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DISCRETIZATION OF THE EQUATION OF RADIATIVE TRANSFER WITH COMPTON SCATTERING

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ABSTRACT

We have attempted to solve the equation of radiative transfer by including Compton and Inverse Compton scattering. We expanded the specific intensity by Taylor series at about $\nu' = \nu_0$. The resulting equation has been discretized for obtaining the solution.

Key Words: Radiative transfer, spherical symmetry, Compton scattering, Inverse Compton scattering

1. Introduction

Compton and Inverse Compton scattering play an important role in the process of emission of x-ray spectrum which has been observed in a variety of objects. However compton scattering is very poorly understood inspite of the fact that many people claim to have successfully explained few observational results. In this paper we shall solve the transfer problem connected with Compton scattering.

2. Formulation of the equation of transfer

From Pomraning (1973), the classical equation of transfer is written (with usual notation).

$$\frac{1}{c}\frac{\partial \underset{\sim}{I}(\nu,\underset{\sim}{\Omega})}{\partial t} + \underset{\sim}{\Omega}\cdot \Delta \underset{\sim}{I} = \sigma_a'(\nu)\,[\underset{\sim}{B}(\nu) - \underset{\sim}{I}(\nu,\underset{\sim}{\Omega})]$$

$$+\int_0^\infty d\nu' \int d\underset{\sim}{\Omega}' \frac{\nu}{\nu'}\,\sigma_s(\nu'\to\nu,\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega})\,\underset{\sim}{I}(\nu',\underset{\sim}{\Omega}')\,[1+\frac{c^2 \underset{\sim}{I}(\nu,\underset{\sim}{\Omega})}{2h\nu^3}]$$

$$\int_0^\infty d\nu' \int_{4\pi} d\Omega'\,\sigma_s(\nu\to\nu',\underset{\sim}{\Omega}\cdot\underset{\sim}{\Omega}')\,\underset{\sim}{I}(\nu,\underset{\sim}{\Omega})\,[1+\frac{c^2 \underset{\sim}{I}(\nu,\underset{\sim}{\Omega})}{2h\nu'^3}] \qquad (2.1)$$

We shall assume that the electron temperatures and photon frequencies are not high, compared to the rest energy of the electron, i.e. Fokker-Planck approximation. We first expand $\underset{\sim}{I}(\nu',\underset{\sim}{\Omega}')$ in equation (2.1) in a Taylor series about $\nu' = \nu$, given by,

$$\underset{\sim}{I}(\nu',\underset{\sim}{\Omega}') = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n \underset{\sim}{I}(\nu,\underset{\sim}{\Omega}')}{\partial \nu^n}(\nu' - \nu)^n \qquad (2.2)$$

Introducing (2.2) into (2.1) we obtain, (Pomraning, 1973)

$$\frac{1}{c}\frac{\partial \underset{\sim}{I}(\nu,\underset{\sim}{\Omega})}{\partial t} + \underset{\sim}{\Omega}\cdot \Delta \underset{\sim}{I}(\nu,\underset{\sim}{\Omega}) = \sigma'_a(\nu)[B(\nu) - \underset{\sim}{I}(\nu,\Omega)]$$

$$- \sigma_{Th}(1-2\gamma) I(\nu,\underset{\sim}{\Omega})$$

$$+ \sigma_{Th} \int_{4\pi} d\underset{\sim}{\Omega}' \sum_{n=0}^{n=3} \left(\frac{2n+1}{4\pi}\right) p_n(\underset{\sim}{\Omega}\cdot\underset{\sim}{\Omega}') S_n I(\nu,\underset{\sim}{\Omega}')$$

$$- \frac{3\sigma_{Th}}{16\pi}\frac{c^2}{h\nu^3} \gamma \underset{\sim}{I}(\nu,\underset{\sim}{\Omega})(1-\nu\frac{\partial}{\partial\nu}) \int d\underset{\sim}{\Omega}'[1-(\underset{\sim}{\Omega}\cdot\Omega')^2 - (\underset{\sim}{\Omega}\cdot\underset{\sim}{\Omega}')^3]\underset{\sim}{I}(\nu,\underset{\sim}{\Omega}') \tag{2.3}$$

where

$$S_o = [1-\gamma(1-\nu\frac{\partial}{\partial\nu}) - \alpha(2\nu\frac{\partial}{\partial\nu} - \nu^2\frac{\partial^2}{\partial\nu^2})]$$

$$S_1 = \frac{2}{5}[\gamma(1-\nu\frac{\partial}{\partial\nu}) - \alpha(1-2\nu\frac{\partial}{\partial\nu} + \nu^2\frac{\partial^2}{\partial\nu^2})]$$

$$S_2 = \frac{1}{10}[1-\gamma(1-\nu\frac{\partial}{\partial\nu}) - \alpha(6+2\nu\frac{\partial}{\partial\nu} - \nu^2\frac{\partial^2}{\partial\nu^2})]$$

$$S_3 = \frac{3}{10}[\gamma(1-\nu\frac{\partial}{\partial\nu}) + \alpha[(4+2\nu\frac{\partial}{\partial\nu} - \nu^2\frac{\partial^2}{\partial\nu^2}] \tag{2.4}$$

and p'_ns are Legendre polynomials.

$$\gamma = \frac{h\nu}{m_o c^2},\quad \alpha = \frac{kT}{m_o c^2} \tag{2.5}$$

After simplification we can write equation (2.3) as

$$\frac{1}{c}\frac{\partial \underset{\sim}{I}(\nu,\Omega)}{\partial t} + \underset{\sim}{\Omega}, \underset{\sim}{I}(\nu,\underset{\sim}{\Omega}) = \sigma'_a(\nu)[\underset{\sim}{B}(\nu) - I(\nu_1\underset{\sim}{\Omega})]$$

$$- \sigma_{Th}(1-2\gamma)\underset{\sim}{I}(\nu,\underset{\sim}{\Omega})$$

$$+ \sigma_{Th}\int_{4\pi}\{A + B\frac{\partial I(\nu,\underset{\sim}{\Omega}')}{\partial\nu} + C\frac{\partial^2 \underset{\sim}{I}(\nu,\underset{\sim}{\Omega}')}{\partial\nu^2}\} d\underset{\sim}{\Omega}'$$

$$- \frac{3\sigma_{Th}}{16\pi}\frac{c^2\gamma}{h\nu^3}\underset{\sim}{I}(\nu,\underset{\sim}{\Omega})(1-\nu\frac{\partial}{\partial\nu})$$

$$\int_{4\pi}(1-\cos\Theta + \cos^2\Theta - \cos^3\Theta)\underset{\sim}{I}(\nu,\underset{\sim}{\Omega}')\, d\Omega' \tag{2.6}$$

Where $\cos\Theta = \underset{\sim}{\Omega} \cdot \underset{\sim}{\Omega}' = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\phi'$ (2.7)

The quantities A, B, C are given by,

$$A = \frac{1}{4\pi}\ \{(1-\gamma)+\frac{6}{5}\ p_1\ (\gamma-\alpha)+\frac{1}{2}\ p_2\ (1-\gamma-6\alpha)+\frac{3}{10}\ p_3\ (\gamma+4\alpha)\} \quad (2.8)$$

$$B = \frac{\nu(\gamma-2\alpha)}{4\pi}\ (1-\frac{6}{5}\ p_1 + \frac{1}{2}\ p_2 - \frac{3}{10}\ p_3) \quad (2.9)$$

$$C = \frac{2\alpha\nu^2}{4\pi}\ (1-\frac{3}{5}\ p_1 + \frac{1}{2}\ p_2 - \frac{3}{20}\ p_3) \quad (2.10)$$

Equation (2.6) will be discretized in the next section. We shall consider the time independent transfer equation.

3. Discretization of Transfer Equation

Let us write the equation from (2.6) as follows:

$$\mu\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{r}\ \frac{\partial I_\nu}{\partial\mu} = \sigma_a'(\nu)\ [B_\nu - I_\nu] - \sigma'_{Th}(1-2\gamma)I_\nu$$

$$+ \sigma_{Th}\int_{4\pi}[A+B\ \frac{\partial I_\nu(\Omega')}{\partial\nu} + C\ \frac{\partial^2 I_\nu(\Omega')}{\partial\nu^2}]\ d\Omega'$$

$$\frac{3\sigma_{Th}}{16\pi}\ \frac{c^2\gamma}{h\nu^3}\ I_\nu(1-\nu\ \frac{\partial}{\partial\nu})\ \int(1-\cos\Theta+\cos^2\Theta-\cos^3\Theta)I_\nu(\Omega')d\Omega' \quad (1)$$

$$\cos\Theta = \cos\theta\ \cos\theta' + \sin\theta\ \sin\theta'\ \cos\phi = \mu\mu' + (1-\mu^2)^{\frac{1}{2}}\ (1-\mu'^2)^{\frac{1}{2}}\ \cos\phi$$

$$d\Omega' = \sin\Theta d\Theta d\varphi$$

$$\cos\Theta = M_1 + M_2\cos\phi$$

$$\cos^2 = (M_1 + M_2\cos\phi)^2 = M_1^2 + M_2^2\cos^2\phi + 2M_1M_2\cos\phi$$

$$\cos^3\Theta = (M_1 + M_2\cos\phi)^3 = M_1^3 + 3M_1^2M_2\cos\phi + 3M_1M_2^2\cos^2\phi + M_2^3\cos^3\phi$$

$$\int\cos\Theta\ d\Omega' = \int_0^\pi\int_0^{2\pi}(M_1 + M_2\cos\phi)\sin\Theta\ d\Theta\ d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} M_1\sin\Theta d\Theta d\varphi + \int_0^{\pi}\int_0^{2\pi} M\cos\phi\sin\Theta\ d\Theta d\varphi$$

$$= 2\pi \int_0^{\pi} M_1 \sin\theta \, d\theta = 2\pi \int_0^{\pi} M_1 \sin\theta \, d\theta$$

$$\int (M_1^2 + M_2^2 \cos^2\phi + 2M_1 M_2 \cos\phi) \, d\phi$$

$$\int (M_1^2 + \pi M_2^2 + 2M_1 M_2) \frac{\sin\theta d\theta}{d\mu} \quad \int (M_1^2 + \pi M_2^2) \, d\mu' \tag{2}$$

$$\int \cos\Theta \, I_\nu(\Omega') \, d\Omega = \int_0^{2\pi} \int_0^{\pi} (M_1 + M_2 \cos\phi) \, d\mu' d\phi$$

$$\int \cos^2\Theta \, d\Omega' I_\nu(\Omega')$$

$$= \int_0^{2\pi} \int_0^{\pi} I_\nu(\Omega')(M_1^2 + M_2^2 \cos^2\phi + 2M_1 M_2 \cos\phi) \, d\phi \, d(\cos\theta')$$

$$= \int_0^{2\pi} I_\nu(\theta', \phi') [2\pi M_1 + \pi M_2^2] \, d(\cos\theta') \tag{3}$$

$$\int \cos^3\Theta \, d\Omega = (M_1^3 + 3M_1 M_2^2 \cos\varphi + 3M_1 M_2^2 \cos^2\varphi + M_2^3 \cos^3\varphi) \, d\varphi$$

$$= 2\pi M_1^3 + 3\pi M_1 M_2^2$$

$$\int (1 - \cos\Theta + \cos^2\Theta - \cos^3\Theta) \, I_\nu(\Omega') \, d\Omega$$

$$\int (1 + \pi M_2^2 - 2\pi M_1^3 - 3\pi M_1 M_2^2) \, I(\mu') \, d\mu'$$

$$= \pi \int (1 + M_2^2 - 2M_1^3 - 3M_1 M_2^2) \, I(\mu') \, d\mu' \tag{4}$$

$$M_2^2 - 2M_1^3 - 3M_1 M_2^2 = M_2^2 - M_1(2M_1^2 - 3M_2^2)$$

$$= \mu_1 \mu_2 \{\mu_1 \mu_2 + 3 - 3(\mu_1^2 + \mu_2^2) + \mu_1^2 \mu_2^2\} \tag{5}$$

Further more,

$$\frac{3}{16} \frac{\sigma_{Th}}{\pi} \frac{c^2 \gamma}{h\nu^3} I_\nu(\Omega)\left(1 - \nu \frac{\partial}{\partial\nu}\right) \int_{4\pi} (1 - \cos\Theta + \cos^2\Theta$$

$$- \cos^3\Theta) \, I_\nu(\Omega') \, d\Omega'$$

$$= \frac{3}{16} \frac{\sigma_{Th}}{\pi} \frac{c^2 \gamma}{h\nu^3} I_\nu(\mu)\left(1 - \nu \frac{\partial}{\partial\nu}\right) \cdot \pi \int_{-1}^{+1} (1 + M_2^2 - 2M_1^3$$

$$3M_1 M_2^2) \, I(\mu') \, d\mu'$$

$$= \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I_\nu(\mu)(1-\nu\frac{\partial}{\partial\nu})\int_{-1}^{+1}(1+M_2^2 - 2M_1^3 - 3M_1M_2^2)I_\nu(\mu')d\mu' \tag{6}$$

$$M_1 = \mu\mu',\ M_2 = [(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}} \tag{7}$$

The equation for Compton scattering can be written as,

$$\frac{1}{c}\frac{\partial I(r,\mu,\nu)}{\partial t} + \mu\frac{\partial I(r,\mu,\nu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(r,\mu,\nu)}{\partial\mu}$$

$$= \sigma_a'(\nu)\{B(r,\mu,\nu) - I(r,\mu,\nu)\} - \sigma_{Th}(1-2\gamma)I(r,\mu,\nu)$$

$$+ \sigma_{Th}\cdot 2\pi\int_{4\pi}\{A + B\frac{\partial I(r,\mu,\nu)}{\partial\nu} + C\frac{\partial^2 I(r,\mu,\nu)}{\partial\nu^2}\}d\mu'$$

$$\frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I(r,\mu,\nu)(1-\nu\frac{\partial}{\partial\nu})\int(1-M_2^2 - 2M_1^3 - 3M_1M_2^2)I(r,\mu',\nu)d\mu' \tag{8}$$

Let us write (Peraiah and Varghese, 1985)

$$I(r,\mu,\nu) = I_o + I_\rho\xi + I_\mu\eta + I_\nu\chi + I_{\rho\mu}\xi\eta + I_{\mu\nu}\eta\chi$$

$$+ I_{\nu\rho}\chi\xi + I_{\rho\mu\nu}\xi\eta\chi \tag{9}$$

$$\Theta_\mu = \frac{1}{\Delta\mu}\int_{\Delta\mu} \ldots\ldots d\mu$$

$$\Theta_V = \frac{1}{V}\int_{\Delta r} \ldots\ldots 4\pi r^2 dr \tag{10}$$

$$\Theta_\nu = \frac{1}{\Delta\nu}\int_{\Delta\nu} \ldots\ldots d\nu$$

$$\xi = \frac{r - \bar{r}}{\Delta r/2},\ \Delta r = (r_{n+1} - r_n)$$

$$\bar{r} = \frac{1}{2}(r_{n+1} + r_n)$$

$$\eta = \frac{\mu - \bar{\mu}}{\Delta\mu/2},\ \Delta\mu = (\mu_{j+1} - \mu_j)$$

$$\bar{\mu} = \frac{1}{2}(\mu_{j+1} + \mu_j) \tag{11}$$

$$\chi = \frac{\nu - \bar{\nu}}{\Delta\nu/2} \qquad \Delta\nu = (\nu_{k+1} - \nu_k)$$

$$\nu = \frac{1}{2}(\nu_{k+1} + \nu_k)$$

We can have I_a, I_b, I_c, I_d, I_e, I_f, I_g, I_h, as the nodal values. Consider the Time-independent transfer only.

Let us calculate $\frac{\partial I}{\partial \nu}$ and $\frac{\partial^2 I}{\partial \nu^2}$

$$\frac{\partial I}{\partial \nu} = \frac{\partial}{\partial \nu}\{I_o + I_\rho\xi + I_\mu\eta + I_\nu\chi + I_{\rho\mu}\xi\eta + I_{\mu\nu}\eta\chi$$

$$+ I_{\nu\rho}\chi\xi + I_{\rho\mu\nu}\xi\eta\chi\}$$

$$\frac{2}{\Delta\nu} I_\nu + \eta\frac{2}{\Delta\nu} I_{\mu\nu} + \xi\frac{2}{\Delta\nu} I_{\nu\rho} + \xi\eta\frac{2}{\Delta\nu} I_{\rho\mu\nu}$$

$$= \frac{2}{\Delta\nu}\{I_\nu + \eta I_{\mu\nu} + \xi I_{\nu\rho} + \xi\eta I_{\rho\mu\nu}\} \tag{12}$$

$$\frac{\partial^2 I(r,\mu,\nu)}{\partial\nu^2} = 0 \tag{13}$$

Let us calculate the integral

$$\int_{-1}^{+1}(1 - M_2^2 - 2M_1^3 - 3M_1M_2^2)\, I(r,\mu',\nu)d\mu'$$

$$\int_{-1}^{+1}[1-(1-\mu^2)(1-\mu'^2)-2\mu^3\mu'^3-3\mu\mu'(1-\mu^2)(1-\mu'^2)]$$

$$[I_o + I_\rho\xi + I_\mu\eta' + I_\nu\chi + I_{\rho\mu}\xi\eta' + I_{\mu\nu}\eta'\chi$$

$$+ I_{\nu\rho}\chi\xi + I_{\rho\mu\nu}\xi\eta'\chi]d\mu'$$

$$-(1-\mu^2-\mu'^2+\mu^2\mu'^2)-2\mu^3\mu'^3-3\mu\mu'(1-\mu^2-\mu'^2+\mu^2\mu'^2)$$

$$= \frac{2}{3}(2\mu^2 + 1) \tag{14}$$

$$[\mu^2 + \mu'^2 - \mu^2\mu'^2 - 5\mu^3\mu'^3 - 3\mu\mu' + 3\mu^3\mu' + 3\mu\mu'^3]\mu'$$

$$= \mu^2\mu' + \mu'^3 - \mu^2\mu'^3 - 5\mu^3\mu'^4 - 3\mu\mu'^2 + 3\mu^3\mu'^2 + 3\mu\mu'^4$$

$$\int_{-1}^{+1}(\mu^2\mu' + \mu'^3 - \mu^2\mu'^3 - 5\mu^3\mu'^4 - 3\mu\mu'^2 + 3\mu^3\mu'^2 + 3\mu\mu'^4)d\mu'$$

$$= \{\tfrac{1}{2}\mu^2\mu'^2 + \tfrac{1}{4}\mu^2\mu'^4 - \mu^3\mu'^5 - \mu\mu'^3 + \mu^3\mu'^3 + \tfrac{3}{5}\mu\mu'^5\}_{-1}^{+1}$$

$$= -\tfrac{4}{5}\mu \quad . \tag{15}$$

Therefore, we have

$$\int_{-1}^{+1}(1-M_2^2 - 2M_1^3 - 3M_1M_2)I(r,\mu',\nu)d\mu'$$

$$= \tfrac{2}{3}(1+2\mu^2)(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi)$$

$$- \frac{2}{\Delta\mu} \cdot \tfrac{4}{5}\mu(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)$$

$$- \frac{2\bar{\mu}}{\Delta\mu}\,\frac{2}{3}\,(1+2\mu^2)(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)$$

$$= \tfrac{2}{3}(1+2\mu^2)[(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi) - \frac{2\bar{\mu}}{\Delta\mu}\,(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)]$$

$$- \frac{8}{5}\frac{\mu}{\Delta\mu}\;)(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi) \tag{16}$$

$$= \frac{2}{3}\,(1+2\mu^2)(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi) - \{\tfrac{2}{3}(1+2\mu^2)\,\frac{2\bar{\mu}}{\Delta\mu} + \frac{8}{5}\,\frac{\mu}{\Delta\mu}\}$$

$$(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi) \tag{17}$$

$$\int_{-1}^{+1}(1 - M_2^2 - 2M_1^3 - 3M_1M_2^2)I(r,\mu'\nu)d\mu'$$

$$= \frac{2}{3}\,(1+2\mu^2)(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi)$$

$$- \{\tfrac{4}{3}\,(1+2\mu^2)\,\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5}\,\frac{\mu}{\Delta\mu}\}(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)$$

$$\frac{3}{16}\,\sigma_{Th}\,\frac{c^2\gamma}{h\nu^3}\,I_\nu(1-\nu\,\frac{\partial}{\partial\nu})\int(1 + M_2^2 - 2M_1^3 - 3M_1M_2^2)I_\nu(\mu')d\mu'$$

$$= \frac{3}{16}\,\sigma_{Th}\,\frac{c^2\gamma}{h\nu^3}\,I_\nu(1-\nu\,\frac{\partial}{\partial\nu})[\tfrac{2}{3}\,(1+2\mu^2)(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi\;)$$

$$- \frac{4}{3}\,(1+2\mu^2)\,\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5}\,\frac{\mu}{\Delta\mu}\;\;(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)]$$

$$= \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I_\nu[\frac{2}{3}(1+2\mu^2)(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi)$$

$$- \{\frac{4}{3}(1+2\mu^2)\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5}\frac{\bar{\mu}}{\Delta\mu}\}(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)]$$

$$= \frac{3}{16}\sigma_{Th}\frac{c^2}{h\nu^2} I_\nu \frac{\partial}{\partial\nu}[\frac{2}{3}(1+2\mu^2)(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi)$$

$$- \{\frac{4}{3}(1+2\mu^2)\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5}\frac{\mu}{\Delta\mu}(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)]$$

$$= \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I(r,\mu,\nu)[\frac{2}{3}(1+2\mu^2)(I_o + I_\rho\xi + I_\nu\chi$$

$$+ I_{\nu\rho}\chi\xi) - \frac{4}{3}(1+2\mu^2)\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5}\frac{\mu}{\Delta\mu}(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)]$$

$$- \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^2} I(r,\mu,\nu)[\frac{2}{3}(1+2\mu^2)\{\frac{2}{\Delta\nu}(I_\nu + I_{\nu\rho}\xi)\}$$

$$- \{\frac{4}{3}(1+2\mu^2)\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5}\frac{\mu}{\Delta\mu}\}\{\frac{2}{\Delta\mu}(I_{\mu\nu} + I_{\rho\mu\nu}\xi)\}]$$

