+1}^d & A_{j+2}^c & & \\ & & \ddots & \ddots & \\ & & & A_{J-1}^d & A_J^c \\ & & & & A_J^d \end{bmatrix} \tag{48}$$

$$\underset{\sim}{A}^{fg} = \begin{bmatrix} A_j^f & A_{j+1}^g & & & \\ & A_{j+1}^f & A_{j+2}^g & & \\ & & \ddots & \ddots & \\ & & & A_{J-1}^f & A_J^g \\ & & & & A_J^f \end{bmatrix} \tag{49}$$

$$\underset{\sim}{A}^{eh} = \begin{bmatrix} A^e_j & A^h_{j+1} & & \\ & A^e_{j+1} & A^h_{j+2} & \\ & & \ddots & \ddots \\ & & A^e_{J-1} & A^h_J \\ & & & A^e_J \end{bmatrix} \tag{50}$$

$$\underset{\sim}{A}'^{ab} = \begin{bmatrix} A'^a_j & A'^b_{j+1} & & \\ & A^a_{j+1} & A'^b_{j+2} & \\ & & \ddots & \ddots \\ & & A'^a_{J-1} & A'^b_J \\ & & & A'^a_J \end{bmatrix} \tag{51}$$

$$\underset{\sim}{A}'^{dc} = \begin{bmatrix} A'^d_j & A'^c_{j+1} & & \\ & A'^d_{j+1} & A'^c_{j+2} & \\ & & \ddots & \ddots \\ & & A'^d_{J-1} & A'^c_J \\ & & & A'^d_J \end{bmatrix} \tag{52}$$

$$\underset{\sim}{A}'^{fg} = \begin{bmatrix} A'^f_j & A'^g_{j+1} & & \\ & A'^f_{j+1} & A'^g_{j+2} & \\ & & \ddots & \ddots \\ & & A'^f_{J-1} & A'^g_J \\ & & & A'^f_J \end{bmatrix} \tag{53}$$

$$\underset{\sim}{A}'^{eh} = \begin{bmatrix} A'^{e}_{j} & A'^{h}_{j+1} & & & \\ & A'^{e}_{j+1} & A'^{h}_{j+2} & & \\ & & \ddots & & \\ & & & A'^{e}_{J-1} & A'^{h}_{J} \\ & & & & A'^{e}_{J} \end{bmatrix} \tag{54}$$

$$I_{k}^{i,+} = \begin{bmatrix} I(r_i,\ +\mu_j,\ X_k) \\ I(r_i,\ +\mu_{j+1},\ X_k) \\ I(r_i,\ +\mu_J,\ X_k) \end{bmatrix} \tag{55}$$

and

$$I_{k}^{i,-} = \begin{bmatrix} I(r_i,\ -\mu_j,\ X_k) \\ I(r_i,\ -\mu_{j+1},\ X_k) \\ I(r_i,\ -\mu_J,\ X_k) \end{bmatrix} \tag{56}$$

Similarly the quantities ϕ, B are defined We shall write that

$$\begin{aligned} \underset{\sim}{A}_q^{ab} &= \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}^{ab} \\ \underset{\sim}{A}_q^{dc} &= \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}^{dc} \\ \underset{\sim}{A}_q^{fg} &= \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}^{fg} \\ \underset{\sim}{A}_q^{eh} &= \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}^{eh} \\ \underset{\sim}{A}_q'^{ab} &= \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}'^{ab} \\ \underset{\sim}{A}_q'^{dc} &= \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}'^{dc} \\ \underset{\sim}{A}_q'^{fg} &= \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}'^{fg} \end{aligned} \tag{57}$$

$$\underset{\sim}{A}{}_q^{\prime eh} = \underset{\sim}{Q}^{-1}\ \underset{\sim}{A}{}^{\prime eh}$$

