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On the saturation of the refractive index structure function – II. Influence of the correlation length on astronomical 'seeing'

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Summary. Loss of phase correlation in long-exposure images could occur on much shorter scales than the scale of saturation of the phase structure function. The departure of the phase structure function from the 5/3 power law at large separations leads to better expectation of 'seeing' at longer wavelengths.

1 Introduction

It is well known that long-exposure images are devoid of detail smaller than an arcsec or thereabouts. Fried (1966) showed that this blurring was due to the integrated effect of refractive index fluctuations encountered by starlight as it travels down through the turbulent layers of the Earth's atmosphere. Using Obukhov's law for the refractive index structure function, Fried (1966) was able to isolate a single parameter r_0 which characterized the loss of detail in a long exposure image. Allowing for the effect of 2-point correlations in the fluctuations of refractive index led to a modified law for the phase structure function (Venkatakrishnan & Chatterjee 1987, Paper I). This law departed from the 5/3 power law for length scales in the wavefront approaching the correlation scale of turbulence. In what follows we will identify a physical length scale in the wavefront corresponding to r_0 and then show the influence of the correlation scale of turbulence as r_0 approaches this scale.

2 The correlation of refractive index fluctuations

Onset of turbulence occurs at the outer scale of turbulence and cascades down to smaller scales until viscous dissipation takes over at the smallest scale known as the inner scale of turbulence. In the inertial range of turbulence, the power spectral density of the fluctuations increases with the size of the eddy following the Kolmogorov spectrum. As the eddy size approaches the outer scale, the power spectral density approaches a constant (Goodman 1984). Correlation of fluctuations between two points separated by r is provided by eddies of sizes $l > r$. Furthermore, since power is concentrated in larger l, the effective correlation length for the fluctuations will be nearly equal to the outer scale.

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Formally, one can write the refractive index structure function as

$$D_N(r) = \sigma_N^2[1-\varrho_N(r)] \tag{1}$$

where σ_N is the position independent variance and ϱ_N the normalized covariance of the refractive index fluctuations, with $\varrho_N \rightarrow 0$ as $r/L_c \rightarrow \infty$, where L_c is the effective correlation length of the fluctuations. The exact functional form for ϱ_N is yet to be established although Takarski (1961) suggests various forms which follow the twin constraints of decay at large r and Obukhov's law at small r. In what follows we will consider the simplest amongst these, namely:

$$\varrho_N(r) = \exp[-(r/L_c)^{2/3}].$$

Note that L_c is *not* the outer scale of turbulence but is related to it.

3 The correlation of phases in the wavefront

Fluctuations in refractive index cause fluctuations in the optical path length traversed by the wave front. As a result, the wavefront will be distorted and will contain random excursions in phase. If $\eta\lambda$ is the rms fluctuation of phase difference across a distance Λ in the wavefront, then the rms error in phase for spatial frequencies $\approx\Lambda/\lambda$ is $2\eta\pi$. If η is a monotonically increasing function of Λ, then it is clear that the phase errors introduced by the atmosphere exceed π rad for $\Lambda(\eta)>\Lambda(\frac{1}{2})$. Thus the smallest angular detail present in long exposure images will be λ/Λ_c, where $\Lambda_c=\Lambda(\frac{1}{2})$. Hence Λ_c may be considered as a correlation length for phase fluctuations and the corresponding angular spread λ/Λ_c is astronomical 'seeing'. Note that the phase structure function would tend to a constant value on another length scale, this saturation being caused by the decay of refractive index correlation in planes normal to the path of starlight (Paper I). However since phase appears as the argument of circular trigonometric functions, the actual scale of the saturation of phase fluctuations in the wavefront has no significance as far as long exposure imaging is concerned. Nevertheless, this scale is meaningful for visibility calculations in long baseline interferometry where the phase fluctuations are compared with the finite bandwidth of the starlight.

4 Relation between refractive index correlation length L_c and phase correlation length

The phase structure function for a wavefront which has traversed a distance L through turbulence having refractive index structure function D_N is given by (Tatarski 1961)

$$D_\phi(\gamma) = k^2L\int_{-\infty}^{\infty} dz\ [D_N(\sqrt{z^2+r^2})-D_N(z)]. \tag{2}$$