Let $p = \frac{2}{3}(1+2\mu^2)$ and $q = \{\frac{4}{3}(1+2\mu^2)\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5}\frac{\mu}{\Delta\mu}\}$

then,

$$= \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I(r,\mu,\nu)[\{p(I_o + I_\rho\xi + I_\nu\chi + I_{\nu\rho}\chi\xi)$$

$$- q(I_\mu + I_{\rho\mu}\xi + I_{\mu\nu}\chi + I_{\rho\mu\nu}\xi\chi)\}$$

$$p\frac{2\nu}{\Delta\nu}(I_\nu + I_{\nu\rho}\xi) - \frac{2\nu}{\Delta\nu} q(I_{\mu\nu} + I_{\rho\mu\nu}\xi)\}] \tag{18}$$

$$= \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I(r,\mu,\nu)\; pI_o + p\xi I_\rho - qI_\mu + p(\chi - \frac{2\nu}{\Delta\nu})I_\nu$$

$$q\xi I_{\rho\mu} - q(\chi - \frac{2\nu}{\Delta\nu})I_{\mu\nu} + p(\chi\xi - \frac{2\nu}{\Delta\nu})I_{\nu\rho} - q(\xi\chi - \frac{2\nu}{\Delta\nu})I_{\rho\mu\nu}]$$

$$= \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I(r,\mu,\nu)[pI_o + p\xi I_p - qI_\mu - 2p\frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- q\xi I_{\rho\mu} + 2q\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} + p(\chi\xi - \frac{2\nu}{\Delta\nu})I_{\nu\rho}$$

$$q(\xi\chi - \frac{2\nu}{\Delta\nu})I \quad] \tag{19}$$

$$\frac{\partial I(r,\mu,\nu)}{\partial \nu} \quad \frac{\partial}{\partial \nu}\{I_o \quad I_\rho \xi \quad I_\mu \eta \quad I_\nu \chi \quad I_{\rho\mu}\xi\eta$$

$$I_{\mu\nu}\eta\chi \quad I_{\nu\rho}\chi\xi \quad I_{\rho\mu\nu}\xi\eta\chi\}$$

$$\frac{2}{\Delta\nu}\{I_\nu \quad \eta I_{\mu\nu} \quad \xi I_{\nu\rho} \quad \xi\eta I_{\rho\mu\nu}\}$$

$$\frac{\partial I(r,\mu,\nu)}{\partial \nu} \quad \frac{2}{\Delta\nu} \quad I_\nu \quad \eta I_{\mu\nu} \quad \xi I_{\nu\rho} \quad \xi\eta I_{\rho\mu\nu}\} \tag{20}$$

$$\mu\,\frac{\partial I(r,\mu,\nu)}{\partial r} \quad \mu\frac{\partial}{\partial r}\{I_o \quad I_\rho \xi \quad I_\mu \eta \quad I_\nu \chi$$

$$I_{\rho\mu}\xi\eta \quad I_{\mu\nu}\eta\chi \quad I_{\nu\rho}\chi\xi \quad I_{\rho\mu\nu}\xi\eta\chi\}$$

$$\mu\frac{2}{\Delta r}\{I_\rho + \eta I_{\rho\mu} + \chi I_{\nu\rho} \quad \eta\chi I_{\rho\mu\nu}\}$$

$$\frac{2\mu}{\Delta r}(I_\rho \quad \eta I_{\rho\mu} \quad \chi I_{\nu\rho} \quad \eta\chi I_{\rho\mu\nu}) \tag{21}$$

$$\frac{1 \quad \mu^2}{r}\;\frac{\partial I(r,\mu,\nu)}{\partial \mu} \quad \frac{1 \quad \mu^2}{r}\frac{\partial}{\partial \mu}\{I_o \quad I_\rho \xi \quad I_\mu \eta \quad I_\nu \chi$$

$$I_{\rho\mu}\xi\eta + I_{\mu\nu}\eta\chi \quad I_{\nu\rho}\chi\xi \quad I_{\rho\mu\nu}\xi\eta\chi\}$$

$$\frac{1 \quad \mu}{r}\;\frac{2}{\Delta\mu}(I_\mu \quad \xi I_{\rho\mu} + \chi I_{\mu\nu} \quad \xi\chi I_{\rho\mu\nu}) \tag{22}$$

We shall replace the specific intensity in equation (8) by that given in equation (9) and rewrite equation (8) as follows:

$$\frac{2\mu}{\Delta r}(I_\rho \quad \eta I_{\rho\mu} \quad \chi I_{\nu\rho} \quad \eta\chi I_{\rho\mu\nu}) \quad \frac{1 \quad \mu^2}{r}\;\frac{2}{\Delta\mu}(I_\mu$$

$$+ \xi I_{\rho\mu} \quad \chi I_{\mu\nu} \quad \xi\chi I_{\rho\mu\nu})$$

$$\sigma_a B(r \;\mu\; \nu) \quad [\sigma_a(\nu) \quad \sigma_{Th}(1 \;\; 2\gamma)][I_o \quad I_\rho \xi + I_\mu \eta$$

$$I_\nu \chi \quad I_{\rho\mu}\xi\eta + I_{\mu\nu}\eta\chi + I_{\nu\rho}\chi\xi + I_{\rho\mu\nu}\xi\eta\chi]$$

$$+ 2\pi\,\sigma_{Th}\int A\,d\mu \quad 2\pi\,\sigma_{Th}\int B\,\frac{2}{\Delta\nu}(I_\nu \quad \eta I_{\mu\nu} \quad \xi I_{\nu\rho}$$

$$+ \xi\eta I_{\rho\mu\nu}) - \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3} I(r,\mu,\nu)[pI_o + p\xi I_\rho$$

$$- qI_\mu - 2p\frac{\bar{\nu}}{\Delta\nu}I_\nu - q\xi I_{\rho\mu} + 2q\frac{\bar{\nu}}{\Delta\nu}I_{\mu\nu}$$

$$+ p(\chi\xi - \frac{2\nu}{\Delta\nu})I_{\nu\rho} - q(\xi\chi - \frac{2\nu}{\Delta\nu})I_{\rho\mu\nu}] \quad (23)$$

We apply Θ_μ on (23):

$$\Theta_\mu[\mu] = \frac{1}{\Delta\mu}\int_{\mu_j}^{\mu_{j+1}} \mu d\mu$$

$$\Theta_\mu[\mu] = \bar{\mu} \quad (24)$$

$$\Theta_\mu[\eta] = 0 \quad (25)$$

$$\Theta_\mu[\mu\eta] = \frac{1}{6}\Delta\mu \quad (26)$$

$$\Theta_\mu[1-\mu^2] = 1-\bar{\mu}^2 \quad (27)$$

$$\bar{\mu}^2 = \frac{1}{3}(\mu_{j+1}^2 + \mu_{j+1}\mu_j + \mu_j^2) \quad (28)$$

Now we apply Θ_μ on (23) and use (24) to (28).

$$\frac{2\bar{\mu}}{\Delta r}(I_\rho + \chi I_{\nu\rho}) + \frac{1}{3}\frac{\Delta\mu}{\Delta r}(I_{\rho\mu} + \chi I_{\rho\mu\nu})$$

$$+ \frac{2(1-\bar{\mu}^2)}{r\Delta\mu}(I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu})$$

$$- \frac{2\bar{\mu}}{r}\{I_o + \xi I_\rho + \chi I_\nu + \chi\xi I_{\nu\rho}\} - \frac{1}{3}\frac{\Delta\mu}{r}\{I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu}\}$$

$$= \sigma_a' B(r,\bar{\mu},\nu) - [\sigma_a'(\nu) + \sigma_{Th}(1-2\gamma)][I_o + \xi I_\rho + \chi I_\nu + \chi\xi I_{\nu\rho}]$$

$$+ 4\pi\sigma_{Th} A + \frac{8\pi\sigma_{Th} B}{\Delta\nu}(I_\nu + \xi I_{\nu\rho})$$

$$- \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3}\Theta_\mu[I(r,\mu,\nu)\{pI_o + p\xi I_\rho - qI_\mu$$

$$- 2p\frac{\bar{\nu}}{\Delta\nu}I_\nu - q\xi I_{\rho\mu} + 2q\frac{\bar{\nu}}{\Delta\nu}I_{\mu\nu} + p(\chi\xi - \frac{2\nu}{\Delta\nu})I_{\nu\rho} - q(\xi\chi - \frac{2\nu}{\Delta\nu})I_{\rho\mu\nu}\}] \quad (29)$$

The last term in (29) should be worked out separately.

$$[I_o + \xi I_\rho + \eta I_\mu + \chi I_\nu + \xi\eta I_{\rho\mu} + \eta\chi I_{\mu\nu}$$

$$+ \chi\xi I_{\nu\rho} + \xi\eta\chi I_{\rho\mu\nu}][pI_o + p\xi I_\rho - qI_\mu$$

$$- 2p \frac{\bar{\nu}}{\Delta\nu} I_\nu - q\xi I_{\rho\mu} + 2q \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} + p(\chi\xi - \frac{2\nu}{\Delta\nu}) I_{\nu\rho}$$

$$- q(\xi\chi - \frac{2\nu}{\Delta\nu}) I_{\rho\mu\nu}] \tag{30}$$

$$p = \frac{2}{3}(1+2\mu^2), \quad q = \frac{4}{3}(1+2\mu^2)\frac{\bar{\mu}}{\Delta\mu} + \frac{8}{5\Delta\mu} \cdot \mu$$

Apply Θ_μ on p and q:

$$\Theta_\mu [p] = \frac{2}{3}[1+2(\bar{\mu}^2)] = p_\mu \tag{31}$$

$$\Theta_\mu [q] = \frac{4\bar{\mu}}{\Delta\mu} \frac{2}{5} + \frac{1}{3}[1+2(\bar{\mu^2})]\} = q_\mu \tag{32}$$

We have to calculate

$$\Theta_\mu [\eta p] \text{ and } \Theta_\mu [\eta q]$$

$$\Theta_\mu [\eta p] = \frac{1}{\Delta\mu} \int_{\Delta\mu} \frac{2(\mu-\bar{\mu})}{\Delta\mu} \frac{2}{3}(1+2\mu^2)d\mu$$

$$\Theta_\mu [\eta p] = \frac{4}{9}\Delta\mu \cdot \bar{\mu} \tag{33}$$

$$= p_{\eta\mu}$$

$$q_{\eta\mu} = \Theta_\mu [\eta q] = 8\left(\frac{\bar{\mu}}{\Delta\mu}\right)^2 \{\frac{7}{5} + \mu_j^2 + \mu_{j+1}^2 + \frac{1}{30}(\frac{\Delta\mu}{\mu}{}^2\} - q_\mu \tag{34}$$

$$\Theta_\mu [p] = \frac{2}{3}\{1+2\bar{\mu}^2\} = p_\mu$$

$$\Theta_\mu [q] = 4\frac{\bar{\mu}}{\Delta\mu}\{\frac{2}{5} + \frac{1}{3}(1+\bar{\mu}^2) = q_\mu$$

$$\Theta_\mu [\eta p] = \frac{4}{9}\Delta\mu . \bar{\mu} = p_{\eta\mu}$$

$$\Theta_\mu [\eta q] = 8\left(\frac{\bar{\mu}}{\Delta\mu}\right)^2 \{\frac{7}{5} + \mu_j^2 + \mu_{j+1}^2 + \frac{1}{30}\left(\frac{\Delta\mu}{\bar{\mu}}\right)^2\} - q_\mu = q_{\eta\mu} \tag{35}$$

From equation (30) and (35) after applying Θ_μ we get, equation (30)

$$= (I_o + \xi I_\rho + \chi I_\nu + \chi\xi I_{\nu\rho})\{p_\mu I_o + p_\mu \xi I_\rho - q_\mu I_\mu$$

$$- 2p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu - q_\mu \xi I_{\rho\mu} + 2q_\mu \frac{\bar{\nu}}{\Delta\mu} I_{\mu\nu} + p_\mu(\chi\xi - \frac{2\nu}{\Delta\nu}) I_{\nu\rho}$$

$$- q_\mu(\xi\chi - \frac{2\nu}{\Delta\nu}) I_{\rho\mu\nu}\}$$

$$+ (I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu})$$

$$\{p_{\eta\mu} I_o + p_{\eta\mu} \xi I_\rho - q_{\eta\mu} I_\mu - 2p_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- q_{\eta\mu} \xi I_{\rho\mu} + 2q_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} + p_{\eta\mu}(\chi\xi - \frac{2\nu}{\Delta\nu}) I_{\nu\rho}$$

$$- q_{\eta\mu}(\xi\chi - \frac{2\nu}{\Delta\nu}) I_{\rho\mu\nu}\} = A_\mu \qquad (36)$$

Let us rewrite equation (29) as follows:

$$\frac{2\bar{\mu}}{\Delta r} (I_\rho + \chi I_{\nu\rho}) + \frac{1}{3} \frac{\Delta\mu}{\Delta r} (I_{\rho\mu} + \chi I_{\rho\mu\nu})$$

$$+ \frac{2}{r} \frac{1-\bar{\mu}^2}{\Delta\mu} (I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu})$$

$$- \frac{2\bar{\mu}}{r} \{I_o + \xi I_\rho + \chi I_\nu + \chi\xi I_{\nu\rho}\} - \frac{1}{3} \frac{\Delta\mu}{r} \{I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu}\}$$

$$= \sigma_a'(\nu) B(r,\mu,\nu) - \sigma_a'(\nu) + \sigma_{Th}(1-2\gamma)][I_o + \xi I_\rho$$

$$+ \chi I_\nu + \chi\xi I_{\nu\rho}] + 4\pi\sigma_{Th} A + \frac{8\pi\sigma_{Th} B}{\Delta\nu} (I_\nu + \xi I_{\nu\rho})$$

$$- \frac{3}{16} \sigma_{Th} \frac{c^2\gamma}{h\nu^3} A_\mu \qquad (37)$$

Where A_μ is given by equation (36). We shall now apply $\Theta_V = \frac{1}{V} \int \ldots\ldots 4\pi r^2 dr$ on equation (37). First let us consider A_μ in equation (36).

$$\Theta_V [\xi/r] = \frac{1}{\Delta r} (2 - \bar{r} . \frac{\Delta A}{V})$$

$$\Theta_V [1/r] = \frac{1}{2} \frac{\Delta A}{V}$$

$$\Theta_V [\ \xi\ ] = \frac{1}{6} \frac{\Delta A}{\bar{A}} \qquad (38)$$

$$\Theta_V\ [\ Y\ ] = Y \text{ where Y is independent of r.}$$

$$\Theta_V\ [\xi^2\ ] = \bar{R}$$

where $\bar{R} = \frac{16\ \pi}{V(\Delta r^2)} [\frac{1}{5} r^5 - \frac{1}{2} \bar{r} r^4 + \frac{1}{3} (\bar{r}) r^3]_{r_i}^{r_{i+1}}$

$$[r^5]_{r_i}^{r_{i+1}} = (r_{i+1} - r_i)(r^4_{i+1} + r^3_{i+1}\ r_i + r^2_{i+1}\ r^2_i + r_{i+1} r^3_i + r^4_i)$$

$$= \Delta r\ r_5$$

$$[r^4]_{r_i}^{r_{i+1}} = (r^2_{i+1} + r^2_i)(r_{i+1} + r_i) r_{i+1} - r_i)$$

$$\frac{1}{2} \bar{r}\quad r^4 + (\bar{r})^2 \Delta r.\ (r^2_{i+1} + r^2_i) = \Delta r (\bar{r})^2 (r^2_{i+1} + r^2_i)$$

$$[r^3_i] = \Delta r . (r^2_{i+1} + r_{i+1}\ r_i + r^2_i) = \Delta r . r_3$$

$$\frac{1}{3} (\bar{r}) r^3 + \frac{1}{3} (\Delta r).\ (\bar{r}).\ r_{3/2}$$

$$\Theta_V\ [\xi^2] = \frac{16\pi}{V.\Delta r} \{\frac{1}{5}\ r_5 - (\bar{r})^2 r_4 + \frac{1}{3} (\bar{r})\ r_3\}$$

or

$$\Theta_V\ [\xi^2] = \frac{16\pi}{V.\Delta r} \{\frac{1}{5}\ r_5 - (\bar{r})^2 r_4 + \frac{1}{3} (\bar{r}) r_3\} = \bar{R}$$

$$r_3 = r^2_{i+1} + r_{i+1}\ r_i + r^2_i$$

$$r_4 = r^2_{i+1} + r^2_i \qquad (39)$$

$$r_5 = r^4_{i+1} + r^3_{i+1}\ r_i + r^2_{i+1}\ r^2_i + r^3_{i+1}\ r^3_i + r^4_i$$

