EQUILIBRIUM OF A THIN, ISOLATED, AXISYMMETRIC FORCE-FREE MAGNETIC FLUX TUBE IN A STRATIFIED ATMOSPHERE

PRASANNA LAKSHMI and M. H. GOKHALE
Indian Institute of Astrophysics, Bangalore 560 034, India

(Received 21 April, in revised form 27 August, 1987)

Abstract. We show that up to second order in the thinness parameter, the equations of equilibrium of a thin, isolated, axisymmetric, force-free magnetic flux tube in a stratified atmosphere and the other essential constraints form a mathematically closed system. Auxiliary assumptions about temperature of the tube plasma are neither needed nor possible.

1. Introduction

Browning and Priest (1984) have modelled the equilibrium of a coronal magnetic flux tube as a thin, isothermal tube of an axisymmetric force-free field embedded in an isothermal atmosphere. While determining the equilibrium configuration, they have used for radial equilibrium in the cross-section planes, an equation given by Parker (1979) which is valid for a twisted but non-force-free tube whose radius of cross-section does not vary with length along the axis. They have also ignored the inhomogeneities of plasma pressure and density in the non-horizontal cross-sections of the tube. In a *force-free* field these inhomogeneities are more important than those of the field *even if* the plasma pressure is much less than the magnetic pressure.

In this research note, we rederive the equations for equilibrium of a thin, isolated, axisymmetric, and force-free magnetic tube taking into account the inhomogeneities of magnetic field, plasma pressure, and plasma density, over the tube's cross-section, and the variation of the cross-section diameter, up to the second-order in the parameter representing the thinness of the tube. The two equations of equilibrium and the three constraints: (i) constancy of longitudinal magnetic flux, (ii) constancy of plasma pressure on each horizontal plane, even within the tube, and (iii) equality of total pressure on both sides of the current sheath at the boundary, form a closed system of equations. Thus there is neither a necessity nor a scope for any auxiliary assumption about the temperature of the plasma inside the tube. The system of equations is quite complicated for obtaining, even numerically, the conditions for existence of arch-like solutions in terms of 'photospheric' boundary conditions.

Solar Physics **114** (1987) 75–80.

2. Equations for Equilibrium and Other Constraints

2.1. Equations for global equilibrium of the tube

The equilibrium of an element ds of a magnetic flux tube in a stratified atmosphere requires that

$$\mathbf{F}_g + \mathbf{F}_{pl} + \mathbf{F}_{em} = 0\,,$$

where the three terms on the left-hand side represent forces due to gravity, plasma stresses, and electromagnetic stresses acting on the element.

The first two terms can be written as

$$\mathbf{F}_g = -\langle \rho_i \rangle g A \mathbf{1}_z\, \mathrm{d}s$$

and

$$\mathbf{F}_{pl} = -\int_\Sigma p_e \mathbf{1}_\Sigma \mathrm{d}\Sigma + \frac{\mathrm{d}}{\mathrm{d}s}[-\langle p_i \rangle A \mathbf{1}_z]\, \mathrm{d}s\,,$$

where g is the gravitational acceleration, s is the arc-length measured along the tube axis, p and ρ are the plasma pressure and density suffixes i and e represent values inside and outside the tube, angular brackets ($\langle\ \rangle$) represent average over area of cross-section A, dΣ represents an element of the curved surface Σ of the tube at which the magnetic field has reached a zero-value, $\mathbf{1}_\Sigma$ represents a unit vector normal to dΣ and $\mathbf{1}_z$ is a unit vector directed vertically upward.

Evaluation of $\mathbf{F}_{em}$ as a volume integral of electromagnetic force density would require integration of infinite force density over the volume of infinitesimally thin current sheath (whose *outer* surface is Σ where the field intensity $\mathbf{B}$ vanishes). Hence, we write, alternatively $\mathbf{F}_{em}$ as a surface integral of the electromagnetic stress tensor $\mathscr{F}_{em}$ over the surfaces which bound the tube element viz.: cross-sections 'A' and '$A + \delta A$' *at* 's' and 's + ds', respectively, and the portion of Σ between s and s + ds. Thus:

$$\mathbf{F}_{em} = \int_A \mathscr{F}_{em} \cdot \mathbf{1}_s\, \mathrm{d}S + \int_{A+\delta A} (\mathscr{F}_{em} + \delta\mathscr{F}_{em}) \cdot (\mathbf{1}_s + \delta\,\mathbf{1}_s)\, \mathrm{d}S + \int_\Sigma \mathscr{F}_{em} \cdot \mathbf{1}_\Sigma \mathrm{d}\Sigma\,,$$

where $\mathscr{F}_{em}$ is the electromagnetic stress tensor and δ represents increments corresponding to the increment ds in s.