Using $D_N(z)=2\sigma_N^2[1-\exp\{-(z/L_c)^{2/3}\}]$, we have

$$D_\phi(r) = 2k^2LL_c\sigma_N^2F(r/L_c), \tag{3}$$

where

$$F(x) = 2\int_0^{\infty} dy[\exp(-y^{2/3})-\exp\{-(x^2+y^2)^{1/3}\}].$$

Using the definition of phase correlation length Λ_c developed in Section 3, we obtain the following relation between Λ_c and L_c by putting $D_\phi(\Lambda_c)=\pi^2$:

$$F(\Lambda_c/L_c) = (8LL_c\sigma_N^2/\lambda^2)^{-1}. \tag{4}$$

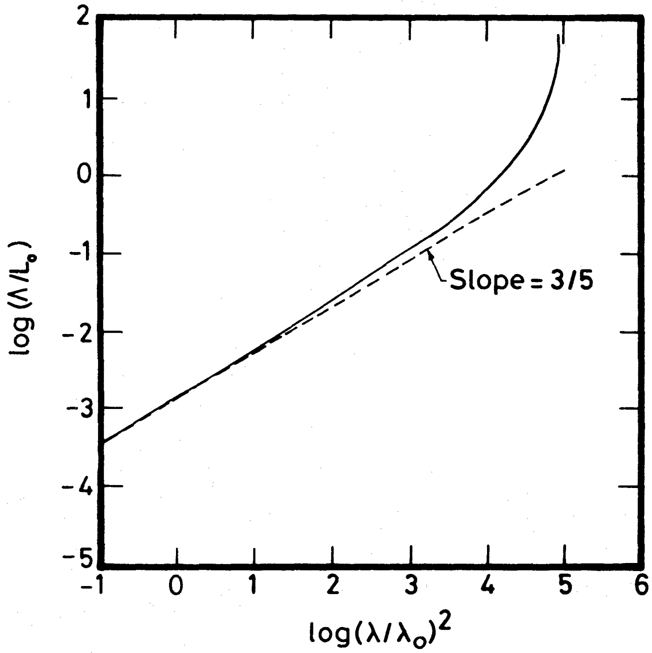

Figure 1. Variation of $\log \Lambda_c/L_c$ is shown as a function of $\log(\lambda/\lambda_0)^2$. This curve is for $\lambda_0 = 0.5\,\mu$m and $L_c = 100$ m.

Let us consider a wavelength λ_0 for which $\Lambda_c/L_c \ll 1$, in which case $F(\Lambda_{c0}/L_c = \alpha(\Lambda_{c0}/L_c)^{5/3}$ (Paper I). Then equation (4) yields

$$\Lambda_{c0} = (8\alpha L L_c \sigma_N^2/\lambda_0^2)^{-3/5}. \tag{5}$$

Resubstituting equation (5) in equation (4) for other values of λ, we have

$$F(\Lambda_c/L_c) = \alpha\left(\frac{\lambda}{\lambda_0}\right)^2\left(\frac{\Lambda_{c0}}{L_c}\right)^{5/3}, \tag{6}$$

or

$$\Lambda_c/L_c = F^{-1}\left[\alpha\left(\frac{\lambda}{\lambda_0}\right)^2\left(\frac{\Lambda_{c0}}{L_c}\right)^{5/3}\right]. \tag{7}$$

Fig. 1 shows the dependence of $\log(\Lambda_c/L_c)$ on $\log(\lambda/\lambda_0)^2$ for $\Lambda_{c0} = 10$ cm and $L_c = 100$ m. One can accommodate a different combination of Λ_{c0}/L_c by a suitable translation of the origin of the abscissa. It can be seen from Fig. 1 that the value for Λ_c/L_c departs considerably from that predicted by the 6/5 power law for longer wavelengths. For instance, the values of Λ_c/L_c depart from that predicted according to the 6/5 law by as much as 20 per cent, 60 and 90 per cent for $\lambda = 10\lambda_0$, $50\lambda_0$ and $100\lambda_0$ respectively. The departures would be more spectacular for smaller values of L_c.

5 Discussion and conclusions

The refractive index fluctuations in the path of the starlight manifest themselves finally as accumulated fluctuations in optical path differences. The ratio of path difference to wavelength yields the phase difference. Thus the longer the wavelength, the larger is the path difference necessary to produce a given phase difference. From the arguments presented in Section 3, one can now understand that a given value of rms phase difference fluctuation is attained for larger separations in the case of longer wavelengths. When this separation approaches L_c, the increase in phase difference practically ceases for any further increase in the separation. Translated into wavelengths, this means that there is a limiting wavelength beyond which the rms phase difference fluctuation can never attain the value π, irrespective of the separation. In Fig. 1, this limiting wavelength is $\lambda/\lambda_0 \approx 300$ corresponding to $\lambda = 150\,\mu$m for $\lambda_0 = 0.5\,\mu$m. Although variation of refractive index with wavelength will modify this result, it serves as an extreme example of the influence of L_c on the phase correlation length Λ_c (or equivalently the Fried's parameter r_0). A less dramatic influence of L_c is, for example, the gain in magnitude by as much as 0.5 mag over that predicted from the 6/5 power law for observations of a point source at $5\,\mu$m. Furthermore, predictions of 'seeing' at longer wavelengths from the observed value in the visible would become more optimistic if the saturation of the phase structure function is taken into account.

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