Expanding equation (36), we obtain by applying Θ_V

$$\{p_\mu I_o + \frac{1}{6} \frac{\Delta A}{A} p_\mu I_\rho - q_\mu I_\mu - 2p_\mu \frac{\bar{\nu}}{\Delta \nu}\ I_\nu$$

$$- \frac{1}{6} \frac{\Delta A}{A} q_\mu I_{\rho\mu} + 2q_\mu \frac{\bar{\nu}}{\Delta \nu}\ I_{\mu\nu} + P_\mu (\frac{1}{6} \frac{\Delta A}{A} \chi - \frac{2\nu}{\Delta \nu}) I_{\nu\rho}$$

$$- q_{\mu} (\frac{1}{6} \frac{\Delta A}{\bar{A}} \chi - \frac{2\nu}{\Delta\nu}) I_{\rho\mu\nu}\}\{I_o + \chi I_{\nu}\}$$

$$+ \{\frac{1}{6} \frac{\Delta A}{\bar{A}} p_{\mu} I_o + \bar{R} p_{\mu} I_{\rho} - \frac{1}{6} \frac{\Delta A}{\bar{A}} q_{\mu} I_{\mu} - \frac{1}{3} \frac{\Delta A}{\bar{A}} p_{\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\nu}$$

$$- \bar{R} q_{\mu} I_{\rho\mu} + \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} + p_{\mu} (\bar{R}\chi - \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{\nu}{\Delta\nu}) I_{\nu\rho}$$

$$- q_{\mu} (\bar{R}\chi - \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{\nu}{\Delta\nu}) I_{\rho\mu\nu}\} \{I_{\rho} + \chi I_{\nu\rho}\}$$

$$+ \{I_{\mu} + \chi I_{\mu\nu}\} \{p_{\eta\mu} I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} p_{\eta\mu} I_{\rho} - q_{\eta\mu} I_{\mu}$$

$$- 2p_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\nu} - \frac{1}{6} \frac{\Delta A}{\bar{A}} q_{\eta\mu} I_{\rho\mu} + 2q_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}$$

$$+ p_{\eta\mu} (\frac{1}{6} \frac{\Delta A}{\bar{A}} \chi - 2 \frac{\nu}{\Delta\nu}) I_{\nu\rho} - q_{\eta\mu} (\frac{1}{6} \frac{\Delta A}{\bar{A}} \chi - 2 \frac{\nu}{\Delta\nu}) I_{\rho\mu\nu}\}$$

$$+ \{I_{\rho\mu} + \chi I_{\rho\mu\nu}\}$$

$$\{\frac{1}{6} \frac{\Delta A}{\bar{A}} p_{\eta\mu} I_o + \bar{R} p_{\eta\mu} I_{\rho} - \frac{1}{6} \frac{\Delta A}{\bar{A}} q_{\eta\mu} I_{\mu} - \frac{1}{3} \frac{\Delta A}{\bar{A}} p_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\nu}$$

$$- \bar{R} q_{\eta\mu} I_{\rho\mu} + \frac{1}{3} \frac{\Delta A}{\bar{A}} q_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} + p_{\eta\mu} (\bar{R}\chi - \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{\nu}{\Delta\nu}) I_{\nu\rho}$$

$$- q_{\eta\mu} (\bar{R}\chi - \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{\nu}{\Delta\nu}) I_{\rho\mu\nu}\} = A_{\mu\xi} \tag{40}$$

The term $\frac{8\pi\sigma_{Th}}{\Delta\nu} B(I_{\nu} + \xi I_{\nu\rho})$ becomes upon Θ_V operation,

$$\Theta_V[\frac{8\pi\sigma_{Th}}{\Delta\nu} B(I_{\nu} + \xi I_{\nu\rho})] = \frac{8\pi\sigma_{Th}}{\Delta\nu} B(I_{\nu} + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho}) \tag{41}$$

$$\text{Let } \sigma'_a(\nu) + \sigma_{Th}(1-2\gamma) = \Sigma(\nu)$$

Then

$$\Theta_V [\Sigma(\nu)(I_o + \xi I_{\rho} + \chi I_{\nu} + \chi\xi I_{\nu\rho})]$$

$$= \Sigma(\nu)(I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\rho} + \chi I_{\nu} + \frac{1}{6} \frac{\Delta A}{\bar{A}} \chi I_{\nu\rho}) \tag{42}$$

$$\theta_V [\sigma'_a(\nu) (B_o + \xi B_{\rho} + \chi B_{\nu} + \chi\xi B_{\nu\rho})]$$

$$= \sigma_a'(\nu)\ \{(B_o + \chi B_\nu) + (B_\rho + B_{\nu\rho}\chi)\ \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\} \tag{43}$$

Equation (36) becomes after Θ_V operation,

$$\frac{2}{\Delta r}\ [\bar{\mu}(I_\rho + \chi I_{\nu\rho}) + \frac{\Delta\mu}{6}\ (I_{\rho\mu} + \chi I_{\rho\mu\nu})]$$

$$+ \frac{2}{\Delta\mu}\ (1-\bar{\mu}^2)\ [(I_\mu + \chi I_{\mu\nu})\ \frac{1}{2}\ \frac{\Delta A}{V} + (I_{\rho\mu} + \chi I_{\rho\mu\nu})\ \frac{1}{\Delta r}\ (2- \frac{\bar{r}\Delta A}{V})]$$

$$- 2\bar{\mu}\ [(I_o + \chi I_\nu)\ \frac{1}{2}\ \frac{\Delta A}{V} + (I_\rho + \chi I_{\nu\rho})\ \frac{1}{\Delta r}\ (2- \frac{\bar{r}\Delta A}{V})]$$

$$- \frac{\Delta\mu}{3}\ [(I_\mu + \chi I_{\mu\nu})\ \frac{1}{2}\ .\ \frac{\Delta A}{V} + (I_{\rho\mu} + \chi I_{\rho\mu\nu})\ \frac{1}{\Delta r}\ (2- \frac{\bar{r}\Delta A}{V})]$$

$$= \sigma_a'(\nu)\ \{(B_o + \chi B_\nu) + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ (B_\rho + \chi B_{\nu\rho})\}$$

$$+ \Sigma(\nu)\ \{(I_o + \chi I_\nu) + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ (I_\rho + \chi I_{\nu\rho})\} + 4\pi\ \sigma_{Th}\ A$$

$$+ \frac{8\pi\sigma_{Th}}{\Delta\nu}\ B\ (I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho}) - \frac{3}{16}\ \sigma_{Th}\ \frac{c^2\gamma}{h\nu^3}\ A_{\mu\xi} \tag{44}$$

We shall now apply

$$\theta_V = \frac{1}{\Delta\nu}\ \int_{\Delta\nu} \ldots\ldots\ d\nu \text{ on equation (44)}$$

First we shall consider $\frac{3}{16}\ \sigma_{Th}\ \frac{c^2\gamma}{h\nu^3}\ A_{\mu\xi}$

$$\gamma = \frac{h\nu}{m_o c^2} = 8.1019\ 10^{-21}\nu = \gamma'\nu$$

$$\frac{3}{16}\ \sigma_{Th}\ \frac{c^2\gamma}{h\nu^3}\ A_{\mu\xi} = \frac{3}{16}\ \sigma_{Th}\ A_{\mu\xi}$$

$$= \frac{3}{16}\ \sigma_{Th}\ \frac{A\mu\xi}{m_o\nu^2} = \frac{3}{16}\ \frac{\sigma_{Th}}{m_o}\ A_{\mu\xi}\nu^{-2}$$

$$\sigma_{Th} = 6.6524 \times 10^{-25}\ cm^2$$

m_o = mass of electron

$$= 9.1095 \times 10^{-28} \text{ gr.}$$

$$\frac{3}{16}\frac{\sigma_{Th}}{m_o} = 136.9257 = k'$$

$$\frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3}A_{\mu\xi} = k'A_{\mu\xi}\nu^{-2}$$

We perform the ν-integration.

$$\theta_\nu\,[\nu^{-2},\ \nu^{-2}\chi,\ \nu^{-1},\nu^{-2}\chi^2,\ \nu^{-1}\chi,\ \nu^{-2}\chi]$$

$$\theta_\nu = \frac{1}{\Delta\nu}\int_{\Delta\nu} \ldots\ldots\, d\nu$$

$$\theta_\nu\,[\nu^{-2}] = \frac{1}{\Delta\nu}\int_{\Delta\nu}\frac{1}{\nu^2}\,d\nu = \frac{1}{\Delta\nu}\left[-\frac{1}{\nu}\right]_{\nu_k}^{\nu_{k+1}}$$

$$= -\frac{1}{\Delta\nu}\left[\frac{1}{\nu_{k+1}} - \frac{1}{\nu_k}\right] = -\frac{1}{\Delta\nu}\frac{(\nu_k - \nu_{k+1})}{\nu_{k+1}\,\nu_k} = \frac{1}{\nu_k\,\nu_{k+1}}$$

$$\theta_\nu\,[\nu^{-2}] = \frac{1}{\nu_k\,\nu_{k+1}} \tag{45}$$

$$\theta_\nu\,[\nu^{-1}] = \frac{1}{\Delta\nu}\int_{\nu_k}^{\nu_{k+1}}\frac{1}{\nu}\cdot d\nu = \frac{1}{\Delta\nu}\,[\ln\nu]_{\nu_k}^{\nu_{k+1}}$$

$$= \frac{1}{\Delta\nu}\,[\ln\nu_{k+1} - \ln\nu_k] = \frac{1}{\Delta\nu}\ln\left[\frac{\nu_{k+1}}{\nu_k}\right]$$

$$\theta_\nu[\nu^{-1}] = \frac{1}{\Delta\nu}\ln\left(\frac{\nu_{k+1}}{\nu_k}\right) \tag{46}$$

$$\nu^{-1}\chi = \frac{1}{\nu}\,\frac{2(\nu) - \bar\nu)}{\Delta\nu} = \frac{2}{\Delta\nu}\left(1-\frac{\bar\nu}{\nu}\right)$$

$$\theta_\nu\,[\nu^{-1}\chi] = \frac{1}{\Delta\nu}\int_{\Delta\nu}\frac{2}{\Delta\nu}\left(1-\frac{\bar\nu}{\nu}\right)d\nu$$

$$= \frac{2}{(\Delta\nu)^2}\int_{\nu_k}^{\nu_{k+1}}\left(1-\frac{\bar\nu}{\nu}\right)d\nu = \frac{2}{(\Delta\nu)^2}\left\{\Delta\nu - \ln\left(\frac{\nu_{k+1}}{\nu_k}\right)\right\}$$

$$\theta_\nu\,[\nu^{-1}\chi] = \frac{2}{(\Delta\nu)^2}\left\{\Delta\nu - \ln\left(\frac{\nu_{k+1}}{\nu_k}\right)\right. \tag{47}$$

$$\nu^{-2}\chi = \frac{1}{\nu^2}\frac{2}{\Delta\nu}(\nu - \bar{\nu}) = \frac{2}{\Delta\nu}\left(\frac{1}{\nu} - \frac{\bar{\nu}}{\nu^2}\right)$$

$$\theta_\nu[\nu^{-2}\chi] = \frac{1}{\Delta\nu}\int\frac{2}{\Delta\nu}\left(\frac{1}{\nu} - \frac{\bar{\nu}}{\nu^2}\right)d\nu = \frac{2}{(\Delta\nu)^2}\left\{\ln\left(\frac{\nu_{k+1}}{\nu_k}\right) - \bar{\nu}.\frac{\Delta\nu}{\nu_k\,\nu_{k+1}}\right\}$$

$$\theta_\nu[\nu^{-2}\chi] = \frac{2}{(\Delta\nu)^2}\left\{\ln\left(\frac{\nu_{k+1}}{\nu_k}\right) - \frac{\bar{\nu} - \Delta\nu}{\nu_k\,\nu_{k+1}}\right\} \tag{48}$$

$$\theta_\nu[\nu^{-3}] = \frac{1}{\Delta\nu}\frac{d\nu}{\nu^3} = \frac{1}{\Delta\nu}\left(\frac{\nu^{-3+1}}{-3+1}\right)_{\nu_k}^{\nu_{k+1}^*}$$

$$= -\frac{1}{2\Delta\nu}\left[\frac{1}{\nu^2}\right]_{\nu_k}^{\nu_{k+1}} = -\frac{1}{2\Delta\nu}\left[\frac{1}{\nu^2_{k+1}} - \frac{1}{\nu^2_k}\right]$$

$$= \frac{1}{2\Delta\nu}\left[\frac{\nu^2_{k+1} - \nu^2_k}{\nu^2_k\,\nu^2_{k+1}}\right]$$

$$= \frac{\bar{\nu}}{\nu^2_k\,\nu^2_{k+1}}$$

$$\theta_\nu[\nu^{-3}] = \frac{\bar{\nu}}{\nu^2_k\,\nu^2_{k+1}} \tag{48a}$$

$$\theta_\nu[\nu^{-3}\chi] = \frac{1}{\Delta\nu}\int\nu^{-3}\chi\,d\nu = \frac{1}{\Delta\nu}\int\nu^{-3}\frac{\nu - \bar{\nu}}{\Delta\nu/2}\,d\nu$$

$$= \frac{2}{(\Delta\nu)^2}\int(\nu^{-2} - \bar{\nu}\,\nu^{-3})d\nu$$

$$= \frac{2}{\Delta\nu}\left[\frac{1}{\nu_k\,\nu_{k+1}} - \left(\frac{\bar{\nu}}{\nu_k\,\nu_{k+1}}\right)^2\right] \tag{48b}$$

$$\nu^{-2}\chi^2 = \frac{\Delta 4}{\nu^2}\left(\frac{\nu - \bar{\nu}}{\Delta\nu}\right)^2$$

$$= \frac{4}{\nu^2}\frac{1}{(\Delta\nu)^2}\{\nu^2 - 2\nu\,\bar{\nu} + (\bar{\nu})^2\}$$

$$= \frac{4}{(\Delta\nu)^2}\left\{1 - \frac{2\bar{\nu}}{\nu} + \left(\frac{\bar{\nu}}{\nu}\right)^2\right\}$$

$$\theta_\nu[\nu^{-2}\chi^2] = \frac{4}{(\Delta\nu)^2}\frac{1}{\Delta\nu}\int_{\nu_k}^{\nu_{k+1}}\cdot\left\{1 - \frac{2\bar{\nu}}{\nu} + \left(\frac{\bar{\nu}}{\nu}\right)^2\right\}d\nu$$

$$= \frac{4}{(\Delta\nu)^3} \{\Delta\nu - 2\bar{\nu} \ln(\frac{\nu_{k+1}}{\nu_k}) + (\bar{\nu})^2 \frac{1}{\nu_k \nu_{k+1}} \}$$

$$\theta_\nu [\nu^{-2}\chi^2] = \frac{4}{(\Delta\nu)^3} \{\Delta\nu - 2\bar{\nu} \ln(\frac{\nu_{k+1}}{\nu_k}) + (\bar{\nu})^2 \frac{1}{\nu_k \nu_{k+1}} \tag{49}$$

Let us write, $\theta_\nu [\nu\chi] = \frac{2}{\Delta\nu} \{\frac{1}{3} (\nu^2_{k+1} + \nu_{k+1} \nu_k + \nu^2_k) - (\bar{\nu})^2\} = a_9$

$$\theta_\nu [\nu^{-1}] = \frac{1}{\Delta\nu} \ln(\frac{\nu_{k+1}}{\nu_k}) = a_1$$

$$\theta_\nu [\nu^{-2}] = \frac{1}{\nu_k \nu_{k+1}} = a_2$$

$$\theta_\nu [\nu^{-1}\chi] = \frac{2}{(\Delta\nu)^2} \{\Delta\nu - \ln(\frac{\nu_{k+1}}{\nu_k})\} = a_3$$

$$\theta_\nu [\nu^{-2}\chi] = \frac{2}{(\Delta\nu)^2} \{\ln(\frac{\nu_{k+1}}{\nu_k}) - \frac{\bar{\nu}\,\Delta\nu}{\nu_k \nu_{k+1}} = a_4 \tag{50}$$

$$\theta_\nu [\nu^{-2}\chi^2] = \frac{4}{(\Delta\nu)^3} \{\Delta\nu - 2\bar{\nu} \ln(\frac{\nu_{k+1}}{\nu_k}) + \frac{(\bar{\nu})^2}{\nu_k \nu_{k+1}} \} = a_5$$

$$\theta_\nu [\nu^{-3}] = \frac{\bar{\nu}}{\nu_k^2 \nu_{k+1}^2} = a_6$$

$$\theta_\nu [\nu^{-3}\chi] = \frac{2}{\Delta\nu} [\frac{1}{\nu_k \nu_{k+1}} - (\frac{\bar{\nu}}{\nu_k \nu_{k+1}})^2] = a_7$$

$$\theta_\nu [\nu] = \bar{\nu} = a_8$$

Upon using the operator θ_ν on $k' A_{\mu\xi}\nu^{-2}$ term we obtain,

$$\{a_2(p_\mu I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} p_\mu I_\rho - q_\mu I_\mu - 2p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- \frac{1}{6} \frac{\Delta A}{\bar{A}} q_\mu I_{\rho\mu} + 2q_\mu \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}) + p_\mu(\frac{1}{6} \frac{\Delta A}{\bar{A}} a_4 - \frac{2a_1}{\Delta\nu}) I_{\nu\rho}$$

$$- q_\mu(\frac{1}{6} \frac{\Delta A}{\bar{A}} a_4 - \frac{2a_1}{\Delta\nu}) I_{\rho\mu\nu}\} I_o$$

$$+ [a_4 p_\mu I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} p_\mu I_\rho - q_\mu I_\mu - 2p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu - \frac{1}{6} \frac{\Delta A}{\bar{A}} q_\mu I_{\rho\mu}$$

$$+ 2q_\mu \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}\} + p_\mu(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2}{\Delta\nu} a_3) I_{\nu\rho}$$

$$- q_\mu(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2}{\Delta\nu} a_3) I_{\rho\mu\nu}] I_\nu +$$

$$\text{Therefore } K'\theta_\nu[\nu^{-2}A_{\mu\xi}] = K'A_{\mu\xi\nu} \tag{51}$$

