GENERAL RELATIVITY OF COMPACT OBJECTS

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1. Introduction

This paper is aimed at illustrating some general relativistic effects concerning one class of compact objects : neutron stars. These are associated with (a) pulsars, believed to be rotating neutron stars and (b) compact X-ray objects, some of which are neutron stars in close binary orbits with more ordinary stars. Pulsar physics, which is two decades old, has seen a recent revival of interest with the discovery (Backer et al. 1982) of a new pulsar, PSR 1937+214, having an astonishingly large rate of rotation (about 640 rotations per second around the axis, corresponding to an observed period P = 1.5578 ms). Since then, four more such rapidly rotating pulsars have (so far) been discovered. The extremely large rotation rates of the neutron stars associated with fast pulsars suggest that besides the large spacetime curvature, general relativistic effects of rotation will be important. Specifically, the aspects of interest are the structure and photon trajectories near the surface (and the consequent implications for the redshift factor and other emission characteristics). Also important in this connection are general relativistic instabilities and gravitational radiation, but these are discussed by other authors in this volume. General relativistic aspects of neutron stars have been reviewed earlier (see e.g. Iyer and Vishveshwara 1985) using the core-envelope model. Here, we emphasize on the general relativistic effects of rotation, namely dragging of inertial frames which will affect the structure as well radiation characteristics of pulsars.

2. The Structure

A first estimate of rotation induced deformation in the structure of fast pulsars can be obtained using Hartle's prescription (Hartle 1967), according to which the spherical deformation terms are the leading correction terms, proportional to Ω^2 (Ω = angular velocity) for a fixed central density. The fractional change in gravitational mass ($\delta M/M$) and radius ($\delta R/R$) are given in terms of radial distributions of the mass and pressure perturbation factors $m_o(r)$ and $P_o(r)$ (Hartle 1967):

$$\delta M = \frac{c^2}{G} m_o(R) + \frac{G J^2}{c^4 R^3} \tag{1}$$

$$\delta R = - \left. \frac{P_o(\rho c^2 + P)}{dP/dr} \right|_{r=R} \tag{2}$$

where J is the angular momentum of the star, and $P(r)$ and $\rho(r)$ the pressure and total mass-energy density at point r. The nonrotating mass M and radius R are obtained by integrating the relativistic equations for hydrostatic equilibrium (see e.g., Arnett & Bowers 1977).

The Hartle prescription is relativistic, but is usually called a 'slow' rotation approximation because it is valid for Ω slow in comparison to the critical $\Omega_c=(GM/R^3)^{1/2}$. Models of rapidly rotating relativistic stars have been calculated based on incompressible fluids and polytropic equations of state, and such formalism has been adopted by Friedman et al. (1986) for neutron stars.

3. Rotational Instability in a Fluid Star

Uniformly rotating self-gravitating bodies with axially symmetric configuration and uniform density (Maclaurin spheroids) become secularly unstable beyond a certain $\Omega = \Omega_S$ (Tassoul 1978):

$$\Omega_S^2 / (2\pi G \bar{\rho}) = 0.18 \tag{3}$$

($\bar{\rho}$ = mean density). Neutron stars are relativistic configurations.. But because their density profiles are remarkably flat, they may be presumed to resemble the Maclaurin sequence in order to serve as an approximate guide to determine the rotational instabilities. Eq.(3) considered for the fast pulsar PSR 1937+214 (making the plausible assumption that it rotates close to Ω_S) implies $\bar{\rho}$ = 2.4×10^{14} g cm^{-3}, which therefore may be taken as the minimum average density for fast pulsars to be rotationally stable. This criterion thus allows one to estimate lower limits on M and moment of inertia (I) and upper limit on R for fast pulsars.

Eq.(3) is a particular case (m = 2) of secular instability to non-axisymmetric modes with angular dependence $e^{im\phi}$ (ϕ = azimuthal angle coordinate). Friedman (1983) suggests that the m = 3 or the m = 4 mode is more likely to set the rotational instability point, and the limiting angular velocity Ω_S can be in the range (0.55 - 0.75) Ω_c instead of 0.52 Ω_c corresponding to Eq.(3). However, if neutron star interiors possess large bulk viscosity (which will be the case if there is a significant concentration of hyperons), then because viscosity is expected to damp out a gravitational wave driven instability (that arises due to non-axisymmetric perturbation modes), the above conclusion regarding the m = 3 or 4 mode may not be valid. In any case, Eq.(3) provides a first estimate of the rotational constraint on the structure.

