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ENERGY TRANSPORT IN INTENSE FLUX TUBES ON THE SUN. I. EQUILIBRIUM ATMOSPHERE

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ABSTRACT

Energy transport in intense magnetic structures on the Sun is studied. The present investigation focuses on the equilibrium problem and attempts to determine the atmosphere within a flux tube in a self-consistent manner. A new feature of the present analysis is the use of the generalized Eddington approximation in three dimensions for modeling radiative transfer in an axisymmetric thin flux tube. Convective energy transport is also included using a mixing length formalism and introducing an efficiency parameter α, which is a crude measure of the magnetic inhibition of convection. The equilibrium equations are solved in the thin flux tube approximation. Model atmospheres corresponding to various magnetic field strengths, parameterized by β_0, are constructed, using the linearization method of Auer and Mihalas and the Feautrier technique. The results indicate that the temperature on the axis of a tube is generally lower than the ambient medium at the same geometric height. However, at equal vertical optical depths, the temperature inside exceeds the outside value. For an optical depth of unity, this difference is typically ~ 500 K. However, in the optically thin layers horizontal exchange of heat is efficient and the temperature inside the tube is found to be insensitive to the magnetic field strength. Finally, it is demonstrated that the equilibrium stratification is almost independent of the degree of convective inhibition.

Subject headings: hydromagnetics — Sun: atmosphere — Sun: magnetic fields

I. INTRODUCTION

The existence of magnetic elements, with field strengths in the range 1–2 kG, in the solar atmosphere appears to be generally accepted (e.g., Beckers and Schröter 1968; Frazier and Stenflo 1972). These elements or intense flux tubes (hereafter IFT) provide a channel for carrying energy from the convection zone to the upper atmosphere. Energy can be transported in IFTs in several ways, such as by mechanical motions, by radiation, and by convection (in regions where the temperature gradient is superadiabatic). This paper forms the first part of a quantitative examination of this subject. In the present investigation we neglect mechanical effects and concentrate solely on radiative and convective energy transport. The inclusion of motions will be considered in a subsequent paper, where we treat the nonlinear time-dependent problem.

The aim of this study is to present model calculations, which shed some light on physical conditions within IFTs when dynamical effects can be neglected. Observations generally suggest that flows, although important, are in a time-averaged sense fairly small (Stenflo and Harvey 1984). This is also supported by theoretical calculations (Hasan 1984*a*, *b*). Thus, our investigation provides an "average" model atmosphere in a flux tube. This is particularly useful for studies involving time-dependent problems, where an initial equilibrium state is often required. Usually the equilibrium is constructed by making the ad hoc assumption that the temperatures inside and outside the flux tube are equal. A self-consistent treatment for determining the thermodynamic state within a flux tube would involve solving the energy equation. We attempt to do this, using a quasi-one-dimensional treatment.

Theoretically, the equilibrium problem for IFTs has been studied by Spruit (1976) and more recently by Ferrari *et al.* (1985) (see also Kalkofen *et al.* 1986). Spruit uses a two-dimensional treatment, in which an IFT is envisaged as a vertical cylinder of uniform thickness. He assumes anisotropic convective heat transport, which within the flux tube is purely vertical and occurs with a reduced efficiency, due to the presence of a strong field. Radiative transport is treated in the diffusion approximation. Ferrari *et al.* use the thin flux tube approximation, i.e., they neglect horizontal variations within the tube, and examine the vertical structure of a tapered flux tube, assuming purely radiative heat transport within the flux tube. Thus, although such an approach is not a satisfactory way to examine the horizontal structure, especially at the interface between the flux tube and the ambient medium, it has the advantage that it permits a more accurate treatment of radiative transport. The problem is particularly acute close to the photospheric surface (i.e., where the continuum optical depth is about unity) in the external atmosphere and in the higher layers, where the diffusion approximation completely breaks down.

In this paper, the thin flux tube approximation is used for reasons of mathematical convenience. Following Spruit (1976), we assume that convective energy transport within the flux tube occurs only along the field, but with a reduced efficiency. Like Ferrari *et al.* (1985), we do not employ the diffusion approximation. However, our treatment of radiative transport is different. Our aim is to combine the attractive features of both models. There are, however, a number of differences in our analyses. These will be discussed in a later section.