The quantity $A_{\mu\xi\nu}$ in (51) can be written as,

$$a_2 p_\mu I_o^2 + \frac{1}{6} a_2 \frac{\Delta A}{\bar{A}} p_\mu I_\rho I_o - q_\mu a_2 I_\mu I_o - 2p_\mu a_2 \frac{\bar{\nu}}{\Delta\nu} I_\nu I_o - \frac{a_2}{6}\frac{\Delta A}{\bar{A}} q_\mu I_{\rho\mu} I_o$$

$$+ 2a_2 q_\mu \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} I_o$$

$$+ p_\mu(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{2a_1}{\Delta\nu}) I_{\nu\rho} I_o - q_\mu(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{2a_1}{\Delta\nu}) I_{\rho\mu\nu} I_o$$

$$+ a_4 p_\mu I_o I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 p_\mu I_\rho I_\nu - a_4 q_\mu I_\mu I_\nu - 2a_4 p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu^2$$

$$\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 q_\mu I_{\rho\mu} I_\nu + 2a_4 q_\mu \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} I_\nu + p_\mu(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2a_3}{\Delta\nu}) I_{\nu\rho} I_\nu$$

$$- q_\mu(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2a_3}{\Delta\nu}) I_{\rho\mu\nu} I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2 p_\mu I_o I_\rho + a_2 \bar{R} p_\mu I_\rho^2$$

$$- \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2 q_\mu I_\mu I_\rho - \frac{1}{3}\frac{\Delta A}{\bar{A}} a_2 p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu I_\rho - \bar{R} a_2 q_\mu I_{\rho\mu} I_\rho$$

$$+ \frac{1}{3}\frac{\Delta A}{\bar{A}} a_2 \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} I_\rho + p_\mu(\bar{R} a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{\Delta\nu}) I_{\nu\rho} I_\rho$$

$$- q_\mu(\bar{R} a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{\Delta\nu}) I_{\rho\mu\nu} I_\rho + \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 p_\mu I_o I_{\nu\rho}$$

$$+ a_4 \bar{R} p_\mu I_\rho I_{\nu\rho} - \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 q_\mu I_\mu I_{\nu\rho}$$

$$- \frac{1}{3}\frac{\Delta A}{\bar{A}} a_4 p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu I_{\nu\rho} - a_4 \bar{R} q_\mu I_{\rho\mu} I_{\nu\rho}$$

$$+ \frac{1}{3}\frac{\Delta A}{\bar{A}} a_4 \frac{\nu}{\Delta\nu} I_{\mu\nu} I_{\nu\rho} + p_\mu(\bar{R} a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I_{\nu\rho}^2$$

$$- q_\mu(\bar{R} a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I_{\rho\mu\nu} I_{\nu\rho} + a_2 p_{\eta\mu} I_o I_\mu$$

$$+ \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2 p_{\eta\mu} I_\rho I_\mu - a_2 q_{\eta\mu} I_\mu^2 - 2a_2 p_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_\nu I_\mu$$

$$- \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2 q_{\eta\mu} I_{\rho\mu} I_\mu + 2a_2 q_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu} I_\mu$$

$$+ p_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{2a_1}{\Delta\nu}) I_{\nu\rho} I_\mu$$

$$\{a_2 [\frac{1}{6}\frac{\Delta A}{\bar{A}} p_\mu I_o + \bar{R} p_\mu I_\rho - \frac{1}{6}\frac{\Delta A}{\bar{A}} q_\mu I_\mu - \frac{1}{3}\frac{\Delta A}{\bar{A}} p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- \bar{R} q_\mu I_{\rho\mu} + \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}] + p_\mu(\bar{R} a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{\Delta\nu}) I_{\nu\rho}$$

$$- q_\mu(\bar{R} a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{}) I_{\rho\mu\nu}\} I_\rho +$$

$$\{a [\frac{1}{6}\frac{\Delta A}{\bar{A}} p_\mu I_o + \bar{R} p_\mu I_\rho - \frac{1}{6}\frac{\Delta A}{\bar{A}} q_\mu I_\mu - \frac{1}{3}\frac{\Delta A}{\bar{A}} p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- \bar{R} q_\mu I_{\rho\mu} + \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}] + p_\mu(\bar{R} a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I_{\nu\rho}$$

$$- q_\mu(\bar{R} a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I_{\rho\mu\nu}\} I_{\nu\rho}$$

$$\{a_2 [p_{\eta\mu} I_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} p_{\eta\mu} I_\rho - q_{\eta\mu} I_\mu - 2p_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- \frac{1}{6}\frac{\Delta A}{\bar{A}} q_{\eta\mu} I_{\rho\mu} + 2q_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}] + p_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{2}{\Delta\nu} a_1) I_{\nu\rho}$$

$$- q_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{2}{\Delta\nu} a_1) I_{\rho\mu\nu}\} I_\mu +$$

$$\{a_4 [p_{\eta\mu} I_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} p_{\eta\mu} I_\rho - q_{\eta\mu} I_\mu - 2p_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- \frac{1}{6}\frac{\Delta A}{\bar{A}} q_{\eta\mu} I_{\rho\mu} + 2q_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}] + p_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2}{\Delta\nu} a_3) I_{\nu\rho}$$

$$- q_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2}{\Delta\nu} a_3) I_{\rho\mu\nu}\} I_{\mu\nu} +$$

$$\{a_2 [\frac{1}{6}\frac{\Delta A}{\bar{A}} p_{\eta\mu} I_o + \bar{R} p_{\eta\mu} I_\rho - \frac{1}{6}\frac{\Delta A}{\bar{A}} q_{\eta\mu} I_\mu - \frac{1}{3}\frac{\Delta A}{\bar{A}} p_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} \, _\nu$$

$$- \bar{R} q_{\eta\mu} I_{\rho\mu} + \frac{1}{3} q_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu}\frac{\Delta A}{\bar{A}} I_{\mu\nu}] + p_{\eta\mu}(\bar{R} a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{\Delta\nu}) I_{\nu\rho}$$

$$- q_{\eta\mu}(\bar{R} a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{\Delta\nu}) I_{\rho\mu\nu}\} I_{\rho\mu} +$$

$$\{a_4 [\frac{1}{6}\frac{\Delta A}{\bar{A}} p_{\eta\mu} I_o + \bar{R} p_{\eta\mu} I_\rho - \frac{1}{6}\frac{\Delta A}{\bar{A}} q_{\eta\mu} I_\mu - \frac{1}{3}\frac{\Delta A}{\bar{A}} p_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_\nu$$

$$- \bar{R}q_{\eta\mu}I_{\rho\mu} + \frac{1}{3}\frac{\Delta A}{\bar{A}} q_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}] + p_{\eta\mu}(\bar{R}a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I_{\nu\rho}$$

$$- q_{\eta\mu}(\bar{R}a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I_{\rho\mu\nu}\} I_{\rho\mu\nu} = \theta_\nu [\nu^{-2}A\mu\xi]$$

$$= A_{\mu\xi\nu}$$

$$- q_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{2a_1}{\Delta\nu}) I_{\rho\mu\nu}I_\mu + a_4 p_{\eta\mu}I_o I_{\mu\nu}$$

$$+ \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 p_{\eta\mu}I_\rho I_{\mu\nu} - a_4 q_{\eta\mu}I_\mu I_{\mu\nu} - 2a_4 p_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_\nu I_{\mu\nu}$$

$$- \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 q_{\eta\mu}I_{\rho\mu}I_{\mu\nu} + 2a_4 q_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I^2_{\mu\nu}$$

$$+ p_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2a_3}{\Delta\nu} I_{\nu\rho}I_{\mu\nu} - q_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_5 - \frac{2a_3}{\Delta\nu}) I_{\rho\mu\nu}I_{\mu\nu}$$

$$+ \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2 p_{\eta\mu}I_o I_{\rho\mu} + a_2\bar{R}p_{\eta\mu}I_\rho I_{\rho\mu} - \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2 q_{\eta\mu}I_\mu I_{\rho\mu}$$

$$- \frac{1}{3}\frac{\Delta A}{\bar{A}} a_2 p_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_\nu I_{\rho\mu} - a_2\bar{R}q_{\eta\mu}I^2_{\rho\mu} + \frac{1}{3}\frac{\Delta A}{\bar{A}} q_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}I_{\rho\mu}$$

$$+ p_{\eta\mu}(\bar{R}a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{\Delta\nu}) I_{\nu\rho}I_{\rho\mu} - q_{\eta\mu}(\bar{R}a_4 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_1}{\Delta\nu}) I_{\rho\mu\nu}I_{\rho\mu}$$

$$+ \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 p_{\eta\mu}I_o I_{\rho\mu\nu} + a_4\bar{R}p_{\eta\mu}I_\rho I_{\rho\mu\nu} - \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 q_{\eta\mu}I_\mu I_{\rho\mu\nu}$$

$$- \frac{1}{3}\frac{\Delta A}{\bar{A}} a_4 p_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_\nu I_{\rho\mu\nu} - a_4\bar{R}q_{\eta\mu}I_{\rho\mu}I_{\rho\mu\nu}$$

$$+ \frac{1}{3}\frac{\Delta A}{\bar{A}} a_4 q_{\eta\mu}\frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}I_{\rho\mu\nu} + p_{\eta\mu}(\bar{R}a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I_{\nu\rho}I_{\rho\mu\nu}$$

$$- q_{\eta\mu}(\bar{R}a_5 - \frac{1}{3}\frac{\Delta A}{\bar{A}}\frac{a_3}{\Delta\nu}) I^2_{\rho\mu\nu} = A_{\mu\xi\nu} \qquad (52)$$

We shall collect the coefficients of the like terms.

Equation (52) is rewritten as

$$A_{\mu\xi\nu} = a_2 p_\mu I^2_o + \frac{1}{3} a_2\frac{\Delta A}{\bar{A}} p_\mu I_o I_\rho + a_2(p_{\eta\mu} - q_\mu) I_o I_\mu$$

$$+ P_\mu(a_4 - 2a_2\frac{\bar{\nu}}{\Delta\nu}) I_o I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2(p_{\eta\mu} - q_\mu) I_o I_{\rho\mu}$$

$$+ (a_4 p_{\eta\mu} + 2a_2 q_\mu\frac{\bar{\nu}}{\Delta\nu}) I_o I_{\mu\nu} + 2p_\mu(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{a_1}{\Delta\nu}) I_o I_{\nu\rho}$$

$$+ [\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 (p_{\eta\mu} - q_\mu) + 2q_\mu \frac{a_1}{\Delta\nu}] I_o I_{\rho\mu\nu} + a_2 \bar{R} p_\mu I_\rho^2$$

$$+ \frac{1}{6}\frac{\Delta A}{\bar{A}} a_2 (p_{\eta\mu} - q_\mu) I_\rho I_\mu + \frac{1}{6}\frac{\Delta A}{\bar{A}} p_\mu (a_4 - 2a_2 \frac{\bar{\nu}}{\Delta\nu}) I_\rho I_\nu$$

$$+ a_2 \bar{R}(p_{\eta\mu} - q_\mu) I_\rho I_{\rho\mu} + \frac{1}{6}\frac{\Delta A}{\bar{A}} (a_4 p_{\eta\mu} + 2a_2 \frac{\bar{\nu}}{\Delta\nu}) I_\rho I_{\mu\nu}$$

$$+ 2p_\mu (a_4 \bar{R} - \frac{1}{6}\frac{\Delta A}{\bar{A}} \frac{a_1}{\Delta\nu}) I_\rho I_{\nu\rho} +$$

$$+ \{a_4 \bar{R}(p_{\eta\mu} - q_\mu) + \frac{1}{3}\frac{\Delta A}{\bar{A}} q_\mu \frac{a_1}{\Delta\nu}\} I_\rho I_{\rho\mu\nu}$$

$$- a_2 q_{\eta\mu} I_\mu^2 - (a_4 q_\mu + 2a_2 q_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu}) I_\mu I_\nu - \frac{1}{3}\frac{\Delta A}{\bar{A}} a_2 q_{\eta\mu} I_\mu I_{\rho\mu}$$

$$+ q_{\eta\mu}(2a_2 \frac{\bar{\nu}}{\Delta\nu} - a_4) I_\mu I_{\mu\nu} + \frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 (p_{\eta\mu} - q_\mu) - 2p_{\eta\mu}\frac{a_1}{\Delta\nu}) I_\mu I_{\nu\rho}$$

$$- 2q_{\eta\mu}(\frac{1}{6}\frac{\Delta A}{\bar{A}} a_4 - \frac{a_1}{\Delta\nu}) I_\mu I_{\rho\mu\nu}$$

$$- 2a_4 p_\mu \frac{\bar{\nu}}{\Delta\nu} I_\nu^2 - \frac{1}{6}\frac{\Delta A}{\bar{A}} (a_4 q_\mu + 2a_2 p_{\eta\mu}) I_\nu I_{\rho\mu}$$

$$+ 2a_4 \frac{\bar{\nu}}{\Delta\nu} (q_\mu - p_{\eta\mu}) I_\nu I_{\mu\nu} +$$

$$+ p_\mu \{\frac{1}{6}\frac{\Delta A}{\bar{A}} (a_5 - 2a_4 \frac{\bar{\nu}}{\Delta\nu}) - \frac{2a_3}{\Delta\nu}\} I_\nu I_{\nu\rho}$$

$$- \frac{1}{6}\frac{\Delta A}{\bar{A}} (a_5 q_\mu + 2a_4 p_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu}) - 2q_\mu \frac{a_3}{\Delta\nu}\} I_\nu I_{\rho\mu\nu}$$

$$- a_2 \bar{R} q_{\eta\mu} I_{\rho\mu}^2 + \frac{1}{6}\frac{\Delta A}{\bar{A}} q_{\eta\mu}(2 \frac{\bar{\nu}}{\Delta\nu} - a_4) I_{\rho\mu} I_{\mu\nu}$$

$$+ \{a_4 \bar{R}(p_{\eta\mu} - a_4) - \frac{1}{3}\frac{\Delta A}{\bar{A}} p_{\eta\mu} \frac{a_1}{\Delta\nu}\} I_{\rho\mu} I_{\nu\rho}$$

$$- 2q_{\eta\mu}(a_4 \bar{R} - \frac{1}{6}\frac{\Delta A}{\bar{A}} \frac{a_1}{\Delta\nu}) I_{\rho\mu} I_{\rho\mu\nu}$$

$$+ 2a_4 q_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu} I_{\mu\nu}^2 + \{\frac{1}{6}\frac{\Delta A}{\bar{A}} (a_5 p_{\eta\mu} + 2a_4 \frac{\bar{\nu}}{\Delta\nu}) - \frac{2a_3}{\Delta\nu} p_{\eta\mu}\} I_{\mu\nu} I_{\nu\rho}$$

$$+ q_{\eta\mu}\{\frac{1}{6}\frac{\Delta A}{\bar{A}} (a_5 + 2a_4 \frac{\bar{\nu}}{\Delta\nu}) + 2\frac{a_3}{\Delta\nu}\} I_{\mu\nu} I_{\rho\mu\nu}$$

$$+ p_\mu (a_5 \bar{R} - \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{a_3}{\Delta\nu}) I^2_{\nu\rho}$$

$$+ \{(p_{\eta\mu} - q_\mu)(a_5 \bar{R} - \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{a_3}{\Delta\nu}) \quad I_{\nu\rho} I_{\rho\mu\nu}$$

$$- q_{\eta\mu}(\bar{R} a_5 - \frac{1}{3} \frac{\Delta A}{\bar{A}} \frac{a_3}{\Delta\nu}) I^2_{\rho\mu\nu} = A_{\mu\xi\nu} \quad (53)$$

Let us introduce the following,

$$f_1 = a_2 p_\mu, \; f_2 = \frac{1}{6} \frac{\Delta A}{\bar{A}}, \; f_3 = p_{\eta\mu} - q_\mu, \; f_4 = a_4 - 2a_2 \frac{\bar{\nu}}{\Delta\nu}$$

$$f_5 = a_4 \bar{R} - \frac{a_1}{\Delta\nu} f_2, \; f_6 = a_4 f_2 - \frac{a_1}{\Delta\nu}$$

$$f_7 = a_4 p_{\eta\mu} + 2a_2 q_\mu \frac{\bar{\nu}}{\Delta\nu}, \; f_8 = a_4 f_2 f_3 + 2q_\mu \frac{a_1}{\Delta\nu}$$

$$f_9 = a_4 \bar{R} - \frac{a_1}{\Delta\nu} \cdot f_2, \; f_{10} = a_4 \bar{R} f_3 + 2f_2 q_\mu \frac{a_1}{\Delta\nu}$$

$$f_{11} = a_2 q_{\eta\mu}, \; f_{12} = a_4 q_\mu + 2a_2 p_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu}$$

$$f_{13} = 2q_{\eta\mu}(f_2 a_4 - \frac{a_1}{\Delta\nu}), \; f_{14} = 2a_4 p_\mu \frac{\bar{\nu}}{\Delta\nu}$$

$$f_{15} = f_2(a_4 q_\mu + 2a_2 p_{\eta\mu}), \; f_{16} = 2a_4 \frac{\bar{\nu}}{\Delta\nu} \cdot f_3$$

$$f_{17} = p_\mu \{f_2 (a_5 - 2a_4 \frac{\bar{\nu}}{\Delta\nu}) - \frac{2a_3}{\Delta\nu}\}$$

$$f_{18} = f_2 (a_5 q_\mu + 2a_4 p_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu}) - 2q_\mu \frac{a_3}{\Delta\nu}, \; f_{19} = a_2 \bar{R} q_{\eta\mu}$$

$$f_{20} = f_2 q_{\eta\mu}(2 \frac{\bar{\nu}}{\Delta\nu} - a_4),$$

$$f_{21} = a_4 \bar{R} f_3 - 2f_2 p_{\eta\mu} \frac{a_1}{\Delta\nu}, \; f_{22} = 2q_{\eta\mu} f_9$$

$$f_{23} = 2a_4 q_{\eta\mu} \frac{\bar{\nu}}{\Delta\nu}, \; f_{24} = f_2(a_5 p_{\eta\mu} + 2a_4 \frac{\bar{\nu}}{\Delta\nu}) - 2p_{\eta\mu} \frac{a_3}{\Delta\nu}$$

$$f_{25} = q_{\eta\mu}\{f_2(a_5 + 2a_4 \frac{\bar{\nu}}{\Delta\nu}) + 2 \frac{a_3}{\Delta\nu}\}, \; f_{26} = (a_5 \bar{R} - 2f_2 \frac{a_3}{\Delta\nu})$$

$$f_{27} = p_\mu f_{26}, \; f_{28} = f_3 f_2, \; f_{29} = q_{\eta\mu} f_{26}$$

With the coefficients defined above, we rewrite equation (53) as follows:

$$
\begin{aligned}
A\mu\xi\nu = {} & f_1 I_0^2 + 2f_1 f_2 I_0 I_\rho + a_2 f_3 I_0 I_\mu + p_\mu f_4 I_0 I_\nu \\
& + f_2 f_3 a_2 I_0 I_{\rho\mu} + f_7 I_0 I_{\mu\nu} + 2p_\mu f_6 I_0 I_{\nu\rho} + f_8 I_0 I_{\rho\mu\nu} \\
& + \bar{R} f_1 I_\rho^2 + a_2 f_2 f_3 I_\rho I_\mu + p_\mu f_2 f_4 I_\rho I_\nu + R a_2 f_3 I_\rho I_{\rho\mu} \\
& + f_2 f_7 I_\rho I_{\mu\nu} + 2p_\mu f_9 I_\rho I_{\nu\rho} + f_{10} I_\rho I_{\rho\mu\nu} - f_{11} I_\mu^2 - f_{12} I_\mu I_\nu \\
& + 2f_2 a_2 q_{\eta\mu} I_\mu I_{\rho\mu} - q_{\eta\mu} f_4 I_\mu I_{\mu\nu} - (a_4 f_2 f_3 - 2p_{\eta\mu} \frac{a}{\Delta\nu}) I_\mu I_{\nu\rho} \\
& - f_{13} I_\mu I_{\rho\mu\nu} - f_{14} I_\nu^2 - f_{15} I_\nu I_{\rho\mu} + f_{16} I_\nu I_{\mu\nu} + f_{17} I_\nu I_{\nu\rho} \\
& - f_{18} I_\nu I_{\rho\mu\nu} - f_{19} I_{\rho\mu}^2 + f_{20} I_{\rho\mu} I_{\mu\nu} + f_{21} I_{\rho\mu} I_{\nu\rho} - f_{22} I_{\rho\mu} I_{\rho\mu\nu} \\
& + f_{23} I_{\mu\nu}^2 + f_{24} I_{\mu\nu} I_{\nu\rho} + f_{25} I_{\mu\nu} I_{\rho\mu\nu} + f_{27} I_{\nu\rho}^2 \\
& + f_{28} I_{\nu\rho} I_{\rho\mu\nu} + f_{29} I_{\rho\mu\nu}^2 = A_{\mu\xi\nu}
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
& g_1 = 2f_1 f_2,\ g_2 = a_2 f_3,\ g_3 = p_\mu f_4,\ g_4 = f_2 f_3 a_2 \\
& g_5 = 2p_\mu f_6,\ g_6 = \bar{R} f_1,\ g_7 = a_2 f_2 f_3,\ g_8 = p_\mu f_2 f_4 \\
& g_9 = R a_2 f_3,\ g_{10} = f_2 f_7,\ g_{11} = 2p_\mu f_9 \\
& g_{12} = 2f_2 a_2 q_{\eta\mu},\ g_{13} = q_{\eta\mu} f_4,\ g_{14} = a_4 f_2 f_3 - 2p_{\eta\mu} \frac{a_1}{\Delta\nu}
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
A_{\mu\xi\nu} = {} & f_1 I_0^2 + g_1 I_0 I_\rho + g_2 I_0 I_\mu + g_3 I_0 I_\nu + g_4 I_0 I_{\rho\mu} + f_7 I_0 I_{\mu\nu} \\
& + g_5 I_0 I_{\nu\rho} + f_8 I_0 I_{\rho\mu\nu} + g_6 I_\rho^2 + g_7 I_\rho I_\mu + g_8 I_\rho I_\nu + g_9 I_\rho I_{\rho\mu} \\
& + g_{10} I_\rho I_{\mu\nu} + g_{11} I_\rho I_{\nu\rho} + f_{10} I_\rho I_{\rho\mu\nu} - f_{11} I_\mu^2 - f_{12} I_\mu I_\nu \\
& - g_{12} I_\mu I_{\rho\mu} - g_{13} I_\mu I_{\mu\nu} - g_{14} I_\mu I_{\nu\rho} - f_{13} I_\mu I_{\rho\mu\nu} - f_{14} I_\nu^2 \\
& - f_{15} I_\nu I_{\rho\mu} + f_{16} I_\nu I_{\mu\nu} + f_{17} I_\nu I_{\nu\rho} - f_{18} I_\nu I_{\rho\mu\nu} - f_{19} I_{\rho\mu}^2 \\
& + f_{20} I_{\rho\mu} I_{\mu\nu} + f_{21} I_{\rho\mu} I_{\nu\rho} - f_{22} I_{\rho\mu} I_{\rho\mu\nu} + f_{23} I_{\mu\nu}^2
\end{aligned}
$$

$$+ f_{24} I_{\mu\nu} I_{\nu\rho} + f_{25} I_{\mu\nu} I_{\rho\mu\nu} + f_{27} I^2_{\nu\rho} + f_{28} I_{\nu\rho} I_{\rho\mu\nu} + f_{29} I^2_{\rho\mu\nu} \qquad (56)$$

