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FLUXVECTOR SPLITTING OF THE INVISCID RADIATION GAS DYNAMIC EQUATIONS

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ABSTRACT

We have analysed various schemes for the iterative simultaneous solution of the inviscid gas dynamic equations. We have considered the problem from the point of view of the stellar atmospheres. The difference between conservative and quasiconservative systems has been analysed by using the flux vector splitting process. The radiation pressure in the atmosphere (due to continuum and resonance lines) has been included in flux vector splitting analysis to obtain best difference schemes.

1. Introduction

The spectra of stars, nebulae and other celestial objects reveal the fact that the matter in the outer layers of these objects is in motion. These motions are the result of a highly complicated interaction of matter and radiation emitted by the central object. In fluid dynamics, one would call this, a non-stationary phenomenon of compressible fluids with small viscosity and thermal conductivity. These phenomenon can be expressed by a set of nonlinear hyperbolic system of partial differential equation. It is quite common that discontinuities may arise. It is customary to express these equations in a conservation-law form and write the difference schemes to solve them. The most widely used among the various difference schemes are due to Lax and wendroff (1960) and its two step version by Richtmyer (Richtmyer and Morton 1967). In addition to these, there are various other schemes both explicit and implicit which are designed to suit the respective given problems. Several authors have utilized these schemes in different physical situations.

Kuo-Petrabic et al (1975) examined the pulsar magnetometer. The dynamical aspects of solar corona have been studied by Steinolfson and Nakagawa (1967) by using Rubin and Burstein's version (1967) of Lax-Wendroff schemes. Seidl and Cameron (1972) and Ikcuti (1968) studied the interaction of two shock waves due to supernovae explosions by using the Lax and Wendroff two step schemes. Hunt (1971, 1975) has used these schemes extensively in his studies of galaxies. There are several others who used these schemes in different physical situations.

One usually writes the equations of motion, energy and mass in a conservation-law form so that one can easily find out the characteristic speeds (eigen values) of a hyperbolic system. Conservation-law form of the inviscid gasdynamic equations has the property that the non linear flux vectors are homogeneous functions of degree one. Steger and Warming (1981) showed that this property permits the splitting of flux vectors into subvectors by similarity transformations so that each subvector has associated with it, a specified eigenvalue spectrum. This flux vector splitting led them to develop new explicit and implicit dissipative finite difference schemes for first order hyperbolic systems of equations. We shall discuss these schemes when the interaction of matter with radiation is taken into consideration. In stellar atmospheric situations the radiation in the lines and continuum help change the momentum of the gases. In such a situation the system of hyperbolic equations need not conform to the conservation-law form. We would like to examine the eigenvalue spectrum when radiation pressure is included and the necessary modifications needed to change the difference schemes.

2. The equations of radiation gas dynamics

In stellar atmospheres one can approximate the equations of gas dynamics in the spherically symmetric geometry. When the equations are written in this geometry, the analysis becomes extremely complicated and therefore, we shall consider the equations in the approximation of plane parallel geometry, then the analysis will be extended to spherically symmetric systems. We shall introduce the radiation and gravity force terms in the equation of conservation of momentum.

Our main interest is to find out the changes introduced by these two terms into the numerical schemes. We shall consider the time dependent terms so that our discussion will be consistent. The equations of conservation of mass, momentum and energy in one dimensional approximation are given in the conservative law form,

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{1}$$

where

$$U = \begin{bmatrix} \rho \\ m \\ e \end{bmatrix}, \quad F(U) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + p + \Gamma \\ (e+p)\frac{m}{\rho} \end{bmatrix} \tag{2}$$

Where ρ, m, e are the mass, momentum and energy per unit volume and p is the pressure and F is called flux vector. The quantity Γ represents the effective gravity term and is given by

$$\Gamma = -\int \rho\,(f_g - f_r)\,dx \tag{3}$$

Where f_g is the gravity force $= \frac{GM}{r^2}$

and f_r is the radiation force in the continuum and in the lines. This is given by

$$= f_r\,(\text{cont}) + \frac{4\pi}{c}\,\Sigma \int F^l_\nu\, \chi^l_\nu\, d\nu \tag{4}$$

Where c is the velocity of light, F^l_ν and χ^l_ν are the line flux and absorption in the line l. The summation extends over all the lines.