The last term vanishes, for $\mathbf{B} = 0$ on Σ, and the first two terms together reduce to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(-\frac{\langle B^2 \rangle}{8\pi} + \frac{\langle B_s^2 \rangle}{4\pi}\right) A\, \mathbf{1}_s\right] \mathrm{d}s\,.$$

Thus the equation of equilibrium of the tube element reduces to:

$$- \langle \rho_i \rangle A g \mathbf{1}_z \,\mathrm{d}s - \int_\Sigma p_e \mathbf{1}_\Sigma \,\mathrm{d}\Sigma + \frac{\mathrm{d}}{\mathrm{d}s} [- \langle p_i \rangle A \mathbf{1}_s] \,\mathrm{d}s + $$

$$+ \frac{\mathrm{d}}{\mathrm{d}s}\left[\left(- \frac{\langle B^2 \rangle}{8\pi} + \frac{\langle B_s^2 \rangle}{4\pi}\right) A \mathbf{1}_s\right] \mathrm{d}s = 0 . \tag{1}$$

Since the flux tube is thin and isolated, the plasma 'displaced' by the tube element would also be in equilibrium in the same surrounding atmosphere with ρ_i and p_i identical to ρ_e and p_e. The equilibrium of the displaced plasma would imply

$$- \langle \rho_e \rangle A g \mathbf{1}_z \,\mathrm{d}s - \int_\Sigma p_e \mathbf{1}_\Sigma \,\mathrm{d}\Sigma + \frac{\mathrm{d}}{\mathrm{d}s} [- \langle p_e \rangle A \mathbf{1}_s] \,\mathrm{d}s = 0 . \tag{2}$$

Subtracting Equation (2) from Equation (1) we obtain the following equations for equilibrium of the element:

$$\frac{\mathrm{d}Q}{\mathrm{d}s} = \langle \Delta\rho \rangle g \pi R^2 \sin\theta \tag{3a}$$

and

$$Q \frac{\mathrm{d}\theta}{\mathrm{d}s} = \langle \Delta\rho \rangle g \pi R^2 \cos\theta , \tag{3b}$$

where

$$Q = \left[\langle \Delta p \rangle + \frac{\langle B_s^2 \rangle}{4\pi} - \frac{\langle B^2 \rangle}{8\pi} \right] \pi R^2 , \tag{4}$$

$$\Delta p = p_e - p_i , \tag{5a}$$

$$\Delta\rho = \rho_i - \rho_e . \tag{5b}$$

R is the radius of the tube at 's' and θ is the angle made by $\mathbf{1}_s$ with horizontal.

2.2. Equations for local equilibrium along the cross-section planes

The plasma parameters p_e and ρ_e are functions of z specified by the external stratification and satisfy

$$\frac{\mathrm{d}p_e}{\mathrm{d}z} = - \rho_e g . \tag{6a}$$

Since $\mathbf{B}$ is force-free inside the tube, p_i and ρ_i are functions of z, to be determined, satisfying

$$\frac{\mathrm{d}p_i}{\mathrm{d}z} = -\rho_i g\,, \tag{6b}$$

and one must have

$$p_i = p_{i*} \tag{7}$$

along *each horizontal line* within any cross-section of the tube, where suffix '$*$' represents the value at a point *just inside the boundary*. The equilibrium of the thin current at the boundary requires

$$p_{i*} + B_*^2/8\pi = p_e\,. \tag{8}$$

Thus

$$p_i = p_e - B_*^2/8\pi \tag{9}$$

on *all horizontal lines* in the cross-section.

Dividing the cross-section into thin horizontal strips and integrating over the cross-section, we obtain

$$\langle \Delta p \rangle = \langle p_e \rangle - \langle p_i \rangle = \frac{B_*^2}{8\pi}\,, \tag{10}$$

where $\langle p_e \rangle$ is the same average as in Equation (2) of the 'displaced' plasma, and B_* has the same value at all points on the circumference of the cross-section (owing to the assumed axisymmetry of the field), and is thus a function of the arc length s alone.