4. Bending of Light Rays in a Rotating Spacetime

In a rotating spacetime the trajectory of a photon will be different from that given by the usual light bending formula in general relativity. The net angle of deflection $|\phi_B|$ will be made up of bending in the azimuthal direction (ϕ) and the polar direction (θ):

$$|\varphi_0| = \int_{r_e}^{D} dr\, \dot{\phi}/\dot{r} + \int_{r_e}^{D} dr\, \dot{\theta}/\dot{r} \qquad (4)$$

where a dot represents differentiation with respect to an affine parameter, r_e is the initial radial location of the photon and D the radial location of a remote observer. The Hartle metric

$$\begin{aligned} ds^2 &= g_{\alpha\beta}\, dx^\alpha dx^\beta \\ &= e^{2\nu} c^2 dt^2 - e^{2\psi}(d\phi - \omega dt)^2 \\ &\quad - e^{2\mu} d\theta^2 - e^{2\lambda} dr^2 \end{aligned} \qquad (5)$$

(here ν, ψ, μ, λ are functions of r) refers to a rotationally perturbed spacetime; so, we may write

$$|\varphi_0| \simeq \int_{r_e}^{D} dr\, \dot{\phi}/\dot{r} \qquad (6)$$

Eq.(6) is conveniently evaluated in terms of the photon impact parameter (q) which is given by (Kapoor & Datta 1984):

$$q = \frac{e^{\psi-\nu}(v_s + \sin\delta)}{1 + e^{\psi-\nu}(\omega v_s + \Omega \sin\delta)}, \qquad (7)$$

$$v_s = e^{\psi-\nu}(\Omega - \omega) \qquad (8)$$

Then,

$$|\varphi_o| = \int_{r_e}^{D} dr \, \frac{\omega(1+q_e) - q_e e^{2(\nu-\psi)}}{e^{\nu-\lambda}\left\{(1+\omega q_e)^2 - q_e^2 e^{2(\nu-\psi)}\right\}^{1/2}} , \quad (9)$$

$$q_e = q\,(r = r_e,\ \theta = \theta_e,\ \phi = \delta) \quad (10)$$

From above it follows that rotation induced inertial frame drag effect will alter the trajectory of even a radially emitted photon (i.e. with $\delta = 0$), unlike in the case of nonrotational (Schwarzschild) spacetime. A schematic illustration of the bending effect is shown in Fig.1.

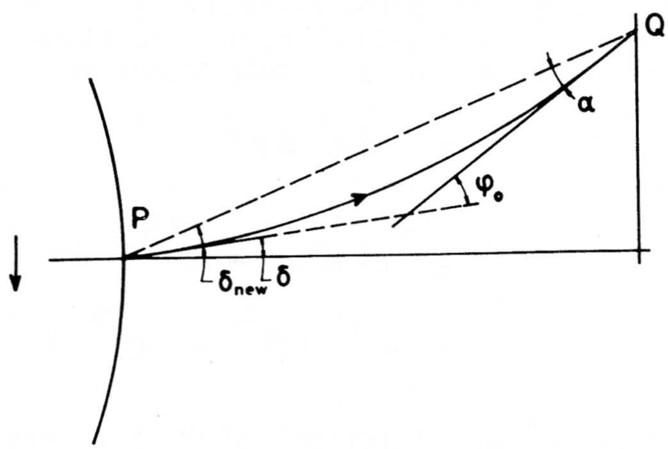

Fig. 1.

Since for a remote observer $\alpha \simeq 0$, the change in the azimuthal angle will be:

$$\delta \longrightarrow \delta_{new} = \delta + |\varphi_o| \quad (11)$$

Because q is not symmetric under the transformation $\delta \to -\delta$, photons with initial azimuthal angles $\pm\delta$ will not end up with $\pm\delta_{new}$ respectively, implying an azimuthal asymmetry of the bending that comes because of rotation.