In terms of (56) we can write equations (44) and (45) as,

$$\begin{aligned}
&[\underset{\sim}{A}{}_q^{ab}\ \underset{\sim}{I}{}_k^{i,+} - \tau^{-}\sigma\underset{\sim}{\phi}{}_k^{i,+}\ \underset{\sim}{Y}{}_k^{i,+}] + [\underset{\sim}{A}{}_q^{dc} - \tau^{-}\sigma\underset{\sim}{\phi}{}^{i,+}\ Y_{k+1}^{i,+}] \\
&+ [\underset{\sim}{A}{}_q^{fg}\ \underset{\sim}{I}{}_k^{i+1.+} - \tau^{+}\sigma\underset{\sim}{\phi}{}_k^{i,+}\ Y_k^{i,+}] + [\underset{\sim}{A}{}_q^{eh}\ \underset{\sim}{I}{}_{k+1}^{i+1,+} - \tau^{+}\sigma\underset{\sim}{\phi}{}_{k+1}^{i+1,+}\ \underset{\sim}{Y}{}_{k+1}^{i+1,+}] \\
&= \tau^{-}\sigma[\underset{\sim}{\phi}{}_k^{i,+}\ \underset{\sim}{Y}{}_k^{i,-} + \underset{\sim}{\phi}{}_{k+1}^{i,+}\ Y_{k+1}^{i,-}] + \tau^{+}\sigma[\underset{\sim}{\phi}{}_k^{i+1,+}\ \underset{\sim}{Y}{}_k^{i+1,-} + \underset{\sim}{\phi}{}_{k+1}^{i+1,+}\ \underset{\sim}{Y}{}_{k+1}^{i+1,-}] \\
&\quad + \tau^{-}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_k^{i,+}]\ \underset{\sim}{B}{}_k^{i,+} + \tau^{-}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_{k+1}^{i,+}]\ \underset{\sim}{B}{}_{k+1}^{i,+} \\
&\quad + \tau^{+}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_k^{i+1,+}]\ \underset{\sim}{B}{}_k^{i+1,+} + \tau^{+}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_{k+1}^{i+1,+}]\ \underset{\sim}{B}{}_{k+1}^{i+1,+}
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
&[\underset{\sim}{A}{}_q^{\prime ab}\ \underset{\sim}{I}{}_k^{i,-} - \tau^{-}\sigma\underset{\sim}{\phi}{}_k^{i,-}\ \underset{\sim}{Y}{}_k^{i,-}] + [\underset{\sim}{A}{}_q^{\prime dc}\ \underset{\sim}{I}{}_{k+1}^{i,-} - \tau^{-}\sigma\underset{\sim}{\phi}{}_{k+1}^{i,-}\ \underset{\sim}{Y}{}_{k+1}^{i,-}] \\
&+ [\underset{\sim}{A}{}_q^{\prime fg}\underset{\sim}{I}{}_k^{i+1,-} - \tau^{+}\sigma\underset{\sim}{\phi}{}_k^{i+1,-}\ \underset{\sim}{Y}{}_k^{i+1,-}] + [\underset{\sim}{A}{}_q^{\prime eh}\ \underset{\sim}{I}{}_{k+1}^{i+1,-} - \tau^{+}\sigma\underset{\sim}{\phi}{}_{k+1}^{i+1,-}\underset{\sim}{Y}{}_{k+1}^{i+1,-}] \\
&= \tau^{-}\sigma[\underset{\sim}{\phi}{}_k^{i,-}\ \underset{\sim}{Y}{}_k^{i,+} + \underset{\sim}{\phi}{}^{i,-}\ \underset{\sim}{Y}{}_{k+1}^{i,+}] + \tau^{+}\sigma[\underset{\sim}{\phi}{}_k^{i+1,-}\ \underset{\sim}{Y}{}_k^{i+1,+} + \underset{\sim}{\phi}{}_{k+1}^{i+1,-}\underset{\sim}{Y}{}_{k+1}^{i+1,+}] \\
&\quad + \tau^{-}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_k^{i,-}]\ \underset{\sim}{B}{}_k^{i,-} + \tau^{-}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_{k+1}^{i,-}]\ \underset{\sim}{B}{}_{k+1}^{i,-} \\
&\quad + \tau^{+}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_k^{i+1,-}]\ \underset{\sim}{B}{}_k^{i+1,-} + \tau^{+}\sigma[\rho\beta+\epsilon\underset{\sim}{\phi}{}_{k+1}^{i+1,-}]\ \underset{\sim}{B}{}_{k+1}^{i+1,-}
\end{aligned} \tag{59}$$

We shall now combine the frequency points. We shall write

$$\underset{\sim}{A}{}_{dc}^{ab} = \begin{bmatrix}
\underset{\sim}{A}{}_{q,k}^{ab} & \underset{\sim}{A}{}_{q,k+1}^{dc} & & \\
 & \underset{\sim}{A}{}_{q,k+1}^{ab} & \underset{\sim}{A}{}_{q,k+2}^{dc} & \\
 & & \ddots & \ddots \\
 & & \underset{\sim}{A}{}_{q,K-1}^{ab} & \underset{\sim}{A}{}_{q,K}^{dc} \\
 & & & \underset{\sim}{A}{}_{q,K}^{ab}
\end{bmatrix} \tag{60}$$

