Mon. Not. R. astr. Soc. (1985) **213**, 381–388

On the dynamical evolution of a spheroidal cluster

G. Som Sunder and R. K. Kochhar *Indian Institute of Astrophysics Bangalore 560 034, India*

Accepted 1984 October 22. Received 1984 October 22; in original form 1984 September 3

Summary. Tensor virial equations are used to investigate the dynamical evolution of a cluster of mass points characterized by a spheroidal (homogeneous or heterogeneous) distribution of matter and an isotropic distribution of velocities. It is known that if the total energy of the system is positive, it will disperse to infinity. We show that in the case of negative total energy, the system executes finite amplitude oscillations between oblate and prolate shapes.

1 Introduction

Chandrasekhar & Elbert (1972) used tensor virial equations to discuss the dynamical evolution of spherical and spheroidal clusters of mass points. Later, Binney (1978) investigated the equilibrium structure of elliptical galaxies in which departures from sphericity are attributed to rotation and anisotropy in velocity distribution.

In this paper we re-examine the dynamical evolution of a spheroidal cluster of mass points. In Section 3 we discuss the case of a homogeneous spheroid, and in Section 5, a heterogeneous spheroid.

2 The tensor virial equations

Following Chandrasekhar & Elbert (1972, Paper I), we consider a cluster of mass points (galaxies or stars) which is initially spheroidal (either oblate or prolate) with semiaxes $a_1 = a_2$ and a_3, and a given total energy E. The system is governed by the virial equations (*cf.* Chandrasekhar 1964)

$$\frac{1}{2}\frac{d^2}{dt^2} I_{11} = 2K_{11} + W_{11} \tag{1}$$

and

$$\frac{1}{2}\frac{d^2}{dt^2} I_{33} = 2K_{33} + W_{33}, \tag{2}$$

where $K_{ij} = T_{ij} + \frac{1}{2}\Pi_{ij}$ denotes the total kinetic energy tensor, and various symbols hav their usual meaning.

We shall suppose that at all times the distribution of velocities is (locally) isotropic so tha

$$K_{11} = K_{22} = K_{33} = \tfrac{1}{3} K. \quad (3$$

From equations (1) and (2) we obtain the equilibrium condition (*cf.* Binney 1978)

$$2K_{11} - 2K_{33} = |W_{11}| - |W_{33}| \quad (4$$

or

$$(2T_{11} - 2T_{33}) + (\Pi_{11} - \Pi_{33}) = |W_{11}| - |W_{33}|. \quad (5$$

One can see from equation (4) that isotropic velocity distribution is not consistent witl the cluster being aspherical and in equilibrium. We now discuss the dynamical evolution of spheroidal cluster.

3 Homogeneous spheroids

If we consider homogeneous spheroids, equations (1) and (2) reduce to (*cf.* Paper I equations 44 and 45).

$$\frac{d^2a_1^2}{dt^2} = \frac{20E}{3M} + \frac{GM}{a_1}\left(\frac{a_1}{a_3} A_1 + 2\frac{a_3}{a_1} A_3\right) \quad (6$$

The results are shown in Figs 1–3.

$$\frac{d^2a_3^2}{dt^2} = \frac{20E}{3M} + \frac{GM}{a_1}\left(4\frac{a_1}{a_3} A_1 - \frac{a_3}{a_1} A_3\right). \quad (7$$

We now make the substitutions

$$a_1 = a_0 z, \qquad a_3^2 = a_1^2(1-y) = a_0^2 z^2 (1-y),$$

$$t = t_0 \tau \qquad \text{with} \quad t_0 = \left(\frac{a_0^3}{2GM}\right)^{1/2},$$

$$Q = E/|W_0| \qquad \text{with} \quad W_0 = -\frac{3}{5}\frac{GM^2}{a_0}. \quad (8$$

Note that W_0 is the potential energy of a sphere of radius a_0. The quantities z and Q define here are related to z and Q of Paper I (equations 46 and 50) by $z_{\text{Paper I}} = z^2$ and $Q_{\text{Paper I}} = 1/(4Q$

