NONLINEAR CURVATURE LAGRANGIANS AND EXTENDED INFLATION IN THE EARLY UNIVERSE

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Abstract. Theories with Lagrangians nonlinear in the curvature scalar can be reduced by appropriate transformations to a form similar to the Brans–Dicke model. This may enable us to have extended inflation in the early universe without the difficulties of introducing scalar fields.

1. Introduction

The inflationary picture while providing an attractive possible solution to the problems confronting the earliest epochs of the standard Big-Bang model (such as the horizon and flatness problems) has run into serious difficulties. Moreover, it introduces massive scalar fields with very special properties and fine-tuned arbitrary small couplings. The masses of the scalar particles can become divergent due to radiative corrections (these difficulties are not there for fermions). As a partial solution to lessen some of these difficulties, recently the so-called extended and hyperextended inflationary models were proposed (Steinhardt and Accetta, 1991).

These models invoke versions of the Brans–Dicke theory of gravity as the appropriate theory for the very early universe, rather than the standard general relativity which is used in the usual inflation formulations. Also the scalar field ϕ is non-minimum coupled to the curvature scalar R. In the initial extended inflation picture only quadratic couplings like $\eta\phi^2 R$ were considered. However, higher-order couplings are expected as well. Again as ϕ is coupled to the curvature its expectation value would contribute corrections to the effective gravitational constant in the early universe as

$$G_{\text{eff}} \approx [f(\phi)]^{-1}, \tag{1}$$

where $f(\phi)$ can be a general interaction $f(\phi)R$ of the form

$$f(\phi) \approx \phi_0^2 + \eta\phi^2 + \eta(\phi^4/\phi_0^2) + \text{higher powers}\ldots, \tag{2}$$

i.e., expansion in small ϕ, for $\phi \ll \phi_0$ the VEV of ϕ_0 corresponding to a mass or energy of $\approx M_{\text{Pl}}$, the Planck mass.

The corresponding Lagrangian was of the form

$$L = -f(\phi)R + \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu + L_{\text{matter}}\,. \tag{3}$$

It was indicated in the above work that the higher-order non-minimum couplings enhance the inflationary scenario and enable its comparison for a wide range of initial

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parameters. Such corrections to Einstein gravity arise in most unified theories coupling gravity to particles physics like superstring models. The analysis is simpler to handle if the model is recast in a form similar to the Brans–Dicke (B–D) theory. If $\phi = f(\phi)$, we can write L as

$$L = -\phi R + \frac{\omega(\phi)}{\phi} \partial_\mu \phi \, \partial^\mu \phi + \cdots + L_{\mathrm{matter}} ; \tag{4}$$

and $\omega(\phi)$ can be written as $\omega(\phi) = f/(f')^2$, where $f' = \mathrm{d}f/\mathrm{d}\phi$. The net inflationary factor in the scale factor $R(t)$ can be expressed as a ratio of $[\phi(t_f)/\phi(t_i)]^{1/kn}$, where t_f and t_i correspond to the start and the end of the inflation and k is a numerical factor ~ 1. The higher-order couplings are found to have the desirable effect of slowing down the drastic expansion.

However, the above picture still invokes a scalar field with its attendant difficulties, independent of its use for inflation. In the next section we shall consider a Lagrangian solely involving a nonlinear function of the curvature scalar R, i.e., of the form $L_R \sim f(R)$. Such corrections to the usual Hilbert action of Einstein gravity (which is linear in R) involving higher powers of the curvature also arise in almost all models involving unification of gravity with particle physics. By suitable transformation L_R can be brought to a form resembling B–D. We shall consider special forms of L_R as appropriate to describe extended inflation in the early universe.

2. Theory with Nonlinear Curvature Lagrangian

If we consider in general a Lagrangian

$$L \sim f(R) , \tag{5}$$

the field equations are of the form (Barrow and Ottewill, 1983; Sivaram, 1979)

$$f'(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R) + \tfrac{1}{2} g_{\alpha\beta}(Rf' - f) - \nabla_\alpha \nabla_\beta f' + g_{\alpha\beta} \Box f' = 0 , \tag{6}$$

where

$$\Box = g_{\alpha\beta} \nabla^\alpha \nabla^\beta . \tag{7}$$

By use of the conformal transformation

$$g'_{\alpha\beta} = \lambda^2 g_{\alpha\beta} \tag{8}$$

or

$$\lambda^2 = [f'(R)]^{1/2} , \tag{9}$$

so that $f'(R) = \lambda^4$, Equations (6) and (7) take the form

$$R'_{\alpha\beta} - \tfrac{1}{2} f_{\alpha\beta} R = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R + \frac{1}{6f'} \nabla_\alpha f' \nabla_\beta f' - \frac{1}{12 f^2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu f' - \frac{1}{f'} \nabla_\alpha \nabla_\beta f' + \frac{1}{f'} g_{\alpha\beta} \Box f' . \tag{10}$$