Here e is related to the internal energy per unit mass ϵ by,

$$e = \rho\epsilon + \frac{\rho u^2}{2} = \rho\epsilon + \frac{m^2}{2\rho} \tag{5}$$

We must consider the equation of state so that the system is complete,

$$P = P(\rho, \epsilon) \tag{6}$$

In the case of perfect gas,

$$P = (\gamma - 1)\,\rho\,\epsilon \tag{7}$$

With the help of equation (5), equation (7) can be written as,

$$P = (\gamma - 1)\left[e - \frac{m^2}{2\rho}\right] \tag{8}$$

Where γ is the ratio of specific heats. Equation in the Jacobian form can be written as,

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \tag{9}$$

where $A_{ij}(U) = \dfrac{\partial F_i}{\partial U_j}$ is the Jacobian

matrix of F. The matrix A is given by,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)\dfrac{u^2}{2} + \Gamma_1 & (3-\gamma)\,u + \Gamma_2 & (\gamma-1) + \Gamma_3 \\ (\gamma-1)\,u^3 - \dfrac{\gamma e u}{\rho} & \dfrac{\gamma e}{\rho} \cdot \dfrac{3u^2}{2}(\gamma-1) & \gamma u \end{bmatrix} \tag{10}$$

where

$$\Gamma_1 = \frac{\partial \Gamma}{\partial \rho}$$

$$\Gamma_2 = \frac{\partial \Gamma}{\partial m} \tag{11}$$

$$\Gamma_3 = \frac{\partial \Gamma}{\partial e}$$

The characteristic equation of A is given by,

$$\lambda^3 - \lambda^2 (F + C) + \lambda (CF - DE - A) + AF - BE = 0 \tag{12}$$

where

$$A = (\gamma-3)\frac{u^2}{2} + \Gamma_1$$

$$B = (\gamma-1)\,u^3 - \frac{\gamma e u}{\rho}$$

$$C = (3-\gamma)\,u + \Gamma_2 \tag{13}$$

$$D = \frac{\gamma e}{\rho} - \frac{3\rho^2}{2}(\gamma-1)$$

$$E = \gamma - 1 + \Gamma_3$$

$$F = \gamma u$$

In principle one should be able to obtain the solution of this equation although one has to go through tedious algebra. If we write equation (12) in the form (see Burington 1965)

$$y^3 + py^2 + qy + r = 0 \tag{14}$$

then

$$y = x - \frac{p}{3} \tag{15}$$

and

$$x^3 + ax + b = 0 \tag{16}$$

where

$$a = \frac{1}{3}(3q - p^2),\ b = (2p^3 - 9pq + 27r) \tag{17}$$

and

$$p = -(F + C),\ q = CF - DE - A,\ \lambda = AF - BE \tag{18}$$

Now the solution of the equation (16) can be written formally as,

$$x_1 = \alpha + \beta$$

$$x_2 = -\tfrac{1}{2}(\alpha + \beta) + \frac{i\sqrt{3}}{2}(\alpha - \beta)$$

$$x_3 = -\tfrac{1}{2}(\alpha + \beta) - \frac{i\sqrt{3}}{2}(\alpha - \beta) \tag{19}$$

where $i^2 = -1$

$$\alpha = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

$$\beta = \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} \tag{20}$$

Now let us estimate a and b, by using equations (17) and (18) First,

$$p = -(3u + \Gamma_2)$$

$$q = \frac{u^2}{2}\left(\gamma^2 - \gamma + 6\right) - \frac{\gamma(\gamma-1)e}{\rho} \quad \Gamma_1 + \Gamma_2 \gamma u$$

$$+ \Gamma_3\left[\frac{3u^2(\gamma-1)}{2} - \frac{\gamma e}{\rho}\right] \tag{21}$$

$$r = -\frac{u^3}{2}(\gamma^2 - \gamma + 2) + \frac{\gamma(\gamma-1)eu}{\rho}$$

$$+ \gamma u \Gamma_1 + \Gamma_3\left\{\frac{\gamma e u}{\rho} - (\gamma - 1)u^3\right\}$$