With the substitutions (8), equations (6) and (7) take the form

$$\frac{d^2z^2}{d\tau^2} = 2Q + \frac{1}{2z} f_1(y) \quad (9$$

and

$$z^2 \frac{d^2y}{d\tau^2} + 2\frac{dz^2}{d\tau}\frac{dy}{d\tau} = -2Qy + \frac{1}{2z} f_2(y), \quad (1$$

where (*cf.* Paper I)

$$f_1(y) = \frac{3}{y}\sqrt{1-y} + \frac{4y-3}{y} S(y), \quad (1$$

$$f_2(y)=\left(\frac{9}{y}-3\right)\sqrt{1-y}-\frac{4y^2-9y+9}{y}S(y), \tag{12}$$

$$S(y)=\begin{cases}\dfrac{\sin^{-1}\sqrt{y}}{\sqrt{y}} & y\geq 0\\[2ex] \dfrac{\sinh^{-1}\sqrt{-y}}{\sqrt{-y}} & y<0.\end{cases} \tag{13}$$

In case of a spherical distribution of matter, equation (10) is identically satisfied, and equation (9) reduces to the spherical case of Paper I.

A comparison of equations (9) and (10) with the corresponding equations (48) and (49) of Paper I shows that the rhs of the latter should be multiplied by $E/|E|$. Thus equations (48) and (49) of Paper I can be used only when E is positive; the system then disperses to infinity. If the total energy of the system, E, is negative i.e. $Q<0$, equations (9) and (10) above should be used.

4 Results

We have numerically solved equations (9) and (10) for various negative values of Q and with the boundary conditions

$$z=1, y=y_0(\text{assigned}), \qquad \frac{dz}{dt}=\frac{dy}{dt}=0 \text{ at } \tau=0. \tag{14}$$

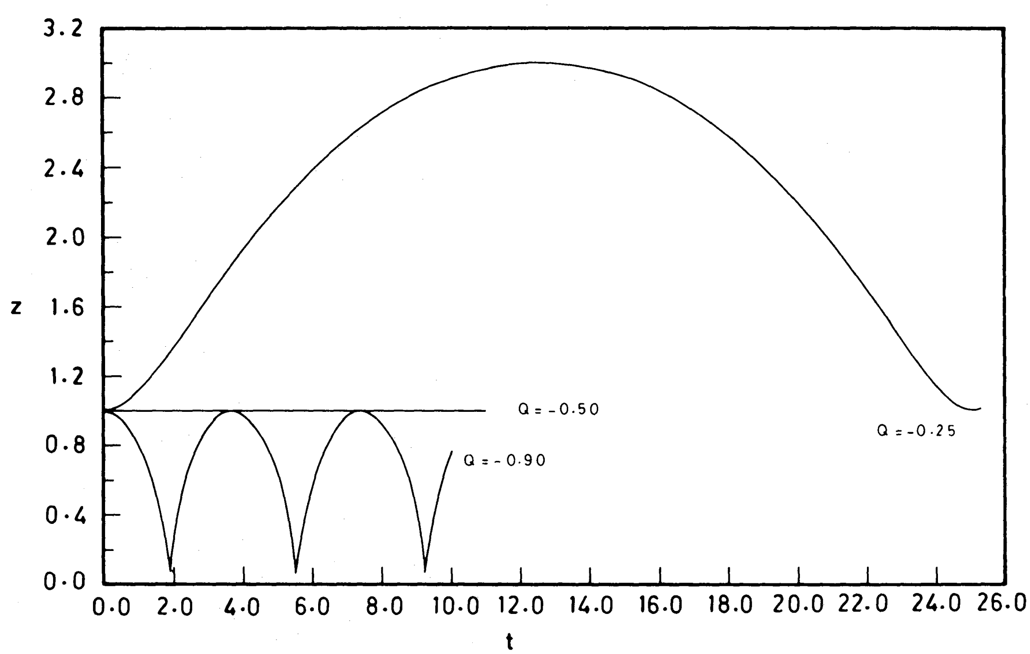

Figure 1. The evolution of homogeneous spheroidal clusters. The ordinate $z\,(=a_1/a_0)$ is the semiaxis of the circular equatorial section and the abscissa measures the time in the units of $t_0=(a_0^3/2GM)^{1/2}$. The curves are labelled by the values of Q to which they belong. The curves are identical on the scale of the figure for various values of y_0.