Moreover, if we define

$$\phi = \tfrac{1}{2} \ln [f'(R)] , \tag{11}$$

Equation (10) becomes

$$R'_{\alpha\beta} - \tfrac{1}{2} g'_{\alpha\beta} R = \tfrac{3}{2} \nabla_\alpha \phi \nabla_\beta \phi - \tfrac{3}{4} g'_{\alpha\beta} (\nabla_\mu \phi \nabla^\mu \phi) - \tfrac{1}{2} g'_{\alpha\beta} (f')^{-2} (Rf' - f) . \tag{12}$$

The effective gravitational constant in Equations (10) and (12) is defined in terms of the derivative of the function f of the curvature scalar as

$$G_{\text{eff}} \sim (f'(R))^{-2} . \tag{13}$$

Since R and ϕ are related through Equation (11), it is possible to express G_{eff} in the form given by Equation (1).

We note that in Equation (6), the function $f(R)$ enters explicitly up to its third derivative. The trace of Equation (6) is

$$f'R - 2f + 3f'' \Box R + 3f''' R_{;\mu} R^{;\mu} = 0 . \tag{14}$$

As indicated in Equations (6) and (12) and elaborated below, constant curvature solutions are obtained for the case of

$$f'R - f = 0 . \tag{15}$$

One can linearize the trace equation around a constant curvature $R = R_0$ solution by putting

$$R = R_0 + \psi .$$

This gives for $f''(R_0) \neq 0$ (Schmidt, 1987; Sivaram, 1987)

$$\Box \psi + m^2 \psi = 0 , \tag{16}$$

with

$$m = \{\tfrac{1}{3} [R_0 - f'(R_0)/f''(R_0)]\}^{1/2} . \tag{17}$$

Equations (16) and (17) show the formal analogy with a theory of a massive scalar field.

3. Equivalence between Theories with $f(R)$ and Ordinary GR with Massive Self-Interacting Non-Minimally Coupled Scalar Fields

To illustrate this equivalence we consider a theory with

$$L \sim f(R) ,$$

where

$$f(R) = R + R^2 + L_{\text{matter}} ; \tag{18}$$

with transformations

$$g'_{mn} = (1 + 2R) g_{mn} \tag{19}$$

and

$$\phi = a \ln(1 + 2R) , \tag{20}$$

a is numerical constant. We have

$$L \sim \{R(g') - (\nabla'\phi) - \tfrac{1}{4}[1 - \exp(\phi/a)^2]\} , \tag{21}$$

with the corresponding field equations

$$G'_{mn} = \nabla'_m\phi\nabla'_n\phi - \tfrac{1}{2}(\nabla'\phi)^2 g'_{mn} - V(\phi)g_{mn} - T_{mn} . \tag{22}$$

In Equation (22), $V(\phi)$ is given by

$$V(y) \approx \tfrac{1}{4}(1 - \exp(\phi/a)^2) , \tag{23}$$

which can be expanded to a form similar to Equation (2), i.e., in powers of ϕ, containing non-minimum terms like ϕ^2, ϕ^4, etc. The constant a would fix the scale of the VEV of ϕ_0. With this expansion, we would have a formal similarity between Equations (3) and (21). Again by defining $\phi = f(\phi)$, we can bring L into the form of Equation (4) which is of the Brans–Dicke form with $\omega(\phi)$ again given by

$$\omega(\phi) = f/(f')^2 . \tag{24}$$

In the general case of $L \sim f(R)$ and the field equations given by Equations (6) and (12), the effective potential $V(\phi)$ is given by

$$V(\phi) = \tfrac{1}{2}(f')^{-2}(Rf' - f) . \tag{25}$$

We do not consider d dimensions here, but the formalism can be easily extended to this case.

Thus the formal equivalence between theories with a nonlinear Lagrangian to a theory with non-minimum massive self-interacting scalar fields coupled to curvature (a generalization of B–D theory) makes these theories suitable as a basis for extended inflation in the early universe.

The net inflationary factor can again be expressed as the ratio of $(\phi(t_f)/\phi(t_i))^{a/k}$.

Or as $R(t_f)/R(t_i) = e^{\omega_f/\omega_i}$, ω defined in terms of Equation (24), ω_f, ω_i values of ω at the beginning and end of inflation.

