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Self Organization in Quasi Two-Dimensional Hydrodynamic Flows

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Abstract

The cascading characteristics of the invariants of 2D and 3D hydrodynamically turbulent media are well known. Analogous to the 2D case, here we set up a variational principle for the 3D case by invoking the differential decay rates of its invariants, the total energy and the mean square helicity density. Analytical solution of the equation so obtained for the velocity field is presented for a special case.

1. Introduction

There is a considerable experimental and observational evidence that self-organization of flow into large structures is not precluded by turbulence and strong nonlinearities. Self-organization is a process by which nonlinear interactions between small fluid elements result in the formation of large ordered coherent structures.

Hasegawa [1] has described the general features of a system capable of exhibiting self-organization. The system is described by a nonlinear partial differential equation with dissipation. The system has two or more quadratic or higher order conserved quantities in the absence of dissipation. If the spectral behaviour of these invariants in the inertial range is such that one of them transfers towards large spatial scales and the other to small spatial scales, they would have differential dissipation rates when the dissipation is introduced.

The formation of large scale structures is a consequence of an inverse cascade of energy. Kraichnan [2, 3] suggested that the formation of large-scale structures in two-dimensional incompressible turbulence is due to an inverse cascade driven by negative viscosity instabilities.

The self-organization processes in two-dimensional hydrodynamic turbulence has been studied by Rhines [4], Bretherton and Haidvogel [5], Hasegawa [6] and by Woltjer [7], Taylor [8], Montgomery *et al.* [9], Mathaeus and Montgomery [10] in magnetohydrodynamic turbulence. An excellent review on self-organization process in continuous media has been presented by Hasegawa [1].

The importance of mean square helicity as an invariant and its role in inverse cascade in three-dimensional turbulent fluids has been brought out by Levich and Tzvetkov [11] in the context of the Earth's atmosphere. Recently Frisch *et al.* [12] and Sulem *et al.* [13] discussed the generation of large scale structures in three-dimensional flows lacking parity-invariance, in terms of kinetic α effect.

A model of solar granulation through inverse cascade has been presented by Krishan [14]. Granules are cellular velocity patterns observed on the solar surface, believed to be manifestations of convective phenomenon occurring in the subphotospheric layers. The formation of these cellular patterns on all scales has been interpreted to be the result of self-organization processes occurring in the turbulent medium of the solar atmosphere. The presence of very large structures in the Universe like the Great Wall indicate the existence of a hierarchy of well-ordered coherent formations up to the very large scales of a few hundred Mpc (Huchra and Geller [15]). Krishan and Sivaram [16] have explained the clustering of galaxies on several scales by inverse cascade in a turbulent medium.

In this paper, we set up a variational principle connecting the invariants E (energy) and I (mean square helicity density) for a three-dimensional incompressible fluid following the arguments given in Hasegawa [1]. The resulting variational equation is highly nonlinear and hence closed form solutions are hard to obtain. However, we present a closed form solution of the equation under some simplifying but physically realizable situations.

2. Derivation of the variational equation

Large helicity fluctuations present in a turbulent medium play an important role in the inverse cascading processes. The helicity density γ, a measure of the knottedness of the vorticity field, is defined as $\gamma = \bar{\mathbf{V}} \cdot \bar{\omega}$, $\bar{\omega} = \nabla \times \bar{\mathbf{V}}$, where $\bar{\mathbf{V}}$ and $\bar{\omega}$ are the velocity and vorticity, respectively. The quantity I defined as

$$I = \int \langle \gamma(x)\gamma(x + r) \rangle \, d^3x, \tag{1}$$

where $\langle \, \rangle$ denote an average over an ensemble, is an invariant of an ideal 3D hydrodynamic system in addition to the total energy E. By assuming a quasi normal distribution of helicities, the invariant I can be expressed as

$$I = C_1 \int [E(k)]^2 \, dk, \tag{2}$$

where C_1 is a constant and $E = \int E(k) \, dk$ is the total energy density.