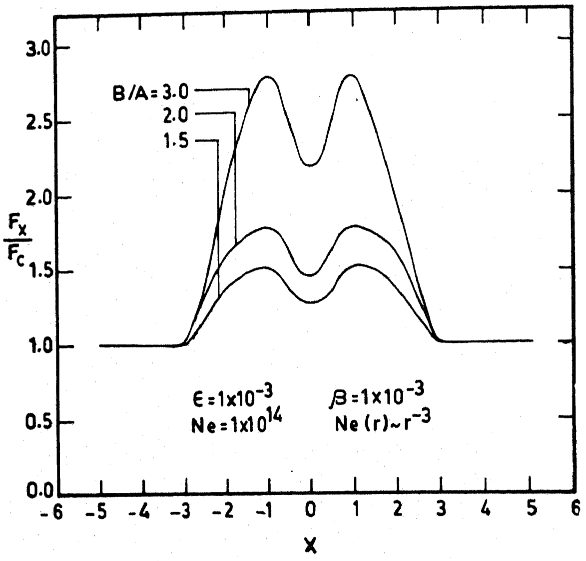


Fig.13 Flux profiles in a medium with line and continuum emission for case 1

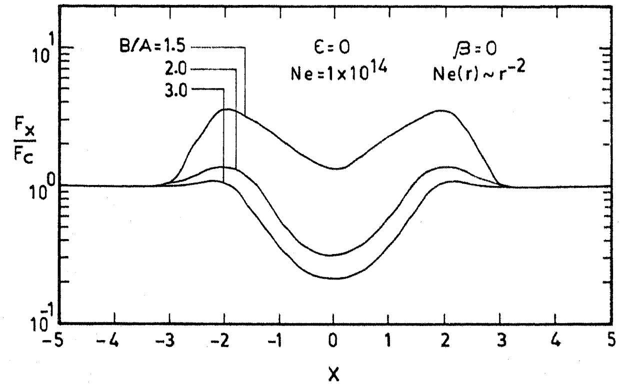


Fig.14 Flux profiles in a scattering medium with case 2

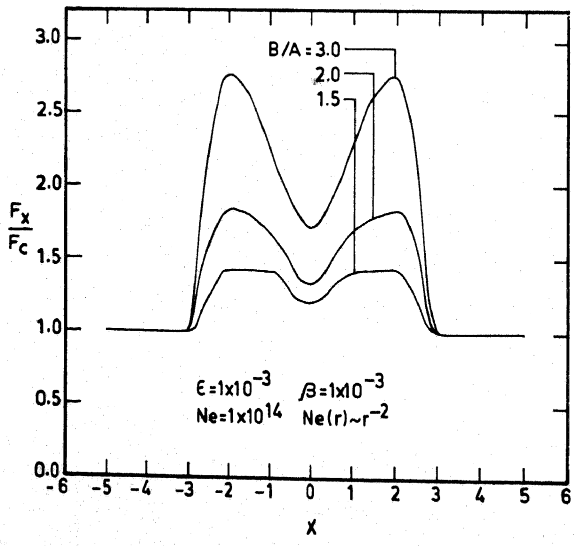


Fig.15 Flux profiles in a line emitting medium for case 2

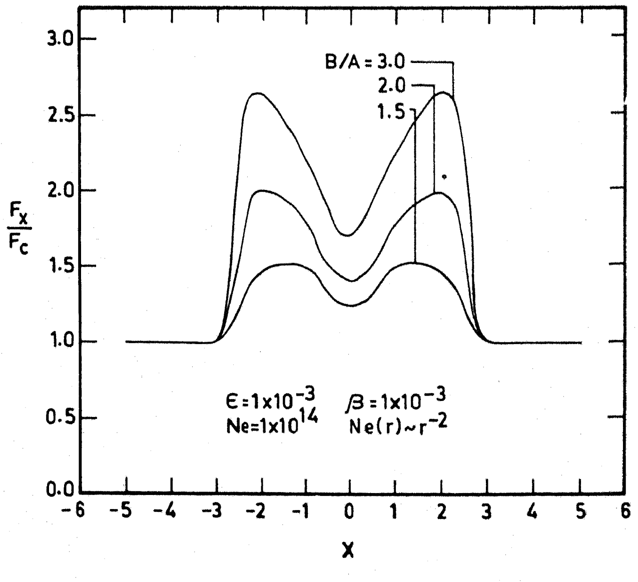


Fig.16 Flux profiles in a medium with both line and continuum emission for case 2

$$\underset{\sim}{A}{}_{eh}^{fg} = \begin{bmatrix} \underset{\sim}{A}{}_{q,k}^{fg} & \underset{\sim}{A}{}_{q,k+1}^{eh} & & \\ & \underset{\sim}{A}{}_{q,k+1}^{fg} & \underset{\sim}{A}{}_{q,k+2}^{eh} & \\ & \ddots & & \ddots \\ & & \underset{\sim}{A}{}_{q,K-1}^{fg} & \underset{\sim}{A}{}_{q,K}^{eh} \\ & & & \underset{\sim}{A}{}_{q,K}^{fg} \end{bmatrix} \tag{61}$$

