THE SECULAR INSTABILITY OF SLIGHTLY VISCOUS MAGNETIC MACLAURIN SPHEROIDS

(Letter to the Editor)

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Abstract. We show that the instability induced by viscosity, at the point of bifurcation where the Jacobi ellipsoids branch off from the sequence of Maclaurin spheroids, is not inhibited by the presence of a magnetic field. It has already been shown that a toroidal magnetic field leaves the point of bifurcation unaffected, whereas a magnetic field along the axis of rotation pushes the point of bifurcation to eccentricities higher than the value that obtains in the absence of a magnetic field.

The recent discovery of a pulsar with a period as short as 1.56 ms (Backer *et al.*, 1982) has aroused great interest in the properties of rotating bodies close to the limits of their stability. Up to a certain critical value of the angular momentum, the only permissible equilibrium figure is a Maclaurin spheroid. At this critical value – known as the point of bifurcation – nonaxisymmetric Jacobi ellipsoids also become possible equilibrium configurations; so that, for higher values of the angular momentum, both Maclaurin and Jacobi figures can coexist. The point of bifurcation, or the point of onset of secular instability, occurs at $e = 0.8127$. The Maclaurin sequence continues to be dynamically stable till $e = 0.9529$ where it becomes unstable (Chandrasekhar, 1969, EFE).

The effect of a magnetic field on the stability of differentially rotating, inviscid Maclaurin spheroids has been investigated. A toroidal magnetic field leaves the point of bifurcation unaffected whereas an axial field shifts it to higher values of eccentricity. Any magnetic field, if sufficiently strong, can suppress the dynamical instability (Kochhar and Trehan, 1971; 1973 = Paper I). However, even if a small amount of viscosity is present, the nonmagnetic Maclaurin-sequence becomes dynamically unstable beyond the point of bifurcation. In this note we investigate whether a magnetic field can inhibit the instability due to viscosity.

We consider a homogenous self-gravitating mass rotating with an angular velocity $\Omega(x)$ about the 3-axis and having a general axisymmetric magnetic field which vanishes at the surface. We shall confine ourselves to the low Reynolds number approximation, in which the effects arising from viscous dissipation are considered as small perturbations on the inviscid flow.

The equilibrium configuration is governed by the condition (Paper I).

$$2(T_{11} - M_{11} + M_{33}) = W_{33} - W_{11}, \tag{1}$$

where the symbols have their usual meaning.

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The perturbations of the spheroid about the equilibrium position are described by a Lagrangian displacement of the form $\xi_i(x)\, e^{\lambda t}$, where ξ_i is written as

$$\xi_i = \sum_{j=1}^{3} X_{ij} x_j \,. \tag{2}$$

Of the nine-second harmonic modes, only the toroidal modes are responsible for the secular and dynamical instabilities. These involve $X_{11} - X_{22}$ and $X_{12} + X_{21}$ and are related by (EFE, p. 98; Kochhar and Trehan, 1971; Paper I).

$$\left\{\left[\lambda^2 + Q + \frac{4M_{33}}{I_{11}}\right] + \frac{10\lambda v}{a_1^2}\right\}(X_{11} + X_{22}) + 2\lambda \langle \Omega \rangle (X_{12} + X_{21}) = 0\,, \tag{3}$$

$$\left\{\lambda^2 + Q + \frac{4M_{33}}{I_{11}} + \frac{10\lambda v}{a_1^2}\right\}(X_{12} + X_{21}) - 2\lambda \langle \Omega \rangle (X_{11} - X_{22}) = 0\,. \tag{4}$$

In writing these two equations, equilibrium condition (1) has been used and we have set (cf. Paper I)

$$Q = \frac{2}{I_{11}} (W_{12;\,12} + W_{11} - W_{33})\,. \tag{5}$$

This pair of equations leads to the characteristic equation

$$\left[\lambda^2 + Q + \frac{4M_{33}}{I_{11}} + \frac{10\lambda v}{a_1^2}\right]^2 + 4\lambda^2 \langle \Omega \rangle^2 = 0\,. \tag{6}$$

Writing $\lambda = i\sigma$, we can factorize Equation (6) to get

$$\sigma^2 - 2\sigma \langle \Omega \rangle - Q - \frac{4M_{33}}{I_{11}} - \frac{10\lambda v}{a_1^2} = 0\,. \tag{7}$$

Since we are working in the low Reynolds number approximation we may write

$$\sigma = \sigma_0 + v\Delta\sigma + 0(v^2)\,, \tag{8}$$

where σ_0 is the characteristic frequency in the inviscid limit. With this substitution, from Equation (6) we obtain

$$\Delta\sigma = i\, \frac{5\sigma_0}{a_1^2(\sigma_0 - \Omega)}\,. \tag{9}$$

In the absence of viscosity, we have for the mode corresponding to the point of bifurcation (Paper I)

$$\sigma_0 = \langle \Omega \rangle - \left[\langle \Omega \rangle^2 + Q + \frac{M_{33}}{I_{11}}\right]^{1/2}. \tag{10}$$

Equation (10) brings out the effect of the axial magnetic field on the onset of secular instability, $Q = 0$ has a root at $e = 0.8127$. If $M_{33} = 0$, that is in case the magnetic field is absent or only a toroidal field is present, the point of bifurcation is unaffected. However, an axial magnetic field ($M_{33} \neq 0$) pushes the point of bifurcation to higher values of e. Substituting Equation (10) in Equation (9), we obtain for the unstable mode

$$i\nu\Delta\sigma = -\frac{5\nu}{a_1^2}\frac{[\langle\Omega\rangle^2 + Q + (4M_{33/I_{11}})]^{1/2} - \langle\Omega\rangle}{[\langle\Omega\rangle^2 + Q + (4M_{33}/I_{11})]^{1/2}}\,. \tag{11}$$

Instability due to viscosity would occur if the right-hand side of Equation (11) is positive. Now $[Q + (4M_{33}/I_{11})]$ has a zero at the point of bifurcation, beyond which it becomes negative (Paper I), making the term under the radical sign less than $\langle\Omega\rangle^2$. Thus, for values of eccentricity beyond the point of bifurcation, $i\nu\Delta\sigma$ would become positive and instability sets in. Consequently, even the slightest viscosity will induce instability beyond the bifurcation point, just as in the nonmagnetic case. The only effect a magnetic field along the axis of rotation has is to shift the point of bifurcation to $e > 0.8127$.

References

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