The inertial range for the energy invariant can be shown to be

$$E(k) \propto k^{-5/3} \quad \text{and} \quad E \propto L^{2/3}, \tag{3}$$

and for the I invariant to be

$$E(k) \propto k^{-1} \quad \text{and} \quad E \propto \log L(t)/l, \tag{4}$$

where $L(t)$ is the largest length scale excited at time t (Levich and Tzvetkov [11]). Hasegawa [1] formulated a variational principle using the two invariants, the energy and entrophy of a 2D system. Along the same lines, we set up a variational equation using the invariants "I" and "E" of a 3D system to get:

$$\delta \int (\bar{V} \cdot \bar{\omega})^2 \, d^3x - \lambda\delta \int V^2 \, d^3x = 0. \tag{5}$$

$$\int (\bar{V} \cdot \bar{\omega})[\delta\bar{V} \cdot \bar{\omega} + \bar{V} \cdot \delta\bar{\omega}] \, d^3x - \lambda \int \bar{V} \cdot \delta\bar{V} \, d^3x = 0. \tag{6}$$

Here λ is the Lagrange multiplier.

It can be shown that

$$\bar{V} \cdot \delta\bar{\omega} = \delta\bar{V} \cdot \bar{\omega} + \nabla \cdot [\delta\bar{V} \times \bar{V}].$$

Equation (6) then becomes:

$$\int (\bar{V} \cdot \bar{\omega}) 2[\delta\bar{V} \cdot \bar{\omega} - \lambda\bar{V}/2(\bar{V} \cdot \bar{\omega})] \, d^3x + \int (\bar{V} \cdot \bar{\omega})[\nabla \cdot (\delta\bar{V} \times \bar{V})] \, d^3x = 0. \tag{7}$$

Manipulating the second integral, eq. (7) can be written as

$$\int 2(\bar{V} \cdot \bar{\omega})\delta\bar{V} \cdot \left[\bar{\omega} - \frac{\lambda\bar{V}}{2(\bar{V} \cdot \bar{\omega})} - \frac{\bar{V} \times \nabla(\bar{V} \cdot \bar{\omega})}{2(\bar{V} \cdot \bar{\omega})}\right] d^3x + \int_S (\bar{V} \cdot \bar{\omega})(\delta\bar{V} \times \bar{V}) \cdot ds = 0. \tag{8}$$

Applying the boundary condition that $\bar{\omega} \cdot \bar{n}$ vanishes on the boundary S, eq. (8) reduces to

$$2\bar{\omega}(\bar{V} \cdot \bar{\omega}) - \lambda\bar{V} - \bar{V} \times \nabla(\bar{V} \cdot \bar{\omega}) = 0. \tag{9}$$

The corresponding equation for a two-dimensional system with entropy and energy as its invariants is

$$\nabla \times \nabla \times \bar{V} - \alpha\bar{V} = 0. \tag{10}$$

This is a linear equation whose solution can be written down immediately. However, eq. (9) is highly nonlinear and more difficult to solve.

The self-organized state described by eq. (9) should be a stationary solution of the Navier–Stokes equation (without dissipation and gravity)

$$\frac{\partial\bar{V}}{\partial t} + (\bar{V} \cdot \nabla)\bar{V} = -\frac{\nabla p}{\rho} \tag{11}$$

$$\nabla \cdot \bar{V} = 0. \tag{12}$$

3. Solution of the variational equation for a special case

3.1. *Quasi two-dimensional case*

The largest dimension of fully 3D structures is given by the ratio $I/E^2 = L = L_z$, where L_z is the characteristic vertical scale. When the correlation length of helicity fluctuations reaches the limit L_z, it can grow only in the horizontal plane. Another consequence of the growth of the correlation length is that the velocity and vorticity become aligned, which reduces the nonlinear term $(\bar{V} \cdot \nabla)\bar{V} = \nabla \times (\bar{V} \times \bar{\omega})$ of the Navier–Stokes equation and thus retards the flow of energy to small spatial scales. With the growth of correlation length only in the horizontal plane, the system becomes more and more anisotropic. In these circumstances, the vertical component of velocity V_z becomes independent of (x, y, z) and the horizontal components V_x and V_y independent of z, leading to $\omega_{x,y} = (\nabla \times \bar{V})_{x,y} = 0$. The invariant I becomes

$$I = \int \langle (V_z\omega_z)^2 \rangle \, dx \, dy \, dz$$

$$L_z\langle V_z^2 \rangle k^2 V_k^2 k^{-2} \propto V_k^2 = kE(k) \propto L^{2/3}, \tag{13}$$

and from $I = \int I(k) \, dk$, one finds

$$I(k) \propto k^{-5/3} \tag{14}$$

Thus $I(k)$ spectrum for the quasi 2D case, coincides with the energy spectrum of 2D turbulence $E(k) \propto k^{-5/3}$, corresponding to the inverse cascade.