Equation (56) can be rewritten as

$$A_{\mu\xi\nu} = f_1 I^2_{o} + g_6 I^2_{\rho} - f_{11} I^2_{\mu} - f_{14} I^2_{\nu} - f_{19} I^2_{\rho\mu} + f_{23} I^2_{\mu\nu}$$

$$+ f_{27} I^2_{\nu\rho} + f_{29} I^2_{\rho\mu\nu} + (g_1 I_{\rho} + g_2 I_{\mu} + g_3 I_{\nu} + g_4 I_{\rho\mu} + f_7 I_{\mu\nu}$$

$$+ g_5 I_{\nu\rho} + f_8 I_{\rho\mu\nu}) I_o + (g_7 I_{\mu} + g_8 I_{\nu} + g_9 I_{\rho\mu} + g_{10} I_{\mu\nu} + g_{11} I_{\nu\rho}$$

$$+ f_{10} I_{\rho\mu\nu}) I_{\rho} - (+f_{12} I_{\nu} + g_{12} I_{\rho\mu} + g_{13} I_{\mu\nu} + g_{14} I_{\nu\rho} + f_{13} I_{\rho\mu\nu}) I_{\mu}$$

$$+ (-f_{15} I_{\rho\mu} + f_{16} I_{\mu\nu} + f_{17} I_{\nu\rho} - f_{18} I_{\rho\mu\nu}) I_{\nu}$$

$$+ (f_{20} I_{\mu\nu} + f_{21} I_{\nu\rho} - f_{22} I_{\rho\mu\nu}) I_{\rho\mu}$$

$$+ (f_{24} I_{\nu\rho} + f_{25} I_{\rho\mu\nu}) I_{\mu\nu} + f_{28} I_{\nu\rho} I_{\rho\mu\nu} \qquad (57)$$

$$\theta_{\nu} [K' A_{\mu\xi} \nu^{-2}] = K' A_{\mu\xi\nu} \qquad (58)$$

= last term in equation (44).

Now apply $\theta_{\nu} [\frac{8\pi\sigma_{Th} B}{\Delta\nu} (I_{\nu} + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho})]$ in (44)

where $B = \frac{\nu(\gamma - 2\alpha)}{4\pi} \{1 - \frac{6}{5} P_1 + \frac{1}{2} P_2 - \frac{3}{10} P_3\}$

$$\gamma = \frac{h\nu}{m_o c^2} = 8.1019 \times 10^{-21} \nu = \gamma' \nu$$

$$B = \{\gamma' \nu^2 - 2\alpha\nu\} \{1 - \frac{6}{5} P_1 + \frac{1}{2} P_2 - \frac{3}{10} P_3\}$$

$$= B_1 \{\gamma' \nu^2 - 2\alpha\nu\}$$

$$\theta_{\nu} [B_1 \{\gamma' \nu^2 - 2\alpha\nu\}] = B_1 \frac{1}{\Delta\nu} \int (\gamma' \nu^2 - 2\alpha\nu) \, d\nu$$

$$= \frac{B_1}{\Delta\nu} \{\gamma' \frac{\nu^3}{3} - \frac{2\alpha}{2} \nu^2\}_{\nu_k}^{\nu_{k+1}}$$

$$= \frac{B_1}{\Delta\nu} \{\frac{1}{3} \gamma' (\nu^3_{k+1} - \nu^3_k) - \alpha(\nu^2_{k+1} - \nu^2_k)$$

$$= \frac{B_1}{\Delta\nu} \{\frac{1}{3} \gamma' \Delta\nu (\nu^2_{k+1} + \nu_{k+1} \nu_k + \nu^2_k) - \alpha(\nu_{k+1} - \nu_k)(\nu_{k+1} + \nu_k)$$

$$= \frac{B_1}{\Delta\nu}\ \{\frac{1}{3}\ \gamma'\Delta\nu(\nu^2_{k+1} + \nu_{k+1}\nu_k + \nu^2_k - \alpha\Delta\nu\ (\nu_{k+1} + \nu_k)$$

$$= B_1\ \frac{1}{3}\ \gamma'(\nu^2_{k+1} + \nu_{k+1}\nu_k + \nu^2_k) - 2\alpha\bar{\nu}\}$$

$$= B_2$$

$$\theta_\nu\ [\frac{8\pi\sigma_{Th}B}{\Delta\nu}\ (I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho})]$$

$$= \frac{8\pi\sigma_{Th}}{\Delta\nu}\ (I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho})\theta_\nu[B]$$

$$= \frac{8\pi\sigma_{Th}}{\Delta\nu}\ (I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho})B_1\ \{\frac{1}{3}\gamma'(\nu^2_{k+1} + \nu_{k+1}\ \nu_k + \nu^2_k) - 2\alpha\bar{\nu}\}$$

Let $\frac{1}{3}\gamma'(\nu^2_{k+1} + \nu_{k+1}\ \nu_k + \nu^2_k) - 2\alpha\bar{\nu}) = B$

$$B_1 = \frac{1}{4\pi}\ \{1 - \frac{6}{5}\ P_1 + \frac{1}{2}\ P_2 - \frac{3}{10}\ P_3\}$$

Then

$$= \frac{2\sigma_{Th}}{\Delta\nu}(I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho})\ \frac{1}{4\pi}\ (1 - \frac{6}{5}\ P_1 + \frac{1}{2}\ P_2 - \frac{3}{10}\ P_3)\{B'\}$$

$$= \frac{2\sigma_{Th}}{\Delta\nu}\ (I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho})(1 - \frac{6}{5}\ P_1 + \frac{1}{2}\ P_2 - \frac{3}{10}\ P_3)$$

$$\{\frac{1}{3}\gamma'(\nu^2_{k+1} + \nu_{k+1}\ \nu_k + \nu^2_k) - 2\alpha\bar{\nu}\}$$

$$\theta_\nu\ [\frac{8\pi\sigma_{Th}B}{\Delta\nu}\ (I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho})]$$

$$= \frac{2\sigma_{Th}}{\Delta\nu}\ (I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho})(1 - \frac{6}{5}\ P_1 + \frac{1}{2}\ P_2 - \frac{3}{10}\ P_3)$$

$$\{\frac{1}{3}\ \gamma'(\nu^2_{k+1} + \nu_{k+1}\ \nu_k + \nu^2_k) - 2\alpha\bar{\nu}\}$$

$$= B_p(I_\nu + \frac{1}{6}\ \frac{\Delta A}{\bar{A}}\ I_{\nu\rho}) \qquad (59)$$

Where

$$B_p = \frac{2\sigma_{Th}}{\Delta\nu}\ (1 - \frac{6}{5}\ P_1 + \frac{1}{2}\ P_2 - \frac{3}{10}\ P_3)\{\ \frac{1}{3}\gamma'(\nu^2_{k+1} + \nu_{k+1}\ \nu_k + \nu^2_k) - 2\gamma\bar{\nu})\} \qquad (60)$$

$$\gamma' = \frac{h}{m_o c^2} = 8.1019 \times 10^{-21} \text{sec.} \tag{61}$$

Now let us consider the term

$$4\pi\sigma_{Th} A$$

Where

$$A = \frac{1}{4\pi} \{(1-\gamma) + \frac{6}{5} p_1(\gamma-\alpha) + \frac{1}{2} p_2(1-\gamma-6\alpha) + \frac{3}{10} p_3(\gamma+4\alpha)$$

$$\frac{1}{4\pi} \{(1+\frac{1}{2} p_2) + \gamma(\frac{6}{5} p_1 - \frac{1}{2} p_2 + \frac{3}{10} p_3 - 1) + \alpha(-\frac{6}{5} p_1 - 3p_2 + \frac{6}{5} p_3)\}$$

$$= \frac{1}{4\pi} \{p_2' + \gamma p_1' + \alpha p_3'\}$$

$$4\pi\sigma_{Th} A = \sigma_{Th} \{p_2' + \gamma p_1' + \alpha p_3'\}$$

$$= \sigma_{Th} \{p_2' + \gamma p_1' + \alpha p_3'\}$$

$$\theta_\nu [4\pi\sigma_{Th} A] = \theta_\nu[\sigma_{Th} \{p_2' + \gamma p_1' + \alpha p_3'\}] \tag{62}$$

$$p_1' = \frac{6}{5} p_1 - \frac{1}{2} p_2 + \frac{3}{10} p_3 - 1$$

$$p_2' = 1 + \frac{1}{2} p_2$$

$$p_3' = -\frac{6}{5} p_1 - 3p_2 + \frac{6}{5} p_3 \tag{63}$$

$$\theta_\nu[\gamma] = \gamma' \frac{1}{\Delta\nu} \int_{\Delta\nu} \nu \, d\nu = \frac{\gamma'}{\Delta\nu} [\tfrac{1}{2}\nu^2]_{\nu_k}^{\nu_{k+1}} = \frac{\gamma'}{\Delta\nu} \bar{\nu}.\Delta\nu$$

$$\gamma'\bar{\nu} = \gamma'\bar{\nu} = \gamma' a_8$$

$$\theta_\nu [4\pi\sigma_{Th} A] = \sigma_{Th} (p_2' = \gamma' p_1' a_8 + \alpha p_3') \tag{64}$$

Now we shall consider the term

$$\Sigma(\nu) \{(I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_\rho) + \chi(I_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho})\}$$

$$\Sigma(\nu) = \sigma'_a(\nu) + \sigma_{Th}(1-2\gamma)$$

$$\sigma'_a(\nu) = \sigma_a(\nu)(1-e^{-h\nu/kT})$$

$$\sigma'_a(\nu) = \alpha_R^{ff} = 1.7 \times 10^{-25} T^{-7/2} Z^2 n_e n_i \bar{g}_R = \alpha_R^{ff} \tag{65}$$

Where α_R^{ff} is the Rosseland mean of the free-free absorption coefficient in CGS units (see page 163 of 'Radiative Processes in Astrophysics' by Rybicki and Lightman. We can choose equation (5.19b) or (5.18b) on page 162). But these are frequency dependent absorption coefficients depending on the criterion $h\nu >> kT$ or $h\nu << kT$. We should consider the three types of absorption coefficients.

$$\alpha_\nu^{ff} = 3.7 \times 10^8 T^{-\frac{1}{2}} Z^2 n_e n_i \nu^{-3}(1-e^{-h\nu/kT})\bar{g}_{ff} \quad \text{(Eq.5.18b)}$$

If $h\nu >> kT$, then, $e^{-h\nu/kT} << 1$ therefore, we have

$$\alpha_\nu^{ff} = 3.7 \times 10^8 T^{-\frac{1}{2}} Z^2 n_e n_i \nu^{-3} \bar{g}_{ff}$$

or

$$\alpha_{\nu>}^{ff} = \alpha_1 \nu^{-3}, \quad \alpha_1 = 3.7 \times 10^8 T^{-\frac{1}{2}} Z^2 n_e n_i \bar{g}_{ff}$$

$$\alpha_{\nu>}^{ff} = \alpha_1 \nu^{-3} \quad (h\nu >> kT) \tag{66}$$

If $h\nu << kT$, then we have

$$\alpha_{\nu<}^{ff} = \alpha_2 \nu^{-2} \quad (h\nu << kT) \tag{67}$$

$$\alpha_2 = 1.8 10^{-2} T^{-3/2} Z^2 n_e n_i \bar{g}_{ff}$$

$$\alpha_1 = 3.7 \times 10^8 T^{-\frac{1}{2}} Z^2 n_e n_i \bar{g}_{ff} \tag{68}$$

Now let us consider the terms

$$\Sigma(\nu) \; \{(I_0 + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_\rho) + \chi(I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_{\nu\rho})\} \tag{69}$$

$$\Sigma(\nu) = \sigma'_a(\nu) + \sigma_{Th}(1-2\gamma)$$

$$\sigma'_a(\nu) = \alpha_{\nu>}^{ff} \text{ or } \alpha_{\nu<}^{ff} \text{ (see equation 66 and 67)}$$

$$\gamma = \gamma'\nu$$

$$\Sigma\ (\nu) = (\alpha^{ff}_{\nu>} \text{ or } \alpha^{ff}_{\nu<} \text{ or } \alpha^{ff}_{R}) + \sigma_{Th}(1-2\gamma'\nu)$$

$$= (\alpha_1 \nu^{-3} \text{ or } \alpha_2 \nu^{-2} \text{ or } \alpha^{ff}_{R}) + \sigma_{Th}(1-2\gamma'\nu)$$

using the operator θ_ν

$$\theta_\nu\ \{[(\alpha_1 \nu^{-3} \text{ or } \alpha_2 \nu^{-2} \text{ or } \alpha^{ff}_{R}) + \sigma_{Th}(1-2\gamma'\nu)]$$

$$[(I_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_\rho) + \chi(I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_{\nu\rho})]\}$$

$$= (I_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_\rho)\ [(\alpha_1 a_6 \text{ or } \alpha_2 a_2 \text{ or } \alpha^{ff}_{R}) + \sigma_{Th}(1-2\gamma' a_8)]$$

$$+ (I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_{\nu\rho})\ [(\alpha_1 a_7 \text{ or } \alpha_2 a_4 \text{ or } 0) - 2\gamma' \sigma_{Th}\ a_9]$$

$$\theta_\nu\ [\Sigma(\nu)\ \{(I_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_\rho) + \chi(I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_{\nu\rho})]$$

$$= [(\alpha_1 a_6 \text{ or } \alpha_2 a_2 \text{ or } \alpha^{ff}_{R}) + \sigma_{Th}(1-2\gamma' a_8)][I_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_\rho]$$

$$+ [(\alpha_1 a_7 \text{ or } \alpha_2 a_4 \text{ or } 0) - 2\gamma' \sigma_{Th} a_9][I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} I_{\nu\rho}] \tag{70}$$

We shall now consider the term

$$\sigma'_a\ (\nu)\ [B_o(\nu) + \chi B_\nu(\nu)] + \frac{1}{6}\frac{\Delta A}{\bar{A}}\ [B_\rho(\nu) + \chi B_{\nu\rho}(\nu)]$$

$$= \sigma'_a\ (\nu)\ \{[B_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_\rho(\nu)] + \chi[B_\nu\ (\nu) + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_{\nu\rho}(\nu)]\}$$

in the equation (44)

Let us assume that

$$\sigma'_a\ (\nu) = [\alpha_1 \nu^{-3} \text{ or } \alpha_2 \nu^{-2} \text{ or } \alpha^{ff}_{R}]$$

$$\theta_\nu\ [\sigma'_a\ (\nu)\ \{(B_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_\rho) + \chi(B_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_{\nu\rho})\}]$$

$$= \theta_\nu\ [(\alpha_1 \nu^{-3} \text{ or } \alpha_2 \nu^{-2} \text{ or } \alpha^{ff}_{R})\{(B_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_\rho) + \chi(B_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_{\nu\rho})\}]$$

$$= (B_o + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_\rho)(\alpha_1 a_6,\ \alpha_2 a_2,\ \alpha^{ff}_{R}) + (B_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}} B_{\nu\rho})(\alpha_1 a_7,\ \alpha_2 a_4,\ 0) \tag{71}$$

Therefore Equation (44) on page 26 is rewritten upon using the operator

$$\theta_\nu = \frac{1}{\Delta\nu} \int \ldots . d\nu$$

$$\frac{2}{\Delta r} (\bar{\mu} I_\rho + \frac{1}{6}\Delta\mu I_{\rho\mu}) + \frac{2}{\Delta\mu} (1-\bar{\mu}^2) [\frac{1}{2} \frac{\Delta A}{V} I_\mu$$

$$+ \frac{1}{\Delta r} (2- \frac{\bar{r}\Delta A}{V}) I_{\rho\mu}] - 2\bar{\mu} [\frac{1}{2} \frac{\Delta A}{V} I_o + \frac{1}{\Delta r} (2- \frac{\bar{r}\Delta A}{V}) I_\rho]$$

$$- \frac{1}{3}\Delta\mu [\frac{1}{2} \frac{\Delta A}{V} I_\mu + \frac{1}{\Delta r} (2- \frac{\bar{r}\Delta A}{V}) I_{\rho\mu}]$$

$$= (\alpha_1 a_6, \alpha_2 a_2, \alpha_R^{ff})(B_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} B_\rho) + (\alpha_1 a_7, \alpha_2 a_4, 0)$$

$$(B_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} B_{\nu\rho})$$

$$+ \{(\alpha_1 a_6, \alpha_2 a_2, \alpha_R^{ff}) + \sigma_{Th} (1-2\gamma' a_8)\} (I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_\rho)$$

$$+ \{(\alpha_1 a_7, \alpha_2 a_4, 0) - 2\sigma_{Th}\gamma' a_9\} (I_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho})$$

$$+ \sigma_{Th} (p_2' + \gamma' p_1' a_8 + \alpha p_3') + B_p (I_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho})$$

$$+ K' A_{\mu\xi\nu} \tag{72}$$

Equation (72) is rewritten as,

$$2(\bar{\mu} I_\rho + \frac{1}{6}\Delta\mu \; I_{\rho\mu}) + \frac{2}{\Delta\mu} (1-\bar{\mu}^2) [\frac{1}{2} \frac{\Delta A}{\bar{A}} I_\mu + (2- \frac{\bar{r}}{\Delta r} \frac{\Delta A}{\bar{A}}) I_{\rho\mu}]$$

$$- 2\bar{\mu} [\frac{1}{2} \frac{\Delta A}{\bar{A}} I_o + (2- \frac{\bar{r}}{\Delta r} \frac{\Delta A}{\bar{A}}) I_\rho] - \frac{1}{3}\Delta\mu [\frac{1}{2} \frac{\Delta A}{\bar{A}} I_\mu + (2- \frac{\bar{r}}{\Delta r} . \frac{\Delta A}{\bar{A}}) I_{\rho\mu}]$$

$$= \Delta r (\alpha_1 a_6, \alpha_2 a_2, \alpha_R^{ff} (B_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} B_\rho) + \Delta r (\alpha_1 a_7, \alpha_2 a_4, 0)$$

$$(B_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} B_{\nu\rho})$$

$$+ \Delta r \; \{(\alpha_1 a_6, \alpha_2 a_2, \alpha_R^{ff}) + \sigma_{Th} (1-2\gamma' a_8)\} \; (I_o + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_\rho)$$

$$+ \Delta r \; \{(\alpha_1 a_7, \alpha_2 a_4, 0) - 2\sigma_{Th}\gamma' a_9 + B_\rho\} (I_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho})$$

$$+ \Delta r \sigma_{Th} (p_2' + \gamma' p_1' a_8 + \alpha p_3') + \Delta r k' A_{\mu\xi\nu} \tag{73}$$

$$\text{Let } \frac{1}{2}\frac{\Delta A}{\bar{A}} = 3f_2,\quad f_{30} = 2 - \frac{\bar{r}}{\Delta r}\frac{\Delta A}{\bar{A}}$$

$$f_{31} = \frac{2}{\Delta\mu}(1-\bar{\mu}^2)$$

$$\tau_1 = \Delta r\,(\alpha_1 a_6,\ \alpha_2 a_2,\ \alpha_R^{ff})$$

$$\tau_2 = \Delta r\,(\alpha_1 a_7,\ \alpha_2 a_4,\ 0)$$

$$\tau_3 = \Delta r\sigma_{Th}(1-2\gamma' a_8)$$

$$\tau_4 = \Delta r.\{(\alpha_1 a_7,\ \alpha_2 a_4,\ 0) - 2\sigma_{Th}\gamma' a_9 + B_p\}$$

$$\tau_5 = \Delta r.\sigma_{Th}(p_2' + \gamma' p_1' a_8 + \alpha p_3')$$

$$\tau_6 = \Delta r.\ k'$$

$$\tau_7 = \tau_2 + \tau_3 \tag{74}$$

With the help of (74), (73) can be rewritten as,

$$2(\bar{\mu} I_\rho + \tfrac{1}{6}\Delta\mu\ I_{\rho\mu}) + f_{31}(3f_2 I_\mu + f_{30} I_{\rho\mu})$$

$$- 2\bar{\mu}(3f_2 I_o + f_{30} I_\rho) - \tfrac{1}{3}\Delta\mu(3f_2 I_\mu + f_{30} I_{\rho\mu})$$

$$= \tau_1(B_o + f_2 B_\rho) + \tau_2(B_\nu + f_2 B_{\nu\rho}) + \tau_7(I_\nu + f_2 I_{\nu\rho}) + \tau_5 + \tau_6 A_{\mu\xi\nu} \tag{75}$$