The plan of this paper is as follows: in § II, the model equations, which describe the equilibrium inside and outside the flux tube, are first presented, followed in § III by a description of the method of solution. In § IV and V we examine and discuss the results, comparing them with the findings of other authors as well as pointing out some observational implications. The main conclusions of the study are finally summarized in § VI.

II. MODEL AND EQUATIONS

Let us consider a flux tube of circular cross section extending vertically through the photosphere and convection zone. We adopt a cylindrical coordinate system (r, θ, z) with the z-axis along the axis of symmetry of the flux tube and pointing into the Sun. It is convenient to treat the internal and external atmospheres separately.

a) Internal Atmosphere

We shall work within the framework of the thin flux tube approximation (Roberts and Webb 1978). The approximation involves expanding all quantities about the axis of the tube in terms of a parameter ϵ, which is the ratio of the tube radius to a typical length scale in the vertical direction (e.g., the pressure scale height). Assuming that none of the quantities has a θ dependence, the time-independent MHD equations to zeroth order are

$$\frac{dp}{dz} = \rho g \,, \tag{1}$$

$$p + \frac{B^2}{8\pi} = p_e \,, \tag{2}$$

$$\text{div}\, \boldsymbol{F} = 0 \tag{3}$$

where p, ρ, B, $\boldsymbol{F}$, and g denote pressure, fluid density, magnetic field, total energy flux, and acceleration due to gravity, respectively. All variables are evaluated on the tube axis ($r = 0$). Equations (1) and (3) express the conditions of hydrostatic and energy equilibrium within the tube, respectively. Equation (2) follows from the condition that the tube is in pressure balance with the external medium. We denote quantities in the external medium by the subscript e. It has been assumed that, to zeroth order, the magnetic field is purely vertical. The radial component B_r, which is a first-order quantity, can be determined from

$$\text{div}\, \boldsymbol{B} = 0 \,. \tag{4}$$

The radius of the tube a is determined using flux conservation as

$$Ba^2 = \text{constant} \,. \tag{5}$$

We relate $\boldsymbol{P}$ and ρ to the temperature, through the equation of state for an ideal gas

$$p = \rho RT/\mu \,, \tag{6}$$

where μ is the mean molecular weight, which is in general a function of density and temperature, and R is the gas constant.

The energy flux $\boldsymbol{F}$ can be written as

$$\boldsymbol{F} = \boldsymbol{F}_R + \boldsymbol{F}_c + \boldsymbol{F}_{\text{ex}} \,,$$

where $\boldsymbol{F}_R$, $\boldsymbol{F}_c$, and $\boldsymbol{F}_{\text{ex}}$ denote the radiative, convective, and extra fluxes, respectively.

b) Radiative Transfer in the Flux Tube

In order to treat radiative transport in the tube, we use the generalization of the transfer equation, in the Eddington approximation, applicable in three dimensions, following Unno and Spiegel (1966). For a gray and static atmosphere in local thermodynamic equilibrium (LTE), the radiative flux defined as

$$\boldsymbol{F}_R = -\frac{4\pi}{3\kappa\rho} \nabla J \tag{7}$$

is related to the mean intensity J through

$$\text{div}\, \boldsymbol{F}_R = 4\pi\kappa\rho(S - J) \,, \tag{8}$$

where $S = \sigma T^4/\pi$, κ is the Rosseland mean opacity, and σ is the Stefan-Boltzmann constant. Expanding J about $r = 0$, equation (8) yields to zeroth order (see Appendix A for details)

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} + \frac{4}{3}\left(\frac{J_e - J}{\tau_a^2}\right) = J - S \,, \tag{9}$$

where $d\tau = \kappa\rho\, dz$, $\tau_a = \kappa\rho a$, and J_e is the mean intensity in the external medium. The second term on the left-hand side of equation (9) is the contribution due to heat exchange between the flux tube and the ambient medium. When $\tau_a \to 0$, which corresponds to the optically thin limit, equation (9) yields $J \to J_e$. This approximation is likely to hold in the upper layers of the photosphere. Deep in the convection zone, which is optically thick, equation (9) reduces to $J = S$. From equation (7), we find

$$\boldsymbol{F}_R = -\frac{4\pi}{3\kappa\rho} \nabla S$$

which is the diffusion approximation.