The quantities a and b are obtained by substituting (21) into (17) and they are given by,

$$a = \gamma(\gamma-1)\left\{\tfrac{1}{2}u^2 - \frac{e}{\rho}\right\} - \Gamma_1 + u(\gamma-2)\Gamma_2 - \frac{1}{3}\Gamma_2^2 + \left[\frac{3u^2(\gamma-1)}{2} - \frac{\gamma e}{\rho}\right]\Gamma_3 \tag{22}$$

and

$$b = \tfrac{1}{2}(\gamma^2 - \gamma + 6)u^2 + (\gamma - \gamma^2 - 3)u^3 + (\gamma - 1)u\Gamma_1$$

$$+ \left\{\frac{1}{3} + \frac{5}{6}\gamma + \frac{1}{6}\gamma^2 - \frac{e\gamma(\gamma-1)}{3\rho}\right\}\Gamma_2$$

$$+\frac{3}{2}u^2(\gamma-1)\left(1-\frac{2}{3}u\right)\Gamma_3+\frac{1}{3}u\left(\gamma-\frac{2}{3}\right)\Gamma_2^2-\frac{2}{29}\Gamma_2^3$$

$$-\frac{1}{3}\Gamma_1\Gamma_2+\frac{1}{3}\left\{\frac{3}{2}u^2(\gamma-1)-\frac{\gamma e}{\rho}\right\}\Gamma_2\Gamma_3 \tag{23}$$

The nature of the roots x_1 x_2 x_3 in (19) depend upon the character of the discriminant $\frac{b^2}{4}+\frac{27}{a^3}$ (see equations (19) and (20.) If the discriminant is positive there are one real and two conjugate imaginary roots, If the discriminant vanishes then the roots are real and atleast two of them are identical and if the discriminant is negative, then the roots are real and unequal,

If $\frac{b^2}{4}+\frac{a^3}{27}=0$, then, the roots are

$$x=\mp 2\sqrt{\frac{-a}{3}},\quad \pm\sqrt{\frac{-a}{3}},\quad \pm\sqrt{\frac{-a}{3}} \tag{24}$$

Where the upper sign is to be used if b is positive and the lower sign if b is negative.

If $\frac{b^2}{4}+\frac{a^3}{27}>0$ and $a>0$ the real roots is given by $x=2\sqrt{\frac{a}{3}}\cot 2\phi$ (25)

and $\tan\phi=(\tan\psi)^{\frac{1}{3}}$ and $\cot 2\psi=\mp\sqrt{\frac{b^2/4}{a^3/27}}$

where again, the upper sign is to be used, if b is positive and the lower sign, if b is negative.

If $\frac{b^2}{4}+\frac{a^3}{27}<0$ then the roots are given by,

$$x_1=2\sqrt{\frac{-a}{3}}\cos\frac{\phi}{3}$$

$$x_2=-\sqrt{-\frac{a}{3}}\left(\cos\frac{\phi}{3}+\sqrt{3}\sin\frac{\phi}{3}\right)$$

$$\text{and } x_3=\sqrt{\frac{-a}{3}}\left(\cos\frac{\phi}{3}-\sqrt{3}\sin\frac{\phi}{3}\right)$$

$$\text{where } \cos\phi=\mp\sqrt{(b^2/4)/(-a^3/27)} \tag{26}$$

In the absence of the effective gravity term Γ in equation (2), we obtain the solution conservative form which has its eigen values given by u, $u\pm c$ where c is the velocity of sound given by

$$c=(\gamma p/\rho)^{\frac{1}{2}} \tag{27}$$

The inclusion of Γ terms in the solution complicates the analysis of the problem. We have to find whether or not the quantities Γ_1, Γ_2 and Γ_3 exist. From (11) we have

$$\Gamma_1=\frac{\partial\Gamma}{\partial\rho}=-\frac{\partial}{\partial\rho}\int(\rho f_g-f_r)\,dx$$

$$\Gamma_2 = \frac{\partial \Gamma}{\partial m} = -\frac{\partial}{\partial m}\int (\rho f_s - f_r)\, dx$$

$$\text{and } \Gamma_3 = \frac{\partial \Gamma}{\partial e} = -\frac{\partial}{\partial e}\int (\rho f_s - f_r)\, dx \tag{28}$$

It would be very difficult to evaluate equations (28) unless we know the functional dependence of f_s f_r on ρ, m and e. This can be done only when we are dealing a particular problem in stellar atmospheres. For the moment, we shall assume that the roots of the characteristic equation (12) exist and that they are real. In such event, we shall write the roots as

$$\begin{aligned} &u + c + \gamma_1 \\ &u + \gamma_2 \\ &u - c + \gamma_3 \end{aligned} \tag{29}$$

Where γ_1, γ_2 and γ_3 represent the contributions corresponding to Γ_1, Γ_2 and Γ_3 given in equation(28).