Equation (75) is rewritten as,

$$- 6\bar{\mu} f_2 I_o + 2\bar{\mu}(1-f_{30}) I_\rho + 3f_2(f_{31} - \tfrac{1}{3}\Delta\mu) I_\mu$$

$$- \tau_7 I_\nu + (\tfrac{1}{3}\Delta\mu.\bar{\mu} + f_{30} f_{31} - \tfrac{1}{3}\Delta\mu.f_{30}) I_{\rho\mu} - \tau_7 f_2 I_{\nu\rho}$$

$$= \tau_1(B_o + f_2 B_\rho) + \tau_2(B_\nu + f_2 B_{\nu\rho}) + \tau_5 + \tau_6 A_{\mu\xi\nu} \tag{76}$$

$$\text{Let } -6\bar{\mu} f_2 = f_{32}$$

$$2\bar{\mu}(1-f_{30}) = f_{33}$$

$$3f_2(f_{31} - \tfrac{1}{3}\Delta\mu) = f_{34} \tag{77}$$

$$f_{30}f_{31} + \frac{1}{3}\Delta\mu(\bar{\mu} - f_{30}) = f_{35}$$

$$f_{32}I_o + f_{33}I_\rho + f_{34}I_\mu - \tau_7 I_\nu + f_{35}I_{\rho\mu} - f_{36}I_{\nu\rho}$$

$$= \tau_1 B_o + \tau_1 f_2 B_\rho + \tau_2 B_\nu + \tau_2 f_2 B_{\nu\rho} + \tau_5 + \tau_6 \Lambda_{\mu\xi\nu} \tag{78}$$

The interpolation coefficients be expressed in terms of the nodal values: (See Fig. 1).

$$\begin{bmatrix} I_o \\ I_\rho \\ I_\mu \\ I_\nu \\ I_{\rho\mu} \\ I_{\mu\nu} \\ I_{\nu\rho} \\ I_{\rho\mu\nu} \end{bmatrix} = 1/8 \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \\ I_d \\ I_e \\ I_f \\ I_g \\ I_h \end{bmatrix} \tag{79}$$

or introducing (79) into (78) we obtain,

$$\begin{aligned} &\frac{1}{8} f_{32}(I_a + I_b + I_c + I_d + I_e + I_f + I_g + I_h) + \\ &+ \frac{1}{8} f_{33}(-I_a - I_b - I_c - I_d + I_e + I_f + I_g + I_h) + \\ &+ \frac{1}{8} f_{34}(-I_a + I_b + I_c - I_d - I_e - I_f + I_g + I_h) - \\ &- \frac{1}{8} \tau_7 \ (I_a + I_b - I_c - I_d - I_e + I_f + I_g - I_h) + \\ &+ \frac{1}{8} f_{35}(I_a - I_b - I_c + I_d - I_e - I_f + I_g + I_h) - \\ &- \frac{1}{8} f_{36}(-I_a - I_b + I_c + I_d - I_e + I_f + I_g - I_h) \\ &= \frac{1}{8} \tau_1 \ (B_a + B_b + B_c + B_d + B_e + B_f + B_g + B_h) \\ &+ \frac{1}{8}\tau_1 f_2(-B_a - B_b - B_c - B_d + B_e + B_f + B_g + B_h) \end{aligned} \tag{80}$$

$$+ \frac{1}{8} \tau_2 \, (-B_a - B_b + B_c + B_d + B_e - B_f - B_g + B_h)$$

$$+ \frac{1}{8} \tau_2 f_2 (B_a + B_b - B_c - B_d + B_e - B_f - B_g + B_h)$$

$$+ \tau_5 + \tau_6 A_{\mu\xi\nu}$$

or

$$I_a(f_{32} - f_{33} - f_{34} + f_{35} - f_{36} + \tau_7)$$

$$+ I_b(f_{32} - f_{33} + f_{34} - f_{35} - f_{36} + \tau_7)$$

$$+ I_c(f_{32} - f_{33} + f_{34} - f_{35} + f_{36} - \tau_7)$$

$$+ I_d(f_{32} - f_{33} - f_{34} + f_{35} + f_{36} - \tau_7)$$

$$+ I_e(f_{32} + f_{33} - f_{34} - f_{35} - f_{36} - \tau_7)$$

$$+ I_f(f_{32} + f_{33} - f_{34} - f_{35} + f_{36} + \tau_7)$$

$$+ I_g(f_{32} + f_{33} + f_{34} + f_{35} + f_{36} + \tau_7)$$

$$+ I_h(f_{32} + f_{33} + f_{34} + f_{35} - f_{36} - \tau_7)$$

$$= 8(\tau_5 + \tau_6 A_{\mu\xi\nu}) +$$

$$+ B_a(\tau_1 - \tau_1 f_2 - \tau_2 + \tau_2 f_2) + B_a\{\tau_1(1-f_2) - \tau_2(1-f_2)\} + B_a(\tau_1 - \tau_2)(1-f_2)$$

$$+ B_b(\tau_1 - \tau_1 f_2 - \tau_2 + \tau_2 f_2) + B_b(\tau_1 - \tau_2)(1-f_2)$$

$$+ B_c(\tau_1 - \tau_1 f_2 + \tau_2 - \tau_2 f_2) + B_c\{\tau_1(1-f_2) + \tau_2(1-f_2)\}$$

$$+ B_c(\tau_1 + \tau_2)(1-f_2)$$

$$+ B_d(\tau_1 - \tau_1 f_2 + \tau_2 - \tau_2 f_2) + B_d(\tau_1 + \tau_2)(1-f_2)$$

$$+ B_e(\tau_1 + \tau_1 f_2 + \tau_2 + \tau_2 f_2) + B_e(\tau_1 + \tau_2)(1+f_2)$$

$$+ B_f(\tau_1 + \tau_1 f_2 - \tau_2 - \tau_2 f_2) \to B_f\{\tau_1(1+f_2) - \tau_2(1+f_2)\}$$

$$\to B_f(\tau_1 - \tau_2)(1+f_2)$$

$$+ B_g(\tau_1 + \tau_1 f_2 - \tau_2 - \tau_2 f_2) \to B_g(\tau_1 - \tau_2)(1+f_2)$$

$$+ B_h(\tau_1 + \tau_1 f_2 + \tau_2 - \tau_2 f_2) \to B_h(\tau_1 + \tau_2)(1+f_2) \qquad (81)$$

Let us write

$$F_a = f_{32} - f_{33} - f_{34} + f_{35} - f_{36} + \tau_7$$

$$F_b = f_{32} - f_{33} + f_{34} - f_{35} - f_{36} + \tau_7$$

$$F_c = f_{32} - f_{33} + f_{34} - f_{35} + f_{36} - \tau_7$$

$$F_d = f_{32} - f_{33} - f_{34} + f_{35} + f_{36} - \tau_7$$

$$F_e = f_{32} + f_{33} - f_{34} - f_{35} - f_{36} - \tau_7$$

$$F_f = f_{32} + f_{33} - f_{34} - f_{35} + f_{36} + \tau_7$$

$$F_g = f_{32} + f_{33} + f_{34} + f_{35} + f_{36} + \tau_7$$

$$F_h = f_{32} + f_{33} + f_{34} + f_{35} - f_{36} - \tau_7 \qquad (82)$$

$$T_a = (\tau_1 - \tau_2)(1-f_2)$$

$$T_b = T_a$$

$$T_c = (\tau_1 + \tau_2)(1-f_2)$$

$$T_d = T_b$$

$$T_e = (\tau_1 + \tau_2)(1+f_2)$$

$$T_f = (\tau_1 - \tau_2)(1+f_2)$$

$$T_g = T_f$$

$$T_h = T_e$$

With (82), (81) can be rewritten as,

$$F_a I_a + F_b I_b + F_c I_c + F_d I_d + F_e I_e + F_f I_f + F_g I_g + F_h I_h$$

$$= 8(\tau_5 + \tau_6 A_{\mu\xi\nu}) +$$

$$+ T_a B_a + T_b B_b + T_c B_c + T_d B_d + T_e B_e + T_f B_f + T_g B_g + T_h B_h \qquad (83)$$

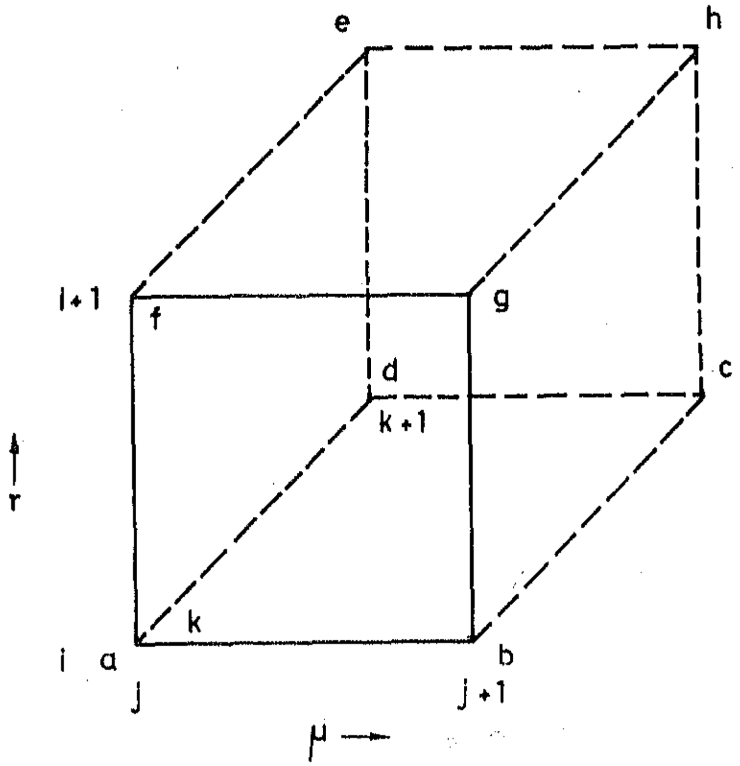

Fig.1 Schematic diagram of the angle-radius-frequency grid.

$$I(r,\mu,\nu) = I_o + \xi I_\rho + \eta I_\mu + \chi I_\nu + \xi\eta I_{\rho\mu}$$

$$+ \eta\chi I_{\mu\nu} + \chi\xi I_{\nu\rho} + \xi\eta\chi I_{\rho\mu\nu}$$

$$\eta = \frac{\mu - \bar{\mu}}{\Delta\mu/2}$$

$$= \frac{-\mu - (-\frac{\mu_j + \mu_{j+1}}{2})}{-(\mu_{j+1} - \mu_j)/2}$$

$$= \frac{\mu-\bar{\mu}}{\Delta\mu/2}$$

We shall now try for the $-\mu$ rays.

$$-\mu \frac{\partial I_\nu^r(-\mu)}{\partial r} - \frac{1-\mu^2}{r}\frac{\partial I_\nu^r(-\mu)}{\partial r} = \sigma_a'(\nu)\ [B_\nu - I_\nu^r(-\mu)]$$

$$- \sigma_{Th}(1-2\gamma)I_\nu^r(-\mu) + \sigma_{Th}\int_{4\pi}[A+B\frac{\partial I_\nu^r(\Omega'')}{\partial r} + C\frac{\partial^2 I_\nu^r(\Omega'')}{\partial \nu^2}\, d\Omega''$$

$$- \frac{3\sigma_{Th}}{16\pi}\ \frac{c^2\gamma}{h\nu^3}\ I_\nu^r(-\mu)(1-\nu\frac{\partial}{\partial\nu})\int(1-\cos\Theta + \cos^2\Theta - \cos^3\Theta)$$

$$I_\nu^r(\Omega'')\ d\Omega'' \qquad (84)$$

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi = (-\mu)(-\mu')+(1-\mu^2)(1-\mu'^2)$$

$$\cos\Theta$$

$$= \mu\mu' + (1-\mu^2)(1-\mu'^2)\cos\varphi$$

From equation (21), for $-\mu$, we have,

$$-\mu\frac{\partial I(r,-\mu,\nu)}{\partial r} = -\frac{2\mu}{\Delta r}(I_\rho + \eta I_{\rho\mu} + \chi I_{\nu\rho} + \eta\chi I_{\rho\mu\nu}) \qquad (85)$$

From equation (22), we have,

$$-\frac{1-\mu^2}{r}\frac{\partial I(r,-\mu,\nu)}{\partial r} = -\frac{1-\mu^2}{r}\frac{2}{\Delta\mu}(I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu}) \quad (86)$$

Therefore equation (23) for $-\mu$ is written as,

$$-\frac{2\mu}{\Delta r}(I_\rho + \eta I_{\rho\mu} + \chi I_{\nu\rho} + \eta\chi I_{\rho\mu\nu}) - \frac{1-\mu^2}{r}\frac{2}{\Delta\mu}(I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu})$$

= same as in equation (23). (87)

Now applying θ_μ on equation (87), we obtain,

$$-\frac{2\bar{\mu}}{\Delta r}(I_\rho + \chi I_{\nu\rho}) - \frac{1}{3}\frac{\Delta\mu}{\Delta r}(I_{\rho\mu} + \chi I_{\rho\mu\nu})$$

$$-\frac{2(1-\bar{\mu}^2)}{r.\Delta\mu}(I_\mu + \xi I_{\rho\mu} + \xi\chi I_{\rho\mu\nu})$$

$$+ 2\frac{\bar{\mu}}{r}(I_o + \xi I_\rho + \chi I_\nu + \chi\xi I_{\nu\rho}) + \frac{1}{3}\frac{\Delta\mu}{r}(I_\mu + \xi I_{\rho\mu} + \chi I_{\mu\nu} + \xi\chi I_{\rho\mu\nu})$$

$$= \sigma_a' B(r,-\mu,\nu) - [\sigma_a'(\nu) + \sigma_{Th}(1-2\gamma)][I_o + \xi I_\rho + \chi I_\nu + \chi\xi I_{\nu\rho}]$$

$$+ 4\pi\sigma_{Th}A + \frac{8\pi\sigma_{Th}B}{\Delta\nu}(I_\nu + \xi I_{\nu\rho})$$

$$- \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3}\theta_\mu[I(r,-\mu,\nu)\ \{PI_o + P\xi I_\rho - qI_\mu - qI_\mu$$

$$- 2P\frac{\bar{\nu}}{\Delta\nu}I_\nu - q\xi I_{\rho\mu} + 2q\frac{\bar{\nu}}{\Delta\nu}I_{\mu\nu} + P(\chi\xi - \frac{2\nu}{\Delta\nu})I_{\nu\rho}$$

$$- q\ (\xi\chi - \frac{2\nu}{\Delta\nu})I_{\rho\mu\nu}\}] \qquad (88)$$

The last term in equation (88) = $\frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3}A_\mu$

When operated by θ_ν, equation (88) becomes,

$$- \frac{2}{\Delta r}[\bar{\mu}(I_\rho + \chi I_{\nu\rho}) + \frac{1}{6}\Delta\mu(I_{\rho\mu} + \chi I_{\rho\mu\nu})]$$

$$\frac{2}{\Delta\mu}(1-\bar{\mu}^2)[(I_\mu + \chi I_{\mu\nu})\frac{1}{2}\frac{\Delta A}{V} + (I_{\rho\mu} + \chi I_{\rho\mu\nu})\frac{1}{\Delta r}(2 - \frac{\bar{r}}{V}\Delta A)]$$

$$+ 2\bar{\mu}\ [(I_o + \chi I_\nu)\frac{1}{2}\frac{\Delta A}{V} + (I_\rho + \chi I_{\nu\rho})\frac{1}{\Delta r}(2 - \frac{\bar{r}}{V}\Delta A)]$$

$$+ \frac{1}{3}\Delta\mu[(I_\mu + \chi I_{\mu\nu})\frac{1}{2}\frac{\Delta A}{V} + (I_{\rho\mu} + \chi I_{\rho\mu\nu})\frac{1}{\Delta r}(2 - \frac{\bar{r}\Delta A}{V})]$$

$$= \sigma_a'(\nu)\ \ (B_o + \chi B_\nu) + \frac{1}{6}\frac{\Delta A}{\bar{A}}(B_\rho + \chi B_{\nu\rho})\}$$

$$+ \Sigma(\nu)\ \{(I_o + \chi I_\nu) + \frac{1}{6}\frac{\Delta A}{\bar{A}}(I_\rho + \chi I_{\nu\rho})\} + 4\pi\sigma_{Th}A$$

$$+ \frac{8\pi\sigma_{Th}}{\Delta\nu}B\ (I_\nu + \frac{1}{6}\frac{\Delta A}{\bar{A}}I_{\nu\rho}) - \frac{3}{16}\sigma_{Th}\frac{c^2\gamma}{h\nu^3}A_{\mu\xi} \qquad (89)$$