$$\underset{\sim}{A}{}_{dc}^{\prime\,ab} = \begin{bmatrix} \underset{\sim}{A}{}_{q,k}^{\prime\,ab} & \underset{\sim}{A}{}_{q,k+1}^{\prime\,dc} & & \\ & \underset{\sim}{A}{}_{q,k+1}^{\prime\,ab} & \underset{\sim}{A}{}_{q,k+2}^{\prime\,dc} & \\ & \ddots & & \ddots \\ & & \underset{\sim}{A}{}_{q,K-1}^{\prime\,ab} & \underset{\sim}{A}{}_{q,K}^{\prime\,dc} \\ & & & \underset{\sim}{A}{}_{q,K}^{\prime\,ab} \end{bmatrix} \tag{62}$$

$$\underset{\sim}{A}{}_{eh}^{\prime\,fg} = \begin{bmatrix} \underset{\sim}{A}{}_{q,k}^{\prime\,fg} & \underset{\sim}{A}{}_{q,k+1}^{\prime\,eh} & & \\ & \underset{\sim}{A}{}_{q,k+1}^{\prime\,fg} & \underset{\sim}{A}{}_{q,k+2}^{\prime\,eh} & \\ & \ddots & & \ddots \\ & & \underset{\sim}{A}{}_{q,K-1}^{\prime\,fg} & \underset{\sim}{A}{}_{q,K}^{\prime\,eh} \\ & & & \underset{\sim}{A}{}_{q,K}^{\prime\,fg} \end{bmatrix} \tag{63}$$

$$\underset{\sim}{Q}{}_{F} = \begin{bmatrix} \underset{\sim}{U} & \underset{\sim}{U} & & \\ & \underset{\sim}{U} & \underset{\sim}{U} & \\ & \ddots & & \ddots \\ & & \underset{\sim}{U} & \underset{\sim}{U} \\ & & & U \end{bmatrix} \tag{64}$$

Where U is the unit matrix.

$$\underset{\sim}{I}_i^+ = \begin{bmatrix} I(r_i, +\mu_j, X_k) \\ I(r_i, +\mu_J, X_k) \\ I(r_i, +\mu_j, X_{k+1}) \\ I(r_i, +\mu_J, X_{k+1}) \\ I(r_i, +\mu_J, X_K \end{bmatrix} \tag{65}$$

and

$$\underset{\sim}{I}_i^- = \begin{bmatrix} I(r_i, -\mu_j, X_k) \\ I(r_i, -\mu_J, X_k) \\ I(r_i, -\mu_j, X_{k+1}) \\ I(r_i, -\mu_J, X_{k+1}) \\ I(r_i, -\mu_J, X_K \end{bmatrix} \tag{66}$$

Equations (58) and (59) can be rewritten as,

$$[A^{ab}_{dc}\ I_i^+ - \tau\ \sigma \underset{\approx}{Q}_F \underset{\sim}{\phi}_i^+\ \underset{\sim}{Y}_i^+] + [\underset{\sim}{A}^{fg}_{eh}\ I^+_{i+1} - \tau^+ \sigma \underset{\approx}{Q} \underset{\sim}{\phi}^+_{i+1}\ \underset{\sim}{Y}^+_{i+1}]$$

$$= \tau^- \sigma \underset{\approx}{Q}_F \underset{\sim}{\phi}_i^+\ \underset{\sim}{Y}_i^- + \tau^+ \sigma \underset{\approx}{Q}_F\ \underset{\sim}{\phi}^+_{i+1}\ \underset{\sim}{Y}^-_{i+1}$$

$$+ \tau^- \sigma \underset{\approx}{Q}_F [\rho\beta + \varepsilon\phi_i^+]\ \underset{\approx}{B}_i^+ + \tau^+ \sigma \underset{\approx}{Q}_F [\rho\beta + \varepsilon\underset{\sim}{\phi}^+_{i+1}]\ \underset{\approx}{B}^+_{i+1} \tag{67}$$

and

$$[\underset{\sim}{A}'^{ab}_{dc}\ \underset{\sim}{I}_i^- - \tau^- \sigma \underset{\approx}{Q}_F \bar{\phi}_i^- \underset{\sim}{Y}_i^-] + [\underset{\approx}{A}'^{fg}_{eh}\ \underset{\sim}{I}^-_{i+1} - \tau^+ \sigma \underset{\approx}{Q}_F \underset{\sim}{\phi}^-_{i+1}\ \underset{\sim}{Y}^-_{i+1}]$$