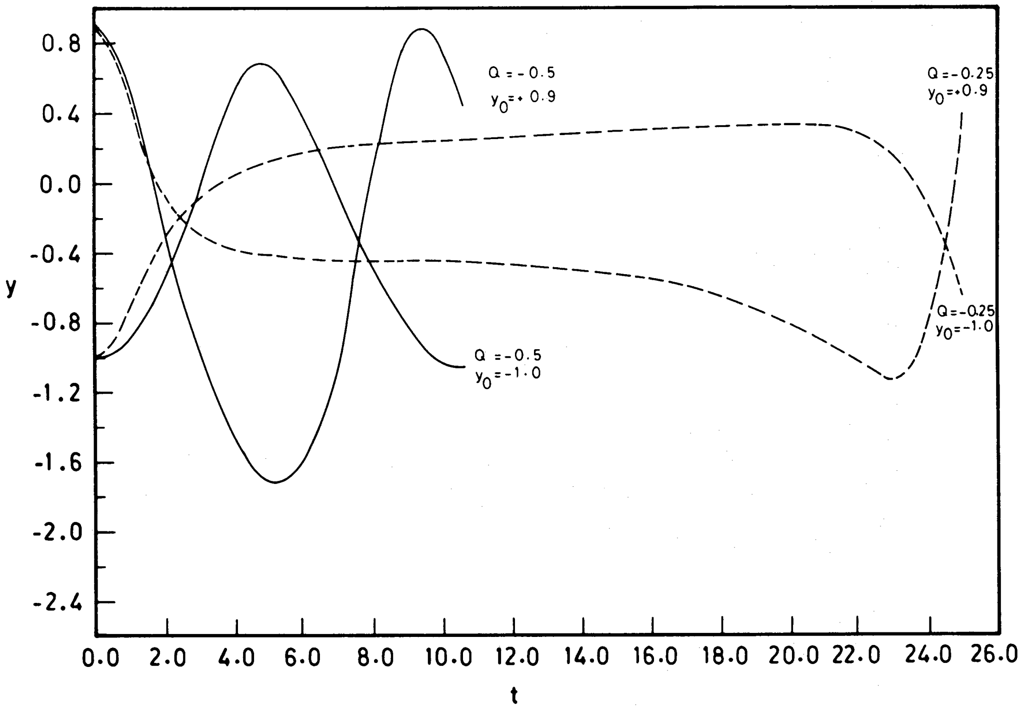

Figure 2. The evolution of homogeneous spheroidal clusters. The ordinate y is a measure of the eccentr city (e^2) positive for oblate spheroids and negative for prolate spheroids. The abscissa measures the tim in the same unit as Fig. 1. The curves are labelled by the values of Q and y_0 to which they belong.