5. The Redshift Factor

The redshift factor can be obtained from the general expression

$$1+z = \lambda_{ob}/\lambda_{em} = (u\cdot p)_{ob}/(u\cdot p)_{em} \tag{12}$$

where p^α is the direction of null ray, $u^\alpha_{em} = dx^\alpha/ds$ and u^α_{ob} = (1,0,0,0). Corresponding to the metric, Eq.(5) one gets

$$1+z = \frac{1+\Omega q}{e^{\nu}(1-v_s^2)^{1/2}} \tag{13}$$

Because of inertial frame dragging, q will not be same for all photons but will be maximum for photons emitted tangentailly backward (that is, from the equatorial limb of the star that moves away from the observer) and minimum for photons emitted tangentially forward (that is, from the equatorial limb of the star that moves towards the observer). The quantity q_{max} will be positive whereas q_{min} will be negative and

$$q_{max} = -q_{min} \tag{14}$$

a manifestation of inertial frame dragging. This implies that, in the rotational spacetime (assuming that the line is localized and isotropic), a line emission will not only get shifted to a lower frequency spectral line, but because of Eq.(14), it will get broadened into a band. This is schematically illustrated in Fig. 2. If the emitted photon has an energy

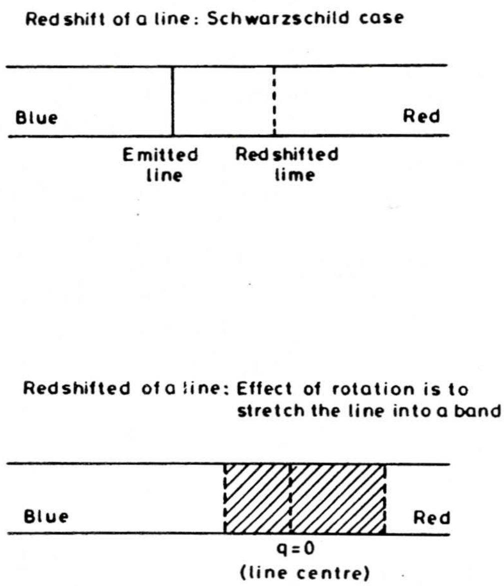

Fig. 2.

E, then the line broadening will be given by

$$W = E\left[\{1+z(q_{min})\}^{-1} - \{1+z(q)_{max}\}^{-1}\right] \tag{15}$$

The line centre will correspond to $q = 0$, and its redshift will be given by Eq.(13). Because of inertial frame dragging, the broadened spectrum will be asymmetric with respect to the redshifted line centre.

6. The Pulsar Pulse Profile in a Rotating Spacetime

Let us make the standard assumption that emission from a pulsar is in the form of a narrow conical beacon (width $\sim 10°$) which corotates with the pulsar (Radhakrishnan and Cooke 1969). From what has been discussed in Section 4, it follows that the axis of the pulsar cone, characterized by $\delta = 0$, will acquire a tilt in the direction of rotation. This tilt angle will be given by the net bending suffered by a $\delta = 0$ photon. Other photons, within the confines of the cone and on either side of $\delta = 0$ will get deflected towards the surface of the pulsar due to spacetime curvature, and in addition, be tilted (like the $\delta = 0$ photon) in the direction of rotation of the pulsar. There will be a gradation in the magnitude of the tilt, depending on δ. The net effect on the pulse cone will be (apart from an overall tilt in the direction of rotation) a widening of the pulse cone width and an overall asymmetry in the final pulse profile (assuming an initial pulse profile symmetric in δ about $\delta = 0$) because the photon impact parameter is not symmetric in $\pm\delta$ as a result of inertial frame dragging. The cone widening, in turn, will imply a reduction in the pulse intensity, because the total energy flux must be conserved.

7. Arrival Times of Pulses with Different Frequencies

The frequency of the coherent radiation from a pulsar depends (assuming the standard polar cap model) on the location (r_e) of emission. A higher frequency pulse has a smaller r_e, and pulses at different frequencies correspond to a set of nested cones with a common axis. This idea is supported by observational data, and is called radius-to-frequency mapping. For two pulses of radiation having frequencies ν_1 and ν_2 ($< \nu_1$) emitted from radial locations r_1 and r_2 ($> r_1$), radius-to-frequency mapping will imply that the arrival time of the former will be delayed with respect to the latter by an amount given by

$$\Delta\tau = \Delta\tau_{ret} + \Delta\tau_{drag} \tag{16}$$

where $\Delta\tau_{ret}$ is the time for the pulse to travel the distance $\Delta r = r_2 - r_1$ (assuming simultaneous emission), called the retardation time and $\Delta\tau_{drag}$ is the time delay arising due to rotation (assuming rigid corotation of the pulses, as in polar cap models), and is called the drag or aberration term. If effects of general relativity are not important, then

$$\Delta\tau_{ret} = \Delta r / c \tag{17}$$

and

$$\Delta\tau_{drag} = \sin\alpha \;\Delta r/c \tag{18}$$

where α is the angle between the rotational and magnetic axes.