The average $\langle p_i \rangle$ can be expressed in terms of the value on the axis, the radius of the cross-section R to its second power, and the inclination θ of the axis to horizontal, in the following manner:

$$\pi R^2 \langle p_i \rangle = \pi R^2 p_{ia} + 2\left(\frac{\mathrm{d}p_{ia}}{\mathrm{d}z}\right)\cos\theta \int_{-R}^{+R} x(R^2 - x^2)^{1/2}\,\mathrm{d}x\, +$$

$$+\left(\frac{\mathrm{d}^2p_{ia}}{\mathrm{d}z^2}\right)\cos^2\theta \int_{-R}^{+R} x^2(R^2 - x^2)^{1/2}\,\mathrm{d}x =$$

$$= \pi R^2 p_{ia} + \frac{\pi R^4}{8}\left(\frac{\mathrm{d}^2p_{ia}}{\mathrm{d}z^2}\right)\cos^2\theta\,, \tag{11}$$

i.e.,

$$\langle p_i \rangle = p_{ia} + \frac{R^2}{8}\left(\frac{\mathrm{d}^2p_{ia}}{\mathrm{d}z^2}\right)\cos^2\theta\,, \tag{11}$$

where the suffix 'a' denotes the *values on the axis*, and x represents the distance of a horizontal strip of thickness $\mathrm{d}\chi$ from the centre of the cross-section. Similar expressions hold for $\langle p_e \rangle$, $\langle \rho_e \rangle$, and $\langle \rho_i \rangle$.

Equation (10) then takes the form

$$\Delta p_a + \frac{R^2}{8}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\Delta p_a)\right]\cos^2\theta = \frac{B_*^2}{8\pi}, \tag{12}$$

where

$$\Delta p_a = p_{ea} - p_{ia}.$$

3. Mathematical Closure of the System of Equations

Following Browning and Priest (1983), the components of a weakly twisted axisymmetric force-free field can be written up to second terms in radial coordinate in a local cylindrical coordinate system r, ψ, s as:

$$B_s = 2f - \frac{r^2}{2}(f'' + b^2 f), \qquad B_\psi = brf, \qquad B_r = -rf', \tag{13}$$

assuming $bR \ll 1$, where

$$f \equiv f(s), \quad f' \equiv \frac{\mathrm{d}f}{\mathrm{d}s}, \quad f'' \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}s^2}. \tag{14}$$

Using these expressions, we obtain up to terms in R^2,

$$B_*^2 = 4f^2 - 2R^2 f(f'' + b^2 f) + b^2 R^2 f^2 + R^2 f'^2, \tag{15a}$$

$$\langle B^2 \rangle = 4f^2 - R^2 f(f'' + b^2 f) + \tfrac{1}{2} b^2 R^2 f^2 + \tfrac{1}{2} R^2 f'^2, \tag{15b}$$

$$\langle B_s^2 \rangle = 4f^2 - R^2 f(f'' + b^2 f). \tag{15c}$$

Equations (4), (10), and (15) yield

$$Q = \frac{R^2}{8}\left[(8 - \tfrac{5}{2} b^2 R^2) f^2 - 3R^2 ff'' + \tfrac{1}{2} R^2 f'^2\right]. \tag{16}$$

Equation (12) takes the form

$$8\pi\Delta p_a + \pi R^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2}(\Delta p_a)\cos^2\theta = (4 - b^2 R^2) f^2 - 2R^2 ff'' + R^2 f'^2. \tag{17}$$

The constancy of the longitudinal magnetic flux in the tube up to second order terms in R^2 requires

$$\int B_s\, \mathrm{d}A \equiv 2f\pi R^2 - \frac{\pi R^4}{4}(f'' + b^2 f) = F, \tag{18}$$

where F is the longitudinal magnetic flux in the tube. Equations (6a) and (6b) yield together:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\Delta p_a) = -\Delta\rho_a g \sin\theta , \tag{19}$$

where

$$\Delta p_a = p_e - p_{ia} , \qquad \Delta\rho_a = \rho_e - \rho_{ia} .$$

By virtue of Equations (4), (5), (11), (15), and (16), Equations (3a), (3b), (17), (18), and (19) define a closed system of five differential equations in the five functions ρ_{ia}, p_{ia}, f, R, and θ. Thus it is not necessary to adopt any auxiliary about the temperature of the plasma in the tube for closing the system of equations. Nor will it be possible to accommodate any *ad hoc* assumption about the plasma temperature since the temperature on the axis T_{ia} will be determined by the equation of state in terms if ρ_{ia} and p_{ia} as given by the above system.

Acknowledgements

We thank an unknown referee for his keen interest and useful comments on the earlier version of this paper.

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