$$= \tau^- \sigma \underset{\approx}{Q}_F \underset{\sim}{\phi}_i^- \underset{\sim}{Y}_i^+ + \tau^+ \sigma \underset{\approx}{Q}_F \underset{\sim}{\phi}^-_{i+1}\ Y^+_{i+1}$$

$$+ \tau^- \sigma \underset{\approx}{Q}_F [\rho\beta + \varepsilon\underset{\sim}{\phi}_i^-]\ \underset{\approx}{B}_i^- + \tau^+ \sigma \underset{\approx}{Q}_F [\rho\beta + \varepsilon\underset{\sim}{\phi}^-_{i+1}]\ \underset{\approx}{B}^-_{i+1} \tag{68}$$

We write the following quantities

$$\bar{\underset{\sim}{A}}{}^{ab}_{dc} = \underset{\approx}{Q}{}_F^{-1}\ \underset{\sim}{A}{}^{ab}_{dc},\quad \bar{\underset{\sim}{A}}{}'^{ab}_{dc} = \underset{\approx}{Q}{}_F^{-1}\ \underset{\sim}{A}{}'^{ab}_{dc}$$

$$\underset{\sim}{A}{}^{fg}_{eh} = \underset{\approx}{Q}{}_F^{-1}\ \underset{\sim}{A}{}^{fg}_{eh},\quad \bar{\underset{\sim}{A}}{}'^{fg}_{eh} = \underset{\approx}{Q}{}_F^{-1}\ \underset{\sim}{A}{}'^{fg}_{eh} \tag{69}$$

and rewrite the equations (67) and (68) as follows:

$$[\bar{\underset{\sim}{A}}{}^{ab}_{dc} - \sigma\tau^-\ \underset{\sim}{F}{}^{++}_i]\ \underset{\sim}{I}{}^+_i + [\bar{\underset{\sim}{A}}{}^{fg}_{eh} - \sigma\tau^+\underset{\sim}{F}{}^{++}_{i+1}]\ \underset{\sim}{I}{}^+_{i+1}$$

$$= \sigma\tau^-\underset{\sim}{F}{}^{+-}_i\underset{\sim}{I}{}^-_i + \sigma\tau^+\underset{\sim}{F}{}^{+-}_{i+1}\ \underset{\sim}{I}{}^-_{i+1} \tag{70}$$

$$+ \sigma\tau^-\underset{\sim}{\phi}{}^+_i\underset{\sim}{B}{}^+_i + \sigma\tau^+\underset{\sim}{\phi}{}^+_{i+1}\ \underset{\sim}{B}{}^+_{i+1}$$

and

$$[\bar{\underset{\sim}{A}}{}^{ab}_{dc} - \sigma\tau^-\underset{\sim}{F}{}^{--}_i]\ \underset{\sim}{I}{}^-_i + [\bar{\underset{\sim}{A}}{}^{fg}_{eh} - \sigma\tau^+F^-_{i+1}]\ I^-_{i+1}$$

$$= \sigma\tau^-\underset{\sim}{F}{}^{-+}_{i+1}\ \underset{\sim}{I}{}^+_i -$$

$$+ \sigma\tau^-\underset{\sim}{\phi}{}^-_i\underset{\sim}{B}{}^-_i + \sigma$$

Where

$$\underset{\sim}{\phi}{}^+_i Y^+_i = [\underset{\sim}{\phi}+\underset{\sim}{\phi}{}^T_+\underset{\sim}{W}{}^+$$

$$\underset{\sim}{\phi}{}^+_{i+1} Y^+_{i+1} = [\underset{\sim}{\phi}_+\underset{\sim}{\phi}{}^T_+\underset{\sim}{W}{}^{++}]$$

$$\underset{\sim}{\Phi}{}^+_i\underset{\sim}{Y}{}^-_i = [\underset{\sim}{\phi}_+\underset{\sim}{\phi}{}^T_-\underset{\sim}{W}{}^{+\cdot}$$

$$\underset{\sim}{\Phi}{}^+_{i+1}\underset{\sim}{Y}{}^-_{i+1} = [\underset{\sim}{\phi}_+\underset{\sim}{\phi}{}^T\underset{\sim}{W}{}^{+-}]_i$$

$$\underset{\sim}{\Phi}{}^+_i = (\rho\beta+\epsilon\phi^f_i)\delta_{kk}, = [\Phi_{kk}] \tag{76}$$

$$\underset{\sim}{B}{}^+_{i,i-1} = B^+_{i,i-1}\ \underset{\sim}{L} \tag{77}$$

$$\underset{\sim}{L} = \begin{bmatrix} 1 \\ 1 \\ \\ \\ 1 \end{bmatrix} \qquad (78)$$