We assume the velocity field to be

$$\bar{V} = \bar{V}[V_x(x, y), V_y(x, y), V_z]$$

where V_z is a constant. The variational equation can be recast

$$\left(\nabla^2 + \frac{\lambda}{V_z^2}\right)\bar{V}_H = 0 \tag{16}$$

where $\bar{V}_H = (V_x V_y)$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial x^2$.

The solution satisfying eqs (12) and (16) is given by

$$V_x = \hat{V} \cos 2\pi X \cos 2\pi Y$$

$$V_y = \hat{V}(b/a) \sin 2\pi X \sin 2\pi Y \tag{17}$$

Here $\hat{V}$, a, b are constants. X and Y are given by $X = x/a$, $Y = y/b$.

The equation for the vorticity $\bar{\omega}$ can be derived by taking the curl of eq. (11) with dissipatation as

$$\frac{\partial\bar{\omega}}{\partial t} + (\bar{V} \cdot \nabla)\bar{\omega} = (\bar{\omega} \cdot \nabla)\bar{V} + \nu\nabla^2\bar{\omega} \tag{18}$$

For a quasi 2D situation, $\bar{V}$ and $\bar{\omega}$ can be expressed by a scalar stream function as

$$\bar{V} = -\nabla\psi \times \hat{z} + V_z\hat{z}$$

$$\bar{\omega} = \nabla^2\psi\hat{z} \tag{19}$$

Here $\hat{z}$ is the unit vector along the z direction. In terms of ψ, eq. (18) can be written as

$$\frac{\partial}{\partial t}\nabla^2\psi\hat{z} + \{-\nabla\psi \times \hat{z} + V_z\hat{z}\} \cdot \nabla(\nabla^2\psi\hat{z}) - \nu\nabla^4\psi = 0. \tag{20}$$

An exact solution of the above equation can be written as

$$\psi(X, Y, t) = \hat{\psi}[1 + \exp\{-\nu\hat{\lambda}t\}] \cos 2\pi X \sin 2\pi Y. \tag{21}$$

Here $\hat{\psi}$, a, b are constants. $\hat{\lambda} = \lambda/V_z^2$. $V_x V_y$ and ω can be written as

$$V_x = \hat{V}[1 + \exp\{-\nu\hat{\lambda}t\}] \cos 2\pi X \cos 2\pi Y$$

$$V_y = \hat{V}(b/a)[1 + \exp\{-\nu\hat{\lambda}t\}] \sin 2\pi X \sin 2\pi Y$$

$$\omega = \hat{\omega}[1 + \exp\{-\nu\hat{\lambda}t\}] \cos 2\pi X \sin 2\pi Y. \tag{22}$$

In the limit $\nu \to 0$, the solution (22) represents the self-organized solution (17) of the variational equation for the quasi two-dimensional case.

Figure 1 presents the 3D plot of the velocity $V = (V_x^2 + V_y^2 + V_z^2)^{1/2}$, corresponding to the solution (17) for $(\hat{V}/V_z) = 0.7$ and $b/a = 0.6$ as a function of X and Y. The maxima and minima of the velocity are attained at the boundaries, the maximum values at the corners of the cells,

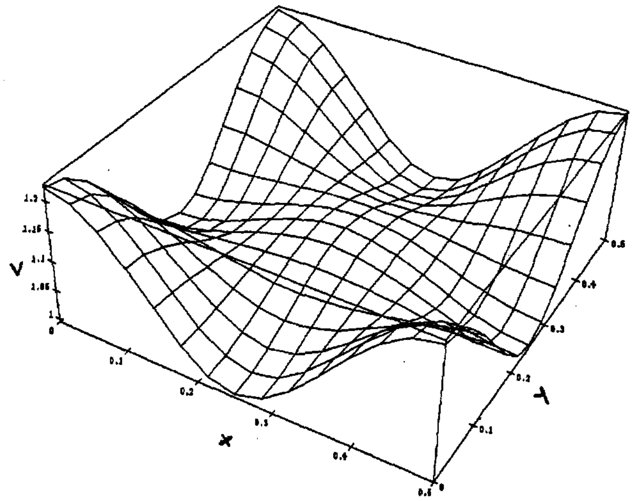

Fig. 1. 3D plot of the velocity profile corresponding to the solution (17)

while the minimum are at the mid points of the boundaries of the cell.

4. Conclusions

In conclusion we have shown that though the variational equation for a fully 3D system does not admit closed form solutions easily, owing to the highly nonlinear behaviour, it does exhibit analytical solutions for a simplified though physically realizable situation.

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