For fast pulsars, general relativistic effects will modify Eqs.(17) and (18) to

$$\Delta\tau_{ret} = \int_{r_1}^{r_2} dr \; \dot{t}/\dot{r}$$

$$= \int_{r_1}^{r_2} dr \frac{(1+\omega q_e)}{e^{2\nu}\left[(1+\omega q_e)^2 - q_e^2 e^{2\nu-2\psi}\right]^{1/2}} \tag{19}$$

$$\Delta\tau_{drag} = \frac{P}{2\pi}\left\{|\varphi_0(r_2,\theta,\delta=0)| - |\varphi_0(r_1,\theta,\delta=0)|\right\} \tag{20}$$

where the angles $|\phi_B|$ are the pulse cone axis tilt in the direction of rotation, and P is the pulsar period.

8. Results and Discussion

Magnitudes of the above effects, calculated for the millisecond pulsar PSR 1937+214, are expected to be generally valid for fast pulsars, since these have a common genesis and evolution. Results of such calculations using available realistic neutron star models are summarized below (for details, see Datta 1988).

Rotation induced spherical deformations in the mass and radius for pulsars rotating at the secular instability limit are 10% and 5% respectively. Lower limit of mass of fast pulsars, estimated on grounds of secular rotational instability, is (0.6-0.8) $M_\odot$ for the softer equation of state models and (1-1.54) $M_\odot$ for the comparatively stiffer models. For the moment of inertia of fast pulsars, the following range is indicated : (0.24-3.44)x10^{45} g cm^2.

Turning to radiation characteristics, a large rotation will transform a line emission into a broadened band with a highly asymmetric intensity profile. In view of this, the concept of detection of a line emission as a line from fast pulsars - and the usual interpretation of the redshift factor in terms of mass-to-radius ratio - will no longer hold good.

Detailed calculations (Datta & Kapoor 1985) indicate that spacetime curvature will make a pulse cone wider by a factor ~ 2, and that the pulse intensity peak will get reduced by an order of magnitude. Also rotation will make an initially symmetric pulse profile asymmetric. The magnitude of this effect turns out to be small. Observationally pulsars are known to possess narrow pulse profiles. Therefore, the above results can be taken to imply that at the emission location, the pulse must be spiky in shape. It would then follow that brightness temperature (proportional to the intensity) of pulsars in general (since the curvature effects dominate over the rotational effects) are larger by an order of magnitude than have been hitherto presumed.

Multifrequency timing analysis of signals from PSR 1937+214 limit time delay discrepancies to within 6 μs, assuming emission region thickness $\Delta r \simeq 4$ Km (Cordes & Stinebring 1984). However, calculations using Eqs.(19) and (20) give the estimate of the net arrival time advancement to be much larger, about (25-50) μs (Kapoor & Datta 1986). Therefore, to conform to the observational data, a possible conclusion is that Δr be much thinner ($\simeq 1$ Km instead of 4 Km). On the other hand, if one insists on retaining $\Delta r = 4$ Km, alternate conclusions could be (a) a polar cap model, but without a radius-to-frequency mapping (Arons 1979) is perhaps a more viable model for fast pulsars, or (b) the radiation mechanism is altogether different from the standard polar cap models, as also suggested by the extremely narrow pulse width of PSR 1937+214 (Backer 1984).

We have confined our discussions to the framework suggested by Hartle (1967) to describe the rotating spacetime geometry. This formalism is fully relativistic, but not valid for arbitrarily high Ω. Even so, this provides a first estimate of a wide variety of fast pulsar properties. Models of relativistic, rapidly rotating stars (see Friedman et al. 1986), which go beyond the Hartle approximation, will reiterate the trends of the results presented here more strongly.

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