The quantities $\underset{\sim}{F}$, $\underset{\sim}{F}^{+}$, $\underset{\sim}{\Phi}$ and $\underset{\sim}{B}$ are similarly defined

We shall introduce the following quantities

$$\underset{\sim}{X}_1 = \underset{\sim}{A}^{fg}_{eh} \quad \sigma\tau^{+}\underset{\sim}{F}^{++}_{i+1} \qquad (79)$$

$$\underset{\sim}{X}_2 = \sigma\tau \ F^{+}_{i} \qquad (80)$$

$$\underset{\sim}{X}_3 - \sigma\tau^{+}F^{\ +}_{i+1} \qquad (81)$$

$$\underset{\sim}{X}_4 \quad \underset{\sim}{A}'^{ab}_{dc} \quad \sigma\tau \ \underset{\sim}{F}_i \qquad (82)$$

$$\underset{\sim}{Y}_1 = \underset{\sim}{A}^{ab}_{dc} \quad \sigma\tau \ \underset{\sim}{F}^{++}_{i} \qquad (83)$$

$$\underset{\sim}{Y}_2 = \sigma\tau^{+}\underset{\sim}{F}^{+}_{i+1} \qquad (84)$$

$$\underset{\sim}{Y}_3 \quad \sigma\tau \ \underset{\sim}{F}_i^{\ +} \qquad (85)$$

$$\underset{\sim}{Y}_4 \quad \underset{\sim}{A}'^{fg}_{eh} \quad \sigma\tau^{+}\underset{\sim}{F}_{i+1} \qquad (86)$$

By using equations (79) to (86), we can rewrite equations (70) and (71) as follows

$$\underset{\sim}{Y}_1\underset{\sim}{I}^{+}_{i} + \underset{\sim}{X}_1\underset{\sim}{I}^{+}_{i+1} - \underset{\sim}{X}_2\underset{\sim}{I}_{i} + \underset{\sim}{Y}_2\underset{\sim}{I}_{i+1} + \sigma\tau \ \underset{\sim}{\Phi}_i\underset{\sim}{B}^{+}_{i} + \sigma\tau \ \underset{\sim}{\Phi}_i\underset{\sim}{B}^{+}_{i} + \sigma\tau \ \underset{\sim}{\Phi}^{+}_{i+1}\underset{\sim}{B}^{+}_{i+1} \qquad (87)$$

$$\underset{\sim}{X}_4\underset{\sim}{I}_{i} + \underset{\sim}{Y}_4\underset{\sim}{I}_{i+1} \quad \underset{\sim}{Y}_3\underset{\sim}{I}^{+}_{i} + \underset{\sim}{X}_3\underset{\sim}{I}^{+}_{i+1} + \sigma\tau \ \underset{\sim}{\Phi}_i\underset{\sim}{B}_{i} + \sigma\tau^{+}\underset{\sim}{\Phi}_{i+1}\underset{\sim}{B}_{i+1} \qquad (88)$$

Equations (87) and (88) are written in the form

$$\begin{bmatrix} \underset{\sim}{I}^{+}_{i+1} \\ \\ \underset{\sim}{I}_{i} \end{bmatrix} - \underset{\sim}{K}^{\ 1} \begin{bmatrix} \underset{\sim}{Y}_1 & \underset{\sim}{Y}_2 \\ \\ \underset{\sim}{Y}_3 & \underset{\sim}{Y}_4 \end{bmatrix} \begin{bmatrix} \underset{\sim}{I}^{+}_{i} \\ \\ \underset{\sim}{I}_{i+1} \end{bmatrix} + \sigma\tau\underset{\sim}{K}^{\ 1} \begin{bmatrix} \underset{\sim}{\Phi}^{+} & \underset{\sim}{B}^{+} \\ \\ \underset{\sim}{\Phi} & \underset{\sim}{B} \end{bmatrix} \qquad (89)$$

We may compare equation (89) with the canonical form of the interaction principle given by Peraiah, 1984.

$$\begin{bmatrix} \underset{\sim}{I}^{+}_{i+1} \\ \underset{\sim}{I}^{-}_{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{t}(i+1,i) & \underset{\sim}{r}(i,i+1) \\ \underset{\sim}{r}(i+1,i) & \underset{\sim}{t}(i,i+1) \end{bmatrix} \begin{bmatrix} \underset{\sim}{I}^{+}_{i} \\ \underset{\sim}{I}^{-}_{i+1} \end{bmatrix} + \begin{bmatrix} \Sigma^{+}_{i+} \\ \Sigma^{-}_{i+} \end{bmatrix} \tag{90}$$

Where the quantities $\underset{\sim}{t}(i+1,i)$, $\underset{\sim}{t}(i,i+1)$, $\underset{\sim}{r}(i+1,i)$ and $\underset{\sim}{r}(i,i+1)$ are the two pairs of transmission and reflection operators. The quantities $\underset{\sim}{\Sigma}^{+}_{i+\frac{1}{2}}$ and $\underset{\sim}{\Sigma}^{-}_{i+\frac{1}{2}}$ are the internal source vectors.