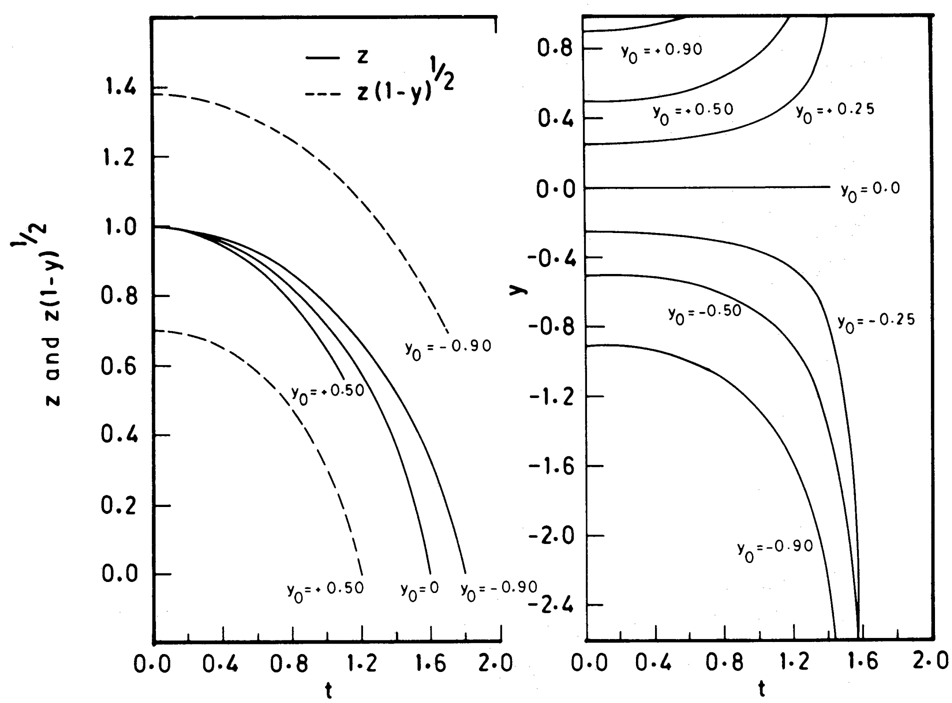

Figure 3. The evolution of homogeneous spheroids with zero initial kinetic energy. (A) depicts tl behaviour of the two semiaxes z and $z(1-y)^{1/2}$, while (B) the behaviour of the eccentricity as the syste evolves. The curves are labelled by the values of y_0 to which they belong.

Fig. 1 shows the behaviour of z with time. For a given Q, the curves are almost identical (on the scale of the graph) for various y_0 and correspond to the solution

$$\tau = \frac{1}{(-Q)}(Qz^2 + z - Q - 1)^{1/2} + \frac{1}{2(-Q)^{3/2}} \cos^{-1}\left(\frac{2Qz+1}{2Q+1}\right), \tag{15}$$

obtained analytically for a spherical system ($y_0 = 0$) (*cf.* Paper I, equation 34). The period is $\pi(-Q)^{-3/2}$ and the amplitude, $|2 + 1/Q|$.

Fig. 2 illustrates the behaviour of y for $Q = -0.25$ and -0.50. In both the cases, y oscillates approximately with the period $\pi(-Q)^{-3/2}$. Initially oblate systems ($y > 0$) become prolate ($y < 0$) and then again oblate. Similarly initially prolate systems become oblate and then prolate in a time $\approx \pi(-Q)^{-3/2}$.

Fig. 3A depicts the behaviour of the two axes z and $z(1-y)^{1/2}$ for systems with zero kinetic energy initially. It is seen that the smaller of the two axes always collapses to a point. The behaviour of y for such systems (Fig. 3B) shows that the initial oblateness or prolateness of the system is enhanced as it evolves. This result is in agreement with that of Lin, Mestel & Shu (1965).

5 Heterogeneous spheroids

We now consider the dynamical evolution of a heterogeneous spheroidal system whose density distribution is of the form

$$\rho(\mathbf{x}) = \rho_c(1-m^2)^{\nu}, \tag{17}$$

where

$$m^2 = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2}. \tag{18}$$

In equations (17) and (18) ρ_c is the central density and ν a non-negative number increasing values of which indicate greater central concentration. Such distributions have been used in the study of galaxies by Perek (1962) and Kerr & de Vaucouleurs (1956). The case $\nu = 0$ corresponds to the homogeneous systems of Section 3. Since equidensity strata for such a distribution is similar to and concentric with the bounding ellipsoid we can use the results of Roberts (1962) to obtain expressions for the moments of inertia and potential energy tensors. For the function $F(m^2)$ we have

$$F(m^2) = \int_{m^2}^{1} \rho(m^2)\, dm^2 = \frac{\rho_c}{\nu+1}(1-m^2)^{\nu+1}. \tag{19}$$

and

$$\int_0^1 (1-m^2)^p\, dm = \frac{\sqrt{\pi}}{2} \frac{\Gamma(p+1)}{\Gamma(p+1+\tfrac{1}{2})} \tag{20}$$