We shall now consider different schemes for the solution of the conservative form equations. As we have already stated above, the most used schemes are those due to Lax and wendroff (see Richtmyer and Morton 1967). McGuire and Morris (1973, 1974, 1976) have generalized the two step L-W Schemes and developed a scheme which is a combination of explicit and implicit procedures. The Lax-Wendroff equations begin with the Taylor's series in t. From equation (1) we have,

$$U_j^{n+1} = U_j^n + \Delta t \left(\frac{\partial u}{\partial t}\right)_j^n + \left(\frac{\Delta t^2}{2}\right)\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n + \;\ldots\ldots\ldots\ldots \tag{30}$$

The x – derivatives are approximated by difference quotients and this gives the Lax-Wendroff scheme,

$$\begin{aligned} U_j^{n+1} &= U_j^n - \tfrac{1}{2}\frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right) \\ &+ \tfrac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[A_{j+\frac{1}{2}}^n\left(F_{j+1}^n - F_j^n\right) - A_{j-\frac{1}{2}}^n\left(F_j^n - F_{j-1}^n\right)\right] \end{aligned} \tag{31}$$

$$\text{where } A_{j+\frac{1}{2}}^n = A\left(\tfrac{1}{2} U_{j+1}^n + \tfrac{1}{2} U_j^n\right) \tag{32}$$

From Steger and Warming (1981), the flux vector F (U) can be split into two parts as

$$F = F^+ + F^- \tag{33}$$

Where F^+ corresponds to the subvector associated with the positive eigenvalues of A, and F^- corresponds to the negative eigenvalues. They are given by,

$$\begin{aligned} F^+ &= A^+ U \\ F^- &= A^- U \end{aligned} \tag{34}$$

The eigenvalues of A^+ are non negative and those of A^- are non positive. The eigenvalues given by (29) are split according to equation

$$\lambda_i = \lambda_i^+ + \lambda_i^-$$

$$\text{where } \lambda_i^+ = \frac{\lambda_i + |\lambda_i|}{2}, \quad \lambda_i^- = \frac{\lambda_i - |\lambda_i|}{2} \tag{35}$$

(Notice that if $\lambda_i \geq 0$, then $\lambda_i^+ = \lambda_i$, $\lambda_i^- = 0$ and with opposite result when $\lambda_i < 0$). The split eigenvalues are given by,

$$\lambda_1^+ = \frac{u+c+\gamma_1 + |u+c+\gamma_1|}{2}$$

$$\lambda_1^- = \frac{u+c+\gamma_1 - |u+c+\gamma_1|}{2}$$

$$\lambda_2^+ = \frac{u+\gamma_2 + |u+\gamma_2|}{2}$$

$$\lambda_2^- = \frac{u+\gamma_2 - |u+\gamma_2|}{2}$$

$$\lambda_3^+ = \frac{u-c+\gamma_3 + |u-c+\gamma_3|}{2} \tag{36}$$

$$\lambda_3^- = \frac{u-c+\gamma_3 - |u-c+\gamma_3|}{2}$$

We shall now consider the iterative schemes with the flux vector splitting.

The simplest explicit first order scheme is given by

$$U_j^{n+1} = U_j^n - \frac{\Delta t \, \nabla_x (F_j^+)^n}{\Delta x} - \frac{\Delta t}{\Delta x} \Delta_x (F_j^-)^n \tag{37}$$

This scheme is stable if and only if

$$\left| \lambda_i^{\pm} \frac{\Delta t}{\Delta x} \right| \leq 1 \tag{38}$$

for all eigenvalues $\lambda_i^{\pm}$ where $\lambda_i = \lambda_i^+ + \lambda_i^-$ are the eigenvalues of the Jacobian matrix. The implicit finite difference scheme is

$$\left(I + \frac{\theta \Delta t}{1+\xi} \delta x \, A_j^n \right) \Delta U_j^n = -\frac{\Delta t}{1+\xi} \delta x \, F_j^n + \frac{\xi}{1+\xi} \Delta U_j^{n-1} \tag{39}$$

where I is the identity matrix,

$\Delta U^n = U^{n+1} - U^n$, δx is an appropriate spatial difference operator. The parameters θ and ξ determine the particular time-differencing approximation used. For example

$\xi = 0$ and $\theta = \frac{1}{2}$ trapezoidal formula

$\xi = 0$ and $\theta = 1$ backward Euler formula

$\xi = \frac{1}{2}$ and $\theta = 1$ three point backward formula

These are some of the schemes that are meant to be used in solving the equations of hydrodynamics in stellar atmospheres. The results will be presented in a forth coming paper.

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