Upon using the operator θ_ν on equation (89), we obtain,

$$- \frac{2}{\Delta r}(\bar{\mu}I_\rho + \frac{1}{6}\Delta\mu I_{\rho\mu}) - \frac{2(1-\bar{\mu}^2)}{\Delta\mu}[\frac{1}{2}\frac{\Delta A}{V}I_\mu + \frac{1}{\Delta r}(2 - \frac{\bar{r}\Delta A}{V})I_{\rho\mu}]$$

$$+ 2\bar{\mu}\ [\frac{1}{2}\frac{\Delta A}{V}I_o + \frac{1}{\Delta r}(2 - \frac{\bar{r}\Delta A}{V})I_\rho]$$

$$+ \frac{1}{3} \Delta\mu [\frac{1}{2} \frac{\Delta A}{V} I_\mu + \frac{1}{\Delta r} (2 - \frac{\bar{r}\Delta A}{V}) I_{\rho\mu}]$$

$$= (\alpha_1 a_6, \alpha_2 a_2, \alpha_R^{ff})(B_0 + \frac{1}{6} \frac{\Delta A}{\bar{A}} B_\rho) + (\alpha_1 a_7, \alpha_2 a_4, 0)$$

$$(B_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} B_{\nu\rho})$$

$$+ \{(\alpha_1 a_6, \alpha_2 a_2, \alpha_R^{ff}) + \sigma_{Th}(1 - 2\gamma' a_8)\} (I_0 + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_\rho)$$

$$+ \{(\alpha_1 a_7, \alpha_2 a_4, 0) - 2 \sigma_{Th} \gamma' a_9\}(I_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho})$$

$$+ \sigma_{Th}(P_2' + \gamma' P_1' a_8 + \alpha P_3') + B_p (I_\nu + \frac{1}{6} \frac{\Delta A}{\bar{A}} I_{\nu\rho}) + K' A_{\mu\xi\nu} \quad (90)$$

Equation (90) can be rewritten as follows; similar to equation (75)

$$-2(\bar{\mu} I_\rho + \frac{1}{6} \Delta\mu I_{\rho\mu}) - f_{31}(3 f_2 I_\mu + f_{30} I_{\rho\mu})$$

$$+ 2\bar{\mu}(3 f_2 I_0 + f_{30} I_\rho) + \frac{1}{3}\Delta\mu(3 f_2 I_\mu + f_{30} I_{\rho\mu})$$

$$= \tau_1 (B_0 + f_2 B_\rho) + \tau_2 (B_\nu + f_2 B_{\nu\rho}) + \tau_7 (I_\nu + f_2 I_{\nu\rho})$$

$$+ \tau_5 + \tau_6 A_{\mu\xi\nu} \quad (91)$$

Equation (91) can be rewritten as,

$$6\bar{\mu} f_2 I_0 - 2\bar{\mu}(1 - f_{30}) I_\rho - 3 f_2 (f_{31} - \frac{1}{3} \Delta\mu) I_\mu - \tau_7 I_\nu$$

$$- (\frac{1}{3} \Delta\mu . \bar{\mu} + f_{30} f_{31} - \frac{1}{3}\Delta\mu . f_{30}) I_{\rho\mu} - \tau_7 f_2 I_{\nu\rho}$$

$$= \tau_1 (B_0 + f_2 B_\rho) + \tau_2 (B_\nu + f_2 B_{\nu\rho}) + \tau_5 + \tau_6 A_{\mu\xi\nu} \quad (92)$$

or

$$- f_{32} I_0 - f_{33} I_\rho - f_{34} I_\mu - \tau_7 I_\nu - f_{35} I_{\rho\mu} - f_{36} I_{\nu\rho}$$

$$= \tau_1 B_0 + \tau_1 f_2 B_\rho + \tau_2 B_\nu + \tau_2 f_2 B_{\nu\rho} + \tau_5 + \tau_6 A_{\mu\xi\nu} \quad (93)$$

Introducing (79) into (93), we obtain,

$$
\begin{aligned}
&- \frac{1}{8} f_{32}(I_a + I_b + I_c + I_d + I_e + I_f + I_g + I_h) - \\
&- \frac{1}{8} f_{33}(-I_a - I_b - I_c - I_d + I_e + I_f + I_g + I_h) - \\
&- \frac{1}{8} f_{34}(-I_a + I_b + I_c - I_d - I_e - I_f + I_g + I_h) - \\
&- \frac{1}{8} \tau_7 (I_a + I_b - I_c - I_d - I_e + I_f + I_g - I_h) - \\
&- \frac{1}{8} f_{35}(I_a - I_b - I_c + I_d - I_e - I_f + I_g + I_h) - \\
&- \frac{1}{8} f_{36}(-I_a - I_b + I_c + I_d - I_e + I_f + I_g - I_h) \\
&= \frac{1}{8} \tau_1 (B_a + B_b + B_c + B_d + B_e + B_f + B_g + B_h) + \\
&+ \frac{1}{8} \tau_1 t_2 (-B_a - B_b - B_c - B_d + B_e + B_f + B_g + B_h) + \\
&+ \frac{1}{8} \tau_2 (-B_a - B_b + B_c + B_d + B_e - B_f - B_g + B_h) + \\
&+ \frac{1}{8} \tau_2 f_2 (B_a + B_b - B_c - B_d + B_e - B_f - B_g + B_h) + \\
&+ \tau_5 + \tau_6 A_{\mu\xi\nu}
\end{aligned}
\tag{94}
$$

Collecting the coefficients of the nodal points and rearranging equation (94) would give,

$$
\begin{aligned}
&I_a(-f_{32} + f_{33} + f_{34} - f_{35} + f_{36} - \tau_7) + \\
&+ I_b(-f_{32} + f_{33} - f_{34} + f_{35} + f_{36} - \tau_7) + \\
&+ I_c(-f_{32} + f_{33} - f_{34} + f_{35} - f_{36} + \tau_7) + \\
&+ I_d(-f_{32} + f_{33} + f_{34} - f_{35} - f_{36} + \tau_7) + \\
&+ I_e(-f_{32} - f_{33} + f_{34} + f_{35} + f_{36} + \tau_7) + \\
&+ I_f(-f_{32} - f_{33} + f_{34} + f_{35} - f_{36} - \tau_7) + \\
&+ I_h(-f_{32} - f_{33} - f_{34} - f_{35} + f_{36} + \tau_7) +
\end{aligned}
$$

$$= 8\,(\tau_5 + \tau_6\, A_{\mu\xi\nu})$$

$$f_{11} = a_2 q_\mu,\ f_{10} = a_4 \bar{R} f_3 + 2 f_2 q_\mu \frac{a_1}{\Delta\nu}$$

$$f_9 = a_4 \bar{R} - \frac{a_1}{\Delta\nu} f_2,\ f_8 = a_4 f_2 f_3 + 2 q_\mu \frac{a_1}{\Delta\nu}$$

$$f_7 = a_4 p_{\eta\mu} + 2 a_2 q_\mu \frac{\bar{\nu}}{\Delta\nu}$$

$$f_6 = a_4 f_2 - \frac{a_1}{\Delta\nu};\ f_5 = a_4 \bar{R} - \frac{a_1}{\Delta\nu} f_2$$

$$f_4 = a_4 - 2 a_2 \frac{\bar{\nu}}{\Delta\nu},\ f_3 = p_{\eta\mu} - q_\mu,\ f_2 = \frac{1}{6}\frac{\Delta A}{\bar{A}},\ f_1 = a_2 p_\mu$$

$$g_1 = 2 f_1 f_2,\ g_2 = a_2 f_3,\ g_3 = p_\mu f_4,\ g_4 = f_2 f_3 a_2$$

$$g_5 = 2 p_\mu f_6,\ g_6 = \bar{R} f_1,\ g_7 = a_2 f_2 f_3,\ g_8 = p_\mu f_2 f_4$$

$$g_9 = \bar{R} a_2 f_3,\ g_{10} = f_2 f_7,\ _{11} = 2 p_\mu f_9$$

$$g_{12} = 2 f_2 a_2 q_{\eta\mu},\ g_{13} = q_{\eta\mu} f_4,\ g_{14} = a_4 f_2 f_3 - 2 p_{\eta\mu} \frac{a_1}{\Delta\nu}$$

$$\tau_1 = \Delta r(\alpha_1 a_6,\ \alpha_2 a_2,\ \alpha_R^{ff})$$

$$\tau_2 = \Delta r(\alpha_1 a_7,\ \alpha_2 a_4,\ 0)$$

$$\tau_3 = \Delta r\ \sigma_{Th}(1 - 2\gamma' a_8)$$

$$\tau_4 = \Delta r.\{(\alpha_1 a_7,\ \alpha_2 a_4,\ 0) - 2\sigma_{Th}\gamma' a_9 + B_p\}$$

$$\tau_5 = \Delta r.\sigma_{Th}(p_2' + \gamma' p_1'\ \ a_8 + \alpha p_3')$$

$$\tau_6 = \Delta r.\ k'$$

$$\tau_7 = \tau_2 + \tau_3$$

$$a_1 = \frac{1}{\Delta\nu} \ln\left(\frac{\nu_{k+1}}{\nu_k}\right)$$

$$a_2 = \frac{1}{\nu_k \nu_{k+1}}$$

$$f_{34} = 3f_2(f_{31} - \frac{1}{3}\Delta\mu)$$

$$f_{33} = 2\bar{\mu}\ (1-f_{30})$$

$$f_{32} = -6\bar{\mu}f_2$$

$$f_{31} = \frac{2}{\Delta\mu}(1-\bar{\mu}^2)$$

$$f_{30} = 2 - \frac{\bar{r}}{\Delta r}\ \frac{\Delta A}{\bar{A}}$$

$$f_{29} = q_{\eta\mu}\ f_{26},\quad f_{28} = f_3 f_{26},\quad f_{27} = p_\mu\ f_{26}$$

$$f_{26} = (a_5\bar{R} - 2f_2\ \frac{a_3}{\Delta\nu})$$

$$f_{25} = q_{\eta\mu}\{f_2(a_5 + 2a_4\ \frac{\bar{\nu}}{\Delta\nu}) + \frac{2a_3}{\Delta\nu}\}$$

$$f_{24} = f_2(a_5 p_{\eta\mu} + 2a_4\ \frac{\bar{\nu}}{\Delta\nu}) - 2p_{\eta\mu}\ \frac{a_3}{\Delta\nu}$$

$$f_{23} = 2a_4 q_{\eta\mu}\ \frac{\bar{\nu}}{\Delta\nu},\quad f_{22} = 2q_{\eta\mu}\ f_3$$

$$f_{21} = a_4\bar{R}f_3 - 2f_2 p_{\eta\mu}\ \frac{a_1}{\Delta\nu}$$

$$f_{20} = f_2 q_{\eta\mu}(2\ \frac{\bar{\nu}}{\Delta\nu} - a_4),\quad f_{19} = a_2\ \bar{R}\ q_{\eta\mu}$$

$$f_{18} = f_2(a_5 q_\mu + 2a_4 p_{\eta\mu}\ \frac{\bar{\nu}}{\Delta\nu}) - 2q_\mu\ \frac{a_3}{\Delta\nu}$$

$$f_{17} = p_\mu\{f_2(a_6 - 2a_4\ \frac{\bar{\nu}}{\Delta\nu}) - \frac{2a_3}{\Delta\nu}\}$$

$$f_{16} = -2a_4\ \frac{\bar{\nu}}{\Delta\nu}\ f_3$$

$$f_{15} = f_2(a_4 q_\mu + 2a_2 p_{\eta\mu}),\quad f_{14} = 2a_4 p_\mu\ \frac{\bar{\nu}}{\Delta\nu}$$

$$f_{13} = 2q_{\eta\mu}\ (f_2 a_4 - \frac{a_1}{\Delta\nu})$$

$$f_{12} = a_4 q_\mu + 2a_2 p_{\eta\mu}\ \frac{\bar{\nu}}{\Delta\nu}$$

$$a_3 = \frac{2}{(\Delta\nu)^2} \{\Delta\nu - \ln (\frac{\nu_{k+1}}{\nu_k})\}$$

$$a_4 = \frac{2}{(\Delta\nu)^2} \{\ln (\frac{\nu_{k+1}}{\nu_k}) - \frac{\bar{\nu}.\Delta\nu}{\nu_k \ \nu_{k+1}}\}$$

$$a_5 = \frac{4}{(\Delta\nu)^3} \{\Delta\nu - 2\bar{\nu} \ \ln(\frac{\nu_{k+1}}{\nu_k}) + \frac{(\bar{\nu})^2}{\nu_k \ \nu_{k+1}}\}$$

$$a_6 = \frac{\bar{\nu}}{\nu_k^2 \ \nu_{k+1}^2}$$

$$a_7 = \frac{2}{\Delta\nu} \{\frac{1}{\nu_k \ \nu_{k+1}} - (\frac{\bar{\nu}}{\nu_k \ \nu_{k+1}})^2\}$$

$$a_8 = \bar{\nu}$$

$$a_9 = \frac{2}{\Delta\nu} \{\frac{1}{3} (\nu_{k+1}^2 + \nu_{k+1} \ \nu_k + \nu_k^2) - (\bar{\nu})^2\}$$

$$B_1 = 1 - \frac{6}{5} p_1 + \frac{1}{2} p_2 - \frac{3}{10} p_3$$

Where p_1 , p_2 and p_3 are Legendre Polynomials.

$$B_2 = B_1 \{\frac{1}{3}\gamma' \ \{(\nu_{k+1}^2 + \nu_{k+1} \ \nu_k + \nu_k^2) - 2\alpha\bar{\nu}\}$$

$$B_p = 2 \frac{\sigma_{Th}}{\Delta\nu} B_1 B_2, \quad \gamma' = \frac{h}{m_o c^2} = 8.1019 \times 10^{-21} \text{ sec.}$$

$$p_1' = \frac{6}{5} p_1 - \frac{1}{2} p_2 + \frac{3}{10} p_3 - 1$$

$$p_2' = 1 + \frac{1}{2} p_2$$

$$p_3' = - \frac{6}{5} p_1 - 3p_2 + \frac{6}{5} p_3$$

$$\alpha_1 = 3.7 \times 10^8 \ T^{-\frac{1}{2}} \ Z^2 n_e n_i \bar{g}_{ff}$$

$$\alpha_2 = 1.8 \times 10^{-2} \ T^{-3/2} \ Z^2 n_e n_i \bar{g}_{ff}$$

$$\alpha_R^{ff} = 1.7 \times 10^{-25} \ T^{-7/2} \ Z^2 n_e n_i \bar{g}_R$$

$$\alpha = \frac{kT}{m_o c^2} = 1.687343 \times 10^{-10} T$$

$$K' = \frac{3}{16} \frac{\sigma_{Th}}{m_o} = 136.9257$$

$$+ B_a(\tau_1 - \tau_2)(1-f_2)$$

$$+ B_b(\tau_1 - \tau_2)(1-f_2)$$

$$+ B_c(\tau_1 + \tau_2)(1-f_2)$$

$$+ B_d(\tau_1 + \tau_2)(1-f_2)$$

$$+ B_e(\tau_1 + \tau_2)(1+f_2)$$

$$+ B_f(\tau_1 - \tau_2)(1+f_2)$$

$$+ B_g(\tau_1 - \tau_2)(1+f_2)$$

$$+ B_h(\tau_1 + \tau_2)(1+f_2) \qquad (95)$$

Equation (95) is rewritten in terms of equations (82) we thus obtain,

$$- F_a I_a - F_b I_b - F_c I_c - F_d I_d - F_e I_e - F_f I_f - F_g I_g - F_h I_h$$

$$= 8(\tau_5 + \tau_6 A_{\mu\xi\nu})$$

$$+ T_a B_a + T_b B_b + T_c B_c + T_d B_d + T_e B_e + T_f B_f + T_g B_g + T_h B_h \qquad (96)$$

Equations (83) and (96) can be written for both the directions as follows

$$F_a I_a^+ + F_b I_b^+ + F_c I_c^+ + F_d I_d^+ + F_e I_e^+ + F_f I_f^+ + F_g I_g^+ + F_h I_h^+$$

$$= 8(\tau_5 + \tau_6 A_{\mu\xi\nu}^+)$$

$$+ T_a B_a^+ + T_b B_b^+ + T_c B_c^+ + T_d B_d^+ + T_e B_e^+ + T_f B_f^+ + T_g B_g^+ + T_h B_h^+ \qquad (97)$$

and,

$$- F_a I_a^- - F_b I_b^- - F_c I_c^- - F_d I_d^- - F_e I_e^- - F_f I_f^- - F_g I_g^- - F_h I_h^-$$

$$= 8(\tau_5 + \tau_6 A_{\mu\xi\nu}^-)$$

$$+ T_a B_a^- + T_b B_b^- + T_c B_c^- + T_d B_d^- + T_e B_e^- + T_f B_f^- + T_g B_g^- + T_h B_h^- \qquad (98)$$

From the notation given in Figure 1 we have

$$I_a \quad I^l_{j\ k} \quad I_b^+ \quad I^l_{j\ 1\ k} \quad I_c \quad I^l_{j\ 1\ k\ 1} \quad I_d \quad I^l_{j\ k\ 1}$$

$$I_e \quad I^{l\ l}_{j\ k\ 1} \quad I_f \quad I^{l\ l}_{j\ k} \quad I_g \quad I^{l\ l}_{j\ 1\ k} \quad I_h \quad I^{l\ l\ +}_{j\ 1\ k\ 1} \qquad (99)$$

and

$$I_a \quad I^l_{j\ k} \quad I_b \quad I^l_{j\ 1\ k} \quad I_c \quad I^l_{j\ 1\ k+1} \quad I_d \quad I^l_{j\ k\ 1}$$

$$I_e \quad I^{l\ l}_{j\ k\ 1} \quad I_f \quad I^{l\ l}_{j\ k} \quad I_g \quad I^{l\ l}_{j\ 1\ k} \quad I_h \quad I^{l\ l}_{j\ 1\ k\ 1} \qquad (100)$$

Similarly the source terms are also written Thus

$$B_a \quad B^{l\ +}_{j\ k} \quad B_b \quad B^l_{j\ 1\ k} \quad B_c \quad B^l_{j\ 1\ k\ 1} \quad B_d \quad B^l_{j\ k+1}$$

$$B_e \quad B^{l\ l}_{j\ k\ 1} \quad B_f \quad B^{l\ l}_{j\ k} \quad B_g \quad B^{l\ l}_{j\ 1\ k} \quad B_h^+ \quad B^{l\ l\ +}_{j\ 1\ k\ 1} \qquad (101)$$

and

$$B_a \quad B^l_{j\ k} \quad B_b \quad B^l_{j\ 1\ k} \quad B_c \quad B^l_{j\ 1\ k+1} \quad B_d \quad B^l_{j\ k\ 1}$$

$$B_e \quad B^{l\ l}_{j\ k\ 1} \quad B_f \quad B^{l+1}_{j\ k} \quad B_g \quad B^{l\ l}_{j\ 1\ k} \quad B_h \quad B^{l\ l}_{j\ 1\ k\ 1} \qquad (102)$$