we obtain

$$M = 2\pi a_1 a_2 a_3 \int_0^1 F(m^2)\, dm = \pi a_1 a_2 a_3 \rho_c \left[\sqrt{\pi} \frac{\Gamma(\nu+1)}{\Gamma(\nu+2+\tfrac{1}{2})}\right], \tag{21}$$

$$I_{ij} = 2\pi a_1 a_2 a_3 a_i^2 \delta_{ij} \int_0^1 F(m^2) m^2\, dm = \frac{1}{5} M a_i^2 \delta_{ij} \phi(\nu), \tag{22}$$

$$W_{ij} = -\pi^2 G a_1 a_2 a_3 a_i^2 A_i \delta_{ij} \int_0^1 [F(m^2)]^2\, dm = -\frac{3}{10} \frac{GM^2}{a_1 a_2 a_3} a_i^2 A_i \delta_{ij} \psi(\nu), \tag{23}$$

where

$$\phi(\nu) = \frac{5}{2\nu + 5} \tag{24}$$

$$\psi(\nu) = \frac{5}{3(\nu+1)^2\sqrt{\pi}}\left[\frac{\Gamma(\nu+2+\frac{1}{2})}{\Gamma(\nu+1)}\right]^2 \frac{\Gamma(2\nu+3)}{\Gamma(2\nu+3+\frac{1}{2})} \tag{25}$$

and the A_i are the index symbols defined in Chandrasekhar (1969). The functions $\phi(\nu)$ and $\psi(\nu)$ are measures of the difference of the moment of inertia and potential energy tensors of a distribution with parameter ν from that of a homogeneous ellipsoid with the same mass and semiaxes. These functions are tabulated in Table 1 for a few values of ν.

One can see from Table 1 that spherical configurations with $\nu = 1$ have potential energy equal to that of a polytrope of index $n = 1$. Similarly $\nu = 4$, $\nu = 12$, $\nu = 55$ have potential energies in fair agreement with the values for polytropes of indices $n = 2$, 3 and 4 respectively.

We now assume that the distribution (17) continues to be maintained as the system evolves i.e. ν and hence $\phi(\nu)$ and $\psi(\nu)$ are independent of time. Substituting equations (22) and (23) into equations (1) and (2) we get

$$\frac{d^2a_1^2}{dt^2} = \frac{20E}{3M\phi(\nu)} + \frac{GM}{a_1}\frac{\psi(\nu)}{\phi(\nu)}\left(\frac{a_1}{a_3}A_1 + 2\frac{a_3}{a_1}A_3\right), \tag{26}$$

$$\frac{d^2a_3^2}{dt^2} = \frac{20E}{3M\phi(\nu)} + \frac{GM}{a_1}\frac{\psi(\nu)}{\phi(\nu)}\left(4\frac{a_1}{a_3}A_1 - \frac{a_3}{a_1}A_3\right). \tag{27}$$

Once again we write

$$a_1 = a_0 z \quad \text{and} \quad a_3^2 = a_1^2(1-y) = a_0^2 z^2 (1-y), \tag{28}$$

but now define our unit of time by

$$t = t_0 \tau \quad \text{where} \quad t_0 = \left(\frac{a_0^3}{2GM}\frac{\phi(\nu)}{\psi(\nu)}\right)^{1/2}. \tag{29}$$

and let

$$Q = E/|W_0| \quad \text{with} \quad W_0 = -\frac{3}{5}\frac{GM^2}{a_0}\psi(\nu). \tag{30}$$

Thus W_0 is the potential energy of a sphere of radius a_0 and density distribution parameter ν. With these definitions equations (26) and (27) reduce to

$$\frac{d^2z^2}{d\tau^2} = 2Q + \frac{1}{2z}f_1(y), \tag{31}$$

Table 1.