By comparing equations (89) and (90), we obtain the transmission and reflecti

$$\underset{\sim}{t}(i+1,i) = \underset{\sim}{R}^{+-}\ \underset{\sim}{X}^{-1}\ [\underset{\sim}{X}_2\underset{\sim}{X}_4^{-1}\underset{\sim}{Y}_3 - \underset{\sim}{Y}_1]$$

$$\underset{\sim}{t}(i,i+1) = \underset{\sim}{R}^{-+}\ X_4^{-1}\ [\underset{\sim}{X}_3\underset{\sim}{X}_1^{-1}\underset{\sim}{Y}_2 - \underset{\sim}{Y}_4]$$

$$\underset{\sim}{r}(i+1,i) = \underset{\sim}{R}^{-+}\ \underset{\sim}{X}_4^{-1}\ [\underset{\sim}{Y}_3 - \underset{\sim}{X}_3\underset{\sim}{X}_1^{-1}$$

$$\underset{\sim}{r}(i,i+1) = \underset{\sim}{R}^{-+}$$

$$\underset{\sim}{R}^{+-} = [\underset{\sim}{I} -$$

$$\underset{\sim}{R}^{-+} = [\underset{\sim}{I} -$$

$$\underset{\sim}{\Sigma}^{+}_{i+} = \sigma\underset{\sim}{R}^{+-}\underset{\sim}{X}_1^{-1}[\tau^{-}$$

$$+ \underset{\sim}{X}_2\underset{\sim}{X}_4^{-1}\ (\tau^{-}\underset{\sim}{\Phi}^{-}_{i}$$

$$\underset{\sim}{\Sigma}_{i+} = \sigma\underset{\sim}{R}^{-+}\ \underset{\sim}{X}_4^{-1}[$$

$$+ \underset{\sim}{X}_3\underset{\sim}{X}_1^{-1}\ (\tau^{-}\underset{\sim}{\Phi}^{+}_{i}$$

The equations (91) to (98) are employed in

$$\underset{\sim}{I}^{+}_{i+1} = \underset{\sim}{r}$$

$$\underset{\sim}{I}^{-}_{i} = \underset{\sim}{t}'(i,i+1)\ \underset{\sim}{I}^{-}_{i+1} + \underset{\sim}{V}^{-}_{i+} \tag{100}$$

With the initial condition $\underset{\sim}{I}^{-}_{N+1} = C'(\text{say})$ (101)

Where $\underset{\sim}{t}'(i,i+1) = \underset{\sim}{T}_{i+\frac{1}{2}}\,\underset{\sim}{t}(i,i+1)$ (102)

Equations (99) to (101) are calculated with $i = N, N-1, N-2, \ldots\ldots 2, 1$ where $i=1$ coincides with $r = B$ (the outermost radius) and $i = N$ with $r = A$ (innermost radius). The quantities $\underset{\sim}{r}(i,i+1)$, $\underset{\sim}{t}(i,i+1)$, $\underset{\sim}{V}^{+}_{i+\frac{1}{2}}$, $\underset{\sim}{V}^{-}_{i+\frac{1}{2}}$, $\underset{\sim}{T}_{i+\frac{1}{2}}$ are calculated initially from i=1,2,.... N-1,N with the following recursive relations:

$$\underset{\sim}{r}(1,i+1) = \underset{\sim}{r}(i,i+1) + \underset{\sim}{t}(i+1,i)\,\underset{\sim}{r}(i,i)\,\underset{\sim}{T}_{i+\frac{1}{2}}\,\underset{\sim}{t}(i,i+1) \tag{103}$$

$$\underset{\sim}{V}^{+}_{i+\frac{1}{2}} = \underset{\sim}{t}'(i+1,i)\,\underset{\sim}{V}^{+}_{i-\frac{1}{2}} + \Sigma^{+}(i+1,i) + \underset{\sim}{R}_{i+\frac{1}{2}}\,\underset{\sim}{\Sigma}_{i+\frac{1}{2}} \tag{104}$$

$$\underset{\sim}{V}^{+}_{i+\frac{1}{2}} = \underset{\sim}{r}'(i+1,i)\,\underset{\sim}{V}^{+}_{i-\frac{1}{2}} + \underset{\sim}{T}_{i+\frac{1}{2}}\,\underset{\sim}{\Sigma}^{-}_{i+\frac{1}{2}} \tag{105}$$