The evolution is of the type

$$H^2 = \frac{8\pi\rho}{3\phi} - \frac{\dot\phi}{\phi} H + \frac{\omega}{6}\left(\frac{\dot\phi}{\phi}\right)^2 . \tag{26}$$

In general case, $R(t)$ grows as a power law of the form

$$R(t) = R_i(1 + H_i t/\omega^2)^{\omega\alpha} , \tag{27}$$

where H_i is the initial value of H.

The density perturbations $\partial\rho/\rho$ so as $\sim H^2/\dot\phi$.

4. Inflation with Inhomogeneous Metrics

In this type of theories one can also get inflationary type behaviour even in the case of an inhomogeneous or anisotropic metric as the starting point.

We can consider a metric of the form

$$ds^2 = -dt^2 + A^2(r, t)\,dr^2 + B^2(r, t)\,d\Omega^2 , \tag{28}$$

where

$$d\Omega^2 = d\vartheta^2 + d\phi^2 \sin^2\vartheta .$$

The Ricci curvature scalar for this metric is

$$R = 2\left[\frac{\ddot{A}}{A} + 2\,\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + 2\,\frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2}\right] - \\ - \frac{4}{A^2}\left(2\,\frac{B''}{B} + \frac{B'^2}{B^2} - 2\,\frac{A'B'}{AB}\right). \tag{29}$$

As an illustration consider the Lagrangian density

$$L \approx R + \alpha R^2 + L_m ;$$

L_m corresponding to a fluid with

$$T_{\mu\nu} = (\rho + p)u^\mu u^\nu + p g_{\mu\nu} , \tag{30}$$

$$p = (\Gamma - 1)\rho . \tag{31}$$

Γ is the adiabatic index. This would give the field equations

$$(1 + 2\alpha R)\left[2\,\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2}\right] - \left(2\,\frac{B''}{B} + \frac{B'^2}{B^2} - 2\,\frac{A'B'}{AB}\right)\Big/ A^2 = \\ = 2\alpha\left(\frac{R^2}{4} + \Box R\right) + 2\alpha R_{;tt} - \rho , \tag{32}$$

$$(1 + 2\alpha R)\left(2\,\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} - \frac{B'}{B^2A^2}\right) = \\ = 2\alpha\left(\frac{R^2}{4} + \Box R\right) - 2\alpha R_{;rr} - p ; \tag{33}$$

$$(1 + 2\alpha R)\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \left(\frac{B''}{B} - \frac{A'B'}{AB}\right)\Big/ A^2\right] = \\ = 2\alpha\left(\frac{R^2}{4} + \Box R\right) - 2\alpha R_{;\vartheta\vartheta} - p ; \tag{34}$$

$$2(1 + 2\alpha R)\left(\frac{\dot{A}B'}{AB} - \frac{\dot{B}'}{B}\right) = 2\alpha R_{;tr} .$$

One can look for solutions of the form

$$A(r, t) = A_r(r)A_t(t)\,, \qquad B(r, t) = B_r(r)B_t(t)\,. \tag{35}$$

We can consider the case $\Gamma = 0$, i.e., for $p = -\rho$.

For this case, by use of the relations (Berkin, 1990; Mijič *et al.*, 1986) we obtain

$$\ddot{R} + 3H\dot{R} + \frac{R}{6\alpha} = (4 - 3\gamma)\,, \tag{36}$$

$$\dot{R} = \frac{R^2}{12H} - HR - \frac{H}{2\alpha} - \frac{K}{6\alpha H A_t^2} - \frac{KR}{3HA_t^2} + \frac{\rho}{6\alpha H}\,. \tag{37}$$

We can find the time part of the solution as

$$\begin{aligned} A_t &= A_0 \exp(\sqrt{\rho/3}\,t) && \text{for} \quad K = 0\,, \\ A_t &= \sqrt{K/\rho}\,\sinh(\sqrt{\rho/3}\,t + A_0) && \text{for} \quad K > 0\,, \\ A_t &= (-K/\rho)^{1/2}\cosh(\sqrt{\rho/3}\,t + A_0) && \text{for} \quad K < 0\,; \end{aligned} \tag{38}$$

A_0 is an integration constant (see also Sivaram *et al.*, 1974, 1975). The solutions for $K \neq 0$ also tends to the exponential form for large $t \gg t_{\text{initial}}$. We thus find inflationary solutions evne in the inhomogeneous non-flat metrics. The metric (28) can be shown to take the form

$$ds^2 = -dt^2 - A_t^2[A_r^2\,dr^2 + B_r^2\,d\Omega^2]\,,$$

which can be put in the generalized R–W form for arbitrary K. For the Lagrangian dominated by cubic curvature terms, the theory is equivalent to chaotic inflation dominated by ϕ^4-type of model (Sivaram, 1987; Schmidt, 1987).

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