ν	$\phi(\nu)$	$\psi(\nu)$	$\sqrt{\phi(\nu)}\,\psi(\nu)$
0	1.00	1.00	1.00
1	0.714	1.19	1.0055
2	0.556	1.36	1.014
4	0.385	1.65	1.024
12	0.172	2.50	1.037
55	0.0435	5.03	1.050

$$z^2 \frac{d^2y^2}{d\tau^2} + 2\frac{dz^2}{d\tau}\frac{dy}{d\tau} = -2\,Qy + \frac{1}{2z} f_2(y) \tag{32}$$

where $f_1(y)$ and $f_2(y)$ are just the functions defined by equations (11) and (12). These equations are identical with those in Section 3 whose conclusions therefore also apply to heterogeneous systems.

Thus heterogeneous ellipsoids with positive energies also expand and are eventually dispersed. Those with negative energies oscillate both in size and in eccentricity with a period $\approx \pi(-Q)^{-3/2}$. The time-scales however, are different for different central concentrations as a consequence of the presence of the parameter ν in the definition (29) of t_0.

With a view to comparing systems with different central concentrations we denote by $t_0^{(\nu)}$ and $Q^{(\nu)}$ the values of t_0 and Q for a system with the density parameter ν. Since

$$t_0^{(\nu)} = \sqrt{\frac{\phi(\nu)}{\psi(\nu)}}\, t_0^{(0)}, \qquad Q^{(\nu)} = \frac{Q^{(0)}}{\psi(\nu)} = \frac{E}{|W_0^{(0)}|\,\psi(\nu)} \tag{33}$$

where

$$W_0^{(0)} = -\frac{3}{5}\frac{GM^2}{a_0} \tag{34}$$

we have for the period

$$P^{(\nu)} = P^{(0)}\,\psi(\nu)\sqrt{\phi(\nu)}. \tag{35}$$

From Table 1 it is seen that $\psi(\nu)\sqrt{\phi(\nu)}$ varies very little over a very wide range of ν. Therefore we can conclude that systems with the same value of E oscillate both in size and eccentricity with approximately the same period whatsoever their central concentration.

The amplitude of the oscillation is given by

$$\left|2 + \frac{1}{Q^{(\nu)}}\right| = \left|2 + \frac{|W^{(0)}|}{E}\psi(\nu)\right|. \tag{36}$$

As can be seen from this equation when $Q^{(\nu)} < -\tfrac{1}{2}$ the amplitude decreases with increasing ν while it increases with increasing ν when $Q^{(\nu)} > -\tfrac{1}{2}$.

6 Conclusions

We have discussed the dynamical evolution of a spheroidal cluster of mass points (homogeneous or heterogeneous) characterized by an isotropic velocity distribution. It is known in the case of a homogeneous spheroid that if its total energy is positive it will expand and disperse to infinity. We have demonstrated that this is true of heterogeneous spheroids also.

We have also shown that if the total energy is negative, the cluster, whether homogeneous or not, executes finite amplitude oscillations between prolate and oblate shapes, with a period that is independent of the central concentration and depends only on the total energy.

Acknowledgments

We thank Professor S. M. Alladin for helpful discussion. GSS would also like to thank the staff of the Department of Astronomy, Osmania University, for their assistance and the University Grants Commission for financial support during the period when this work was started. We thank Professor S. Chandrasekhar for going through the manuscript.

References

Binney, J., 1978. *Mon. Not. R. astr. Soc.*, **183**, 501.
Chandrasekhar, S., 1964. *Lectures in theoretical Physics, Vol. 6*, p. 1, ed. Brittin, W. E., University c Colorado Press, Boulder.
Chandrasekhar, S., 1969. Ellipsoidal figures of equilibrium, Yale University Press, New Haven.
Chandrasekhar, S. & Elbert, D. D., 1972. *Mon. Not. R. astr. Soc.*, **155**, 435 (Paper I).
Kerr, F. J. & de Vaucouleurs, G., 1956. *Aust. J. Phys.*, **9**, 90.
Lin, C. C., Mestel, L. & Shu, F. H., 1965. *Astrophys. J.*, **142**, 1431.
Perek, L., 1962. *Adv. Astr. Astrophys.*, **1**, 165.
Roberts, P. H., 1962. *Astrophys. J.*, **136**, 1108.