With the initial conditions

$$\underset{\sim}{r}(1,1) = 0 \tag{106}$$

$$\underset{\sim}{V}^{+}_{\frac{1}{2}} = C\ (\text{say}) \tag{107}$$

and

$$\underset{\sim}{t}'(i+1,i) = \underset{\sim}{t}(i+1,i)\,K_{i+\frac{1}{2}} \tag{108}$$

$$\underset{\sim}{r}'(i+1,i) = \underset{\sim}{r}(i+1,i)\,\underset{\sim}{K}_{i+\frac{1}{2}} \tag{109}$$

$$\underset{\sim}{R}_{i+\frac{1}{2}} = \underset{\sim}{t}'(i+1,i)\,\underset{\sim}{r}(1,i) \tag{110}$$

$$\underset{\sim}{T}_{i+\frac{1}{2}} = [\underset{\sim}{E} - \underset{\sim}{r}(i+1,r(1,i))]^{-1} \tag{111}$$

$$\underset{\sim}{K}_{i+\frac{1}{2}} = [\underset{\sim}{E} - \underset{\sim}{r}(1,i)\,\underset{\sim}{r}(i+1,i)]^{-1} \tag{112}$$

Where $\underset{\sim}{E}$ is the identity matrix.

4. Discussion of the Results

The solution of the transfer equation derived in the previous section is employed to calculate the line profiles in an extended medium. We have considered several cases in which the density and velocity vary according to the equation of conservation of mass in spherical symmetry. That is, For the purpose of calculating the optical depth we have assumed an electron density Ne. The optical depth is calculated using the two variations of electron density, that is,

$$(1)\ v \sim r, \qquad Ne\ (r) \sim r^{-3} \tag{114}$$

and

$$(2)\ v = \text{const}, \quad Ne\ (r) \sim r^{-2} \tag{115}$$

The optical depth is given by

$$\tau\ (r) = \int_{r_1}^{r_2} \sigma_{Th}\ Ne\ (r)\ dr \tag{116}$$

For case 1, we have

$$\tau\ (r) = \frac{\sigma_{Th}\ Ne\ (r_o)\ r_o^3\ \bar{r}\ \Delta r}{r_1^2\ r_2^2} \tag{117}$$

Where r_o is the star's radius, r_1 and r_2 are the inner and outer radii of a given shell. σ_{Th} is the Thomson coefficient for scattering of electrons. And for case 2, we have

$$\tau\ (r) = \frac{\sigma_{Th}\ Ne\ (r_o)\ r_o^2\ \Delta r}{r_1\ r_2} \tag{118}$$

Ne (r_o) is the electron density at the radius r_o. The quantity B/A is taken to be 1.5, 2 and 3, where B and A are the outer and inner radii of the extended atmosphere. We have set $N_e\ (r_o)$ to be equal to 10^{14}cm^{-3} . We considered only a stationary atmosphere. We have plotted the optical depth for the two cases in Figure 1 for B/A = 1.5, in Figure 2 for B/A = 2 and in Figure 3 for B/A = 3. The optical depth increases suddenly in the outer layers and then increases gradually towards the center of the star. The increase is steep for B/A = 3 and after the first few shells the optical depth changes linearly. In Figures 4 to 6 the source function is plotted against the optical depth. We have considered three types of atmospheres (1) Pure scattering $\varepsilon = 0$ and $\beta = 0$. (2) Line emission with $\varepsilon = 10^{-3}$ and $\beta = 0$. (3) With both line emission and continuum emission $\varepsilon = 10^{-3}$, $\beta = 10^{-3}$.

The boundary conditions in pure scattering medium are given by the intensity incident at the bottom of the atmosphere and no incident radiation is given on top of the atmosphere. For the other two media we have set a constant emission source in the medium and no incident radiation is given on either side of the atmospheres. In Figure 4, the source function corresponding to these three media are plotted with respect to the optical depth and for the case 1.

In the case of scattering medium the source function keeps increasing towards the higher optical depth whereas in the media with emission appears to be almost parallel to the optical depth axis In Figure 5 the source functions are given for B/A = 2 and in Figure 6 for B/A = 3. The behaviour of the source function is very similar to that given in Figure 4. In Figures 7 to 9 the same quantity as in Figures 4 to 6 are plotted but for the case 2 The variation of the source function appears to be similar to that in Figures 4 to 6 In Figures 10 to 16 we have plotted the line profiles corresponding to parameters shown in the Figure in the scattering medium for case 2 The profiles in Figures 14 and 15 are also similar to those shown in Figures 11 and 12

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