Equations (97) and (98) can be rewritten using equations (99 102) as follows:

$$F_a I^l_{j\ k} \quad F_b I^l_{j\ 1\ k} \quad F_c I^l_{j+1\ k\ 1} \quad F_d I^{l\ +}_{j\ k\ 1} \quad F\ I^{l\ l}_{j\ k+1}$$

$$F_f I^{l\ l}_{j\ k} \quad F_g I^{l+1}_{j+1\ k} + F_h I^{l\ l}_{j\ 1\ k\ 1} \quad \delta(\tau_s \quad \tau\ A_{\mu\xi\nu})$$

$$T_a B^l_{j\ k} + T_b B^{l\ +}_{j\ 1\ k} \quad T_c B^l_{j\ 1\ k\ 1} \quad T_d B^l_{j\ k+1} + T_e B^{l\ l}_{j\ k\ 1}$$

$$T_f B^{l\ l\ +}_{j\ k} + T_g B^{l\ l}_{j\ 1\ k} + T_h B^{l\ l}_{j\ 1\ k+1} \qquad (103)$$

and

$$F_a I^l_{j\ k} \quad F_b I^l_{j+1\ k} \quad F_c I^l_{j+1\ k\ 1} \quad F_d I^l_{j\ k+1} \quad F_e I^{l+1}_{j\ k\ 1}$$

$$F_f I^{l\ l}_{j\ k} \quad F_g I^{l\ l}_{j+1\ k} \quad F_h I^{l+1}_{j\ 1\ k\ 1} \quad \delta(\tau_s \quad \tau_s A_{\mu\xi\nu})$$

$$+ T_a B^{i,-}_{j,k} + T_b B^{i,-}_{j+1,k} + T_c B^{i,-}_{j+1,k+1} + T_d B^{i,-}_{j,k+1} + T_e B^{i+1,-}_{j,k+1}$$

$$+ T_f B^{i+1,-}_{j,k} + T_g B^{i+1,-}_{j+1,k} + T_h B^{i+1,-}_{j+1,k+1} \qquad (104)$$

Equations (103) and (104) can be written as follows:

$$[F_a^{j,k} \; I^{i,+}_{j,k} + F_b^{j+1,k} \; I^{i,+}_{j+1,k}]_k$$

$$+ [F_d^{j,k+1} \; I^{i,+}_{j,k+1} + F_c^{j+1,k+1} \; I^{i,+}_{j+1,k+1}]_{k+1}$$

$$+ [F_f^{j,k} \; I^{i+1,+}_{j,k} + F_g^{j+1,k} \; I^{i+1,+}_{j+1,k}]_k$$

$$+ [F_e^{j,k+1} \; I^{i+1,+}_{j,k+1} + F_h^{j+1,k+1} \; I^{i+1,+}_{j+1,k+1}]_{k+1}$$

$$= 8(\tau_5 + \tau_6 A^+_{\mu\xi\nu})$$

$$+ [T_a^j \; B^{i,+}_{j,k} + T_b^{j+1} \; B^{i,+}_{j+1,k}]_k +$$

$$[T_d^j \; B^{i,+}_{j,k+1} + T_c^{j+1} \; B^{i,+}_{j+1,k+1}]_{k+1}$$

$$+ [T_f^j \; B^{i+1,+}_{j,k} + T_g^{j+1} \; B^{i+1,+}_{j+1,k}]_k +$$

$$[T_e^j \; B^{i+1,+}_{j,k+1} + T_h^{j+1} \; B^{i+1,+}_{j+1,k+1}]_{k+1} \qquad (105)$$

and

$$- [F_a^{j,k} \; I^{i,-}_{j,k} + F_b^{j+1,k} \; I^{i,-}_{j+1,k}]_k - [F_d^{j,k+1} \; I^{i,-}_{j,k+1} + F_c^{j+1,k+1} \; I^{i,-}_{j+1,k+1}]_{k+1}$$

$$- [F_f^{j,k} \; I^{i+1,-}_{j,k} + F_g^{j+1,k} \; I^{i+1,-}_{j+1,k}]_k -$$

$$[F_e^{j,k+1} \; I^{i+1,-}_{j,k+1} + F_h^{j+1,k+1} \; I^{i+1,-}_{j+1,k+1}]_{k+1}.$$

$$= 8(\tau_5 + \tau_6 A^-_{\mu\xi\nu})$$

$$+ [T_a^j \; B^{i,-}_{j,k} + T_b^{j+1} \; B^{i,-}_{j+1,k}]_k + [T_d^j \; B^{i,-}_{j,k+1} + T_c^{j+1} \; B^{i,-}_{j+1,k+1}]_{k+1}$$

$$+ [T_f^j B_{j,k}^{i+1,-} + T_g^{j+1} B_{j+1,k}^{i+1,-}]_k +$$

$$[T_e^j B_{j,k+1}^{i+1,-} + T_h^{j+1} B_{j+1,k+1}^{i+1,-}]_{k+1} \quad (106$$

s group all the coefficients:

$$T_a = (\tau_1 - \tau_2)(1-f_2),\ T_b = T_a,\ T_c = (\tau_1 + \tau_2)(1-f_2),$$

$$T_d = T_b,\ T_e = (\tau_1 + \tau_2)(1+f_2),\ T_f = (\tau_1 - \tau_2)(1+f_2),$$

$$T_g = T_f,\ T_h = T_e$$

$$F_a = f_{32} - f_{33} - f_{34} + f_{35} - f_{36} + \tau_7$$

$$F_b = f_{32} - f_{33} + f_{34} - f_{35} - f_{36} + \tau_7$$

$$F_c = f_{32} - f_{33} + f_{34} - f_{35} + f_{36} - \tau_7$$

$$F_d = f_{32} - f_{33} - f_{34} + f_{35} + f_{36} - \tau_7$$

$$F_e = f_{32} + f_{33} - f_{34} - f_{35} - f_{36} - \tau_7$$

$$F_f = f_{32} + f_{33} - f_{34} - f_{35} + f_{36} + \tau_7$$

$$F_g = f_{32} + f_{33} + f_{34} + f_{35} + f_{36} + \tau_7$$

$$F_h = f_{32} + f_{33} + f_{34} + f_{35} - f_{36} - \tau_7$$

$$f_{36} = \tau_7 f_2,\ f_{35} = f_{30} f_{31} + \frac{1}{3} \Delta\mu(\bar{\mu} - f_{30})$$

$$r_3 = r_{i+1}^2 + r_{i+1} r_i + r_i^2$$

$$r_4 = r_{i+1}^2 + r_i^2$$

$$r_5 = r_{i+1}^4 + r_{i+1}^3 r_i + r_{i+1}^2 r_i^2 + r_{i+1} r_i^3 + r_i^4$$

$$\bar{R} = \frac{16\pi}{V\ \Delta r} \{\frac{1}{5} r_5 - (\bar{r})^2 r_4 + \frac{1}{3} (\bar{r}) r_3\}$$

$$p_\mu = \frac{2}{3}\,(1+2\bar{\mu}^2)$$

$$q_\mu = 4\,\frac{\bar{\mu}}{\Delta\mu}\,\{\frac{2}{5} + \frac{1}{3}\,(1+\bar{\mu}^2)\}$$

$$p_{\eta\mu} = \frac{4}{9}\,\bar{\mu}\,\Delta\mu$$

$$q_{\eta\mu} = 8\,(\frac{\bar{\mu}}{\Delta\mu})^2\,\{\frac{7}{5} + \mu_j^2 + \mu_{j+1}^2 + \frac{1}{30}\,(\frac{\Delta\mu}{\bar{\mu}})^2 - q_\mu$$

$$\mu^2 = \frac{1}{3}\,(\mu_{j+1}^2 + \mu_{j+1}\,\mu_j + \mu_j^2)$$

Equation (105) can be rewritten as,

$$\underset{\sim}{\Gamma}{}_k^{ab}\,\underset{\sim}{I}{}_k^{i,+} + \underset{\sim}{\Gamma}{}_{k+1}^{dc}\,\underset{\sim}{I}{}_{k+1}^{i,+} + \underset{\sim}{F}{}_k^{fg}\,\underset{\sim}{I}{}_k^{i+1,+} + \underset{\sim}{F}{}_{k+1}^{eh}\,\underset{\sim}{I}{}_{k+1}^{i+1,+}$$

$$= 8\,(\tau_5 + \tau_6 A_{\mu\xi\nu}^{+})\,\underset{\sim}{H}$$

$$+ \underset{\sim}{T}{}_k^{ab}\,\underset{\sim}{B}{}_k^{i,+} + \underset{\sim}{T}{}_{k+1}^{dc}\,\underset{\sim}{B}{}_{k+1}^{i,+} + \underset{\sim}{T}{}_k^{fg}\,\underset{\sim}{B}{}_k^{i+1,+} + \underset{\sim}{T}{}_{k+1}^{eh}\,\underset{\sim}{B}{}_{k+1}^{i+1,+} \qquad (107)$$

Similarly equation (106) is written as,

$$- F_k^{ab}\,\underset{\sim}{I}{}_k^{i,-} - F_{k+1}^{dc}\,I_{k+1}^{i,-} - \underset{\sim}{F}{}_k^{fg}\,\underset{\sim}{I}{}_k^{i+1,-} - \underset{\sim}{F}{}_{k+1}^{eh}\,\underset{\sim}{I}{}_{k+1}^{i+1,-}$$

$$= 8\,(\tau_5 + \tau_6 A_{\mu\xi\nu}^{-})\underset{\sim}{H}$$

$$+ \underset{\sim}{T}{}_k^{ab}\,\underset{\sim}{B}{}_k^{i,-} + \underset{\sim}{T}{}_{k+1}^{dc}\,\underset{\sim}{B}{}_{k+1}^{i,-} + \underset{\sim}{T}{}_k^{fg}\,\underset{\sim}{B}{}_k^{i+1,-} + \underset{\sim}{T}{}_{k+1}^{eh}\,\underset{\sim}{B}{}_{k+1}^{i+1,-} \qquad (108)$$

Where we have for J angles,

$$\underset{\sim}{F}{}_k^{ab} = \begin{bmatrix} F_a^{j,k} & F_b^{j+1,k} & & & \\ & F_a^{j+1,k} & F_b^{j+2,k} & & \\ & & \ddots & \ddots & \\ & & & F_a^{J-1,k} & F_b^{J,k} \\ & & & & F_a^{J,k} \end{bmatrix} \qquad (109)$$

$$\underset{\sim}{F}^{dc}_{k+1} = \begin{bmatrix} F_d^{j,k+1} & F_c^{j+1,k+1} & & \\ & F_d^{j+1,k+1} & F_c^{j+2,k+1} & \\ & \ddots & \ddots & \\ & & F_d^{J-1,k+1} & F_c^{J,k+1} \\ & & & F_d^{J,k+1} \end{bmatrix} \tag{110}$$

$$\underset{\sim}{F}^{fg}_{k} = \begin{bmatrix} F_f^{j,k} & F^{j+1,k} & & \\ & F_f^{j+1,k} & F_g^{j+2,k} & \\ & \ddots & \ddots & \\ & & F_f^{J-1,k} & F_g^{J,k} \\ & & & F_f^{J,k} \end{bmatrix} \tag{111}$$

$$\underset{\sim}{F}^{eh}_{k+1} = \begin{bmatrix} F_e^{j,k+1} & F_h^{j+1,k+1} & & \\ & F_e^{j+1,k+1} & F_h^{j+2,k+1} & \\ & \ddots & \ddots & \\ & & F_e^{J-1,k+1} & F_h^{J,k+1} \\ & & & F_e^{J,k+1} \end{bmatrix} \tag{112}$$

$$\underset{\sim}{T}^{ab}_{k} = \begin{bmatrix} T_a^{j,k} & T_b^{j+1,k} & & \\ & T_a^{j+1,k} & T_b^{j+2,k} & \\ & \ddots & \ddots & \\ & & T_a^{J-1,k} & T_b^{J,k} \\ & & & T_a^{J,k} \end{bmatrix} \tag{113}$$

$$\underset{\sim}{T}^{dc}_{k+1} = \begin{bmatrix} T_d^{j,k+1} & T_c^{j+1,k+1} & & \\ & T_d^{j+1,k+1} & T_c^{j+2,k+1} & \\ & \ddots & \ddots & \\ & & T_d^{J-1,k+1} & T_c^{J,k+1} \\ & & & T_d^{J,k+1} \end{bmatrix} \tag{114}$$

$$\underset{\sim}{T}^{fg}_{k} = \begin{bmatrix} T_f^{j,k} & T_g^{j+1,k} & & \\ & T_f^{j+1,k} & T_g^{j+2,k} & \\ & \ddots & \ddots & \\ & & T_f^{J-1,k} & T_g^{J,k} \\ & & & T_f^{J,k} \end{bmatrix} \tag{115}$$

$$\underset{\sim}{T}^{eh}_{k+1} = \begin{bmatrix} T_e^{j,k+1} & T_h^{j+1,k+1} & & \\ & T_e^{j+1,k+1} & T_h^{j+2,k+1} & \\ & \ddots & \ddots & \\ & & T_e^{J-1,k+1} & T_h^{J,k+1} \\ & & & T_e^{J,k+1} \end{bmatrix} \tag{116}$$

$$\underset{\sim}{H} = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \tag{117}$$

We shall incorporate all K frequency points

Therefo e equations (107) and (108) are rew itten as

$$\underset{\sim}{F}^{ab}_{dc}\ \underset{\sim}{I}_{1}\quad \underset{\sim}{F}^{fg}_{eh}\ I_{1\ 1}\quad 8\ \underset{\sim}{Q}(\tau\qquad \tau\ A_{\mu\xi\nu}) + \underset{\sim}{T}^{ab}_{dc}\ \underset{\sim}{B}_{1}\quad \underset{\sim}{T}^{fg}_{eh}\ \underset{\sim}{B}_{1\ 1} \tag{118}$$

$$\underset{\sim}{F}^{ab}_{dc}\ I_{1}\quad \underset{\sim}{F}^{fg}_{eh}\ I_{1\ 1}\quad 8\ \underset{\sim}{Q}(\tau\qquad \tau\ A_{\mu\xi\nu})\quad T^{ab}_{dc}\ \underset{\sim}{B}_{1}\quad \underset{\sim}{T}^{fg}_{eh}\ \underset{\sim}{B}_{1\ 1} \tag{119}$$

Wh re

$$\underset{\sim}{F}^{ab}_{dc}\quad \begin{bmatrix} \underset{\sim}{F}^{ab}_{k} & \underset{\sim}{F}^{dc}_{k\ 1} & & \\ & \underset{\sim}{F}^{ab}_{k\ 1} & \underset{\sim}{F}^{dc}_{k\ 2} & \\ & & F^{ab}_{K\ 1} & F^{dc}_{K} \\ & & & F^{ab}_{K} \end{bmatrix} \tag{120}$$

$$\underset{\sim}{F}^{fg}_{eh}\quad \begin{bmatrix} \underset{\sim}{F}^{fg}_{k} & \underset{\sim}{F}^{eh}_{k\ 1} & & \\ & \underset{\sim}{F}^{fg}_{k\ 1} & \underset{\sim}{F}^{eh}_{k\ 2} & \\ & & \underset{\sim}{F}^{fg}_{K\ 1} & \underset{\sim}{F}^{eh}_{K} \\ & & & \underset{\sim}{F}^{fg}_{K} \end{bmatrix} \tag{121}$$

$$\underset{\sim}{Q}\quad \begin{bmatrix} \underset{\sim}{H} & \underset{\sim}{H} & & \\ & \underset{\sim}{H} & \underset{\sim}{H} & \\ & & \underset{\sim}{H} & \underset{\sim}{H} \\ & & & \underset{\sim}{H} \end{bmatrix} \tag{122}$$

Equations (118) and (119) are written in the form of Interaction Principle.

$$\underset{\sim}{F}^{fg}_{eh}\, I^{+}_{l+1} = -\underset{\sim}{F}^{ab}_{dc}\, I^{+}_{l} + 8\, \underset{\sim}{Q}(\tau_5 + \tau_6 A^{+}_{\mu\xi\nu}) + \underset{\sim}{T}^{ab}_{dc}\, B^{+}_{l} + \underset{\sim}{T}^{fg}_{eh}\, B^{+}_{l+1} \tag{123}$$

$$-\underset{\sim}{F}^{ab}_{dc}\, I^{-}_{l} = \underset{\sim}{F}^{fg}_{eh}\, I^{-}_{l+1} + 8\, \underset{\sim}{Q}(\tau_5 + \tau_6 A^{-}_{\mu\xi\nu}) + \underset{\sim}{T}^{ab}_{dc}\, B^{-}_{l} + \underset{\sim}{T}^{fg}_{eh}\, B^{-}_{l+1} \tag{124}$$

If we write that

$$\hat{\underset{\sim}{F}}^{ab}_{dc} = \underset{\sim}{Q}^{-1}\, \underset{\sim}{F}^{ab}_{dc},\quad \hat{\underset{\sim}{F}}^{fg}_{eh} = \underset{\sim}{Q}\, \underset{\sim}{F}^{fg}_{eh} \tag{125}$$

$$\hat{\underset{\sim}{T}}^{ab}_{dc} = \underset{\sim}{Q}^{-1}\, \underset{\sim}{T}^{ab}_{dc},\quad \hat{\underset{\sim}{T}}^{fg}_{eh} = \underset{\sim}{Q}\, \underset{\sim}{T}^{fg}_{eh} \tag{126}$$

and

$$\dot{\underset{\sim}{E}} = \begin{bmatrix} \underset{\sim}{E} & \\ & \underset{\sim}{E} \end{bmatrix} \tag{127}$$

$\underset{\sim}{E}$ is the identity matrix.

Then equations (123) and (124) are rewritten as,

$$\begin{bmatrix} \hat{\underset{\sim}{F}}^{fg}_{eh} & 0 \\ 0 & -\hat{\underset{\sim}{F}}^{ab}_{dc} \end{bmatrix} \begin{bmatrix} \underset{\sim}{I}^{+}_{l+1} \\ \underset{\sim}{I}^{-}_{l} \end{bmatrix} = \begin{bmatrix} -\hat{\underset{\sim}{F}}^{ab}_{dc} & 0 \\ 0 & \hat{\underset{\sim}{F}}^{fg}_{eh} \end{bmatrix} \begin{bmatrix} \underset{\sim}{I}^{+}_{l} \\ \underset{\sim}{I}^{-}_{l+1} \end{bmatrix} + \hat{T}^{ab}_{dc}\, \dot{E} \begin{bmatrix} B^{+}_{l} \\ B^{-}_{l} \end{bmatrix} + \hat{T}^{fg}_{eh}\, \dot{E} \begin{bmatrix} B^{+}_{l+1} \\ B^{-}_{l+1} \end{bmatrix} + 8 \begin{bmatrix} \tau_5 + \tau_6 A^{+}_{\mu\xi\nu} \\ \tau_5 + \tau_6 A^{-}_{\mu\xi\nu} \end{bmatrix} \tag{128}$$

or,

$$\begin{bmatrix} \underset{\sim}{I}^{+}_{l+1} \\ \underset{\sim}{I}^{-}_{l} \end{bmatrix} = \underset{\sim}{K}^{-1} \begin{bmatrix} -\hat{\underset{\sim}{F}}^{ab}_{dc} & 0 \\ 0 & \hat{\underset{\sim}{F}}^{fg}_{eh} \end{bmatrix} \begin{bmatrix} I^{+}_{l} \\ I^{-}_{l+1} \end{bmatrix} + \underset{\sim}{K}^{-1}\, \hat{\underset{\sim}{T}}^{ab}_{dc}\, \dot{\underset{\sim}{E}} \begin{bmatrix} \underset{\sim}{B}^{+}_{l} \\ \underset{\sim}{B}^{-}_{l} \end{bmatrix}$$

$$+ \underset{\sim}{K}^{-1} \bar{\underset{\sim}{T}}^{fg}_{eh} \bar{E} \begin{bmatrix} \underset{\sim}{B}^{+}_{i+1} \\ B^{-}_{i+1} \end{bmatrix} + 8 \underset{\sim}{K}^{-1} \begin{bmatrix} \tau_5 + \tau_6 A^{+}_{\mu\xi\nu} \\ \tau_5 + \tau_6 A^{-}_{\mu\xi\nu} \end{bmatrix} \tag{129}$$

$$\underset{\sim}{K}^{-1} = \begin{bmatrix} (\bar{\underset{\sim}{F}}^{fg}_{eh})^{-1} & 0 \\ 0 & -(\bar{\underset{\sim}{F}}^{ab}_{dc})^{-1} \end{bmatrix} \tag{130}$$

From equations (129) and (130), we obtain the transmission and reflection operators.

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