c) Convective Transport within theFlux Tube

In regions of the flux tube, where the temperature gradient is superadiabatic, convective energy transport can occur. Using a mixing length approach, the vertical convective energy flux is given by (Cox and Giuli 1968, pp. 281–325)

$$F_{cz} = -\frac{\alpha(B^2)}{4\sqrt{2}}(gQ\lambda p)^{1/2}(\rho C_p T)\left(\frac{\Lambda}{\lambda_p}\right)^2(\nabla - \nabla')^{3/2}\,, \tag{10}$$

where $\lambda_p^{-1} = d \ln p/dz$, Λ is the mixing length, $\nabla = d \ln T/d \ln z$, C_p is the specific heat at constant pressure, $Q = 1 - (\partial \ln \mu/\partial \ln T)_p$, and $\alpha(B^2)$ parameterizes the suppression of convection by the magnetic field. The quantity α in general depends inversely upon the magnitude of the field strength. The prime denotes a value within a convective element. We neglect radial convective energy transport in the tube, following Spruit (1976), so that

$$F_{cr} = 0\,, \tag{11}$$

We eliminate ∇' by using a further relation that defines Γ, the efficiency of convective transport. This yields (e.g., Mihalas 1978)

$$F_{cz} = 4\sigma T^4\left(\frac{\Lambda}{\lambda p}\right)\alpha(B^2)\frac{\tau_c}{1 + \tau_c^2/2}\frac{\Gamma^3}{A^2}\,, \tag{12}$$

with

$$\Gamma = \tfrac{1}{2}[\sqrt{1 + 4A^2(\nabla - \nabla_a)} - 1]\,, \tag{13}$$

$$A = \frac{\rho C_p(gQ\lambda p)^{1/2}}{16\sqrt{2}\sigma T^3}\left(\frac{\Lambda}{\lambda_p}\right)\frac{1 + \tau_c^2/2}{\tau_c}\,, \tag{14}$$

where $\tau_c = \kappa\rho\Lambda$ and ∇_a denotes the adiabatic gradient.

Combining equations (3), (8), and (11), we obtain the following equation for the energy equilibrium within the flux tube

$$\frac{\partial F_c}{\partial \tau} + Q_{ex} = 4\pi(J - S)\,, \tag{15}$$

where $F_c = F_{cz}$ is given by equation (10), J by equation (9), and Q_{ex} denotes any additional heat inputs. This additional term corresponds to a flux F_{ex}, defined as

$$Q_{ex} = \frac{\partial F_{ex}}{\partial \tau}\,.$$

d) External Atmosphere

In the external medium, we assume a plane-parallel atmosphere in hydrostatic and energy equilibrium. Following Ferrari *et al.* (1985), we neglect the effect of the flux tube on the external medium; i.e., we disregard the boundary layer which separates the interior of the flux tube with the exterior.

Equations (1), (3), and (8) also hold for the external medium. Assuming only a z variation (i.e., a plane-parallel atmosphere), equation (3) can be integrated to yield

$$F_R^{(e)} + F_c^{(e)} + F_{ex}^{(e)} = F_{Sun}(\text{constant})\,, \tag{16}$$

where $F_{ex}^{(e)}$ is any additional flux and F_{Sun} denotes the observed flux in the photosphere. We have dropped the z index for convenience. The radiative flux in the Eddington approximation is

$$F_R^{(e)} = -\frac{4\pi}{3\kappa_e\rho_e}\frac{\partial J_e}{\partial z}\,. \tag{17}$$

The mean radiation intensity J_e satisfies

$$\frac{1}{3}\frac{\partial^2 J_e}{\partial \tau_e^2} = (J_e - S_e)\,, \tag{18}$$

where

$$d\tau_e = \kappa_e\rho_e dz\,. \tag{19}$$

The external convective flux $F_c^{(e)}$ is given by a similar expression as equation (13), with $\alpha = 1$.

III. METHOD OF SOLUTION

The procedure we employ is essentially the partial linearization method of Auer and Mihalas (1968), as adapted by Gustafsson (1971), apart from a few modifications. We shall describe the method briefly (further details can be found in the original references).

Starting from an initial "guess" atmosphere, the equations are solved iteratively. Let us first consider the medium outside the tube.

a) External Atmosphere

In order to begin the iteration we need to input guess values for all quantities. For the first iteration we use the combined model atmospheres of Vernazza, Avrett, and Loeser (1976) (VAL for short) and Spruit (1977, pp. 26–34). The mean radiation intensity can be determined by solving equations (17)–(19). This is best carried out numerically, using finite differences with the following upper and lower boundary conditions

$$\frac{1}{\sqrt{3}}\frac{\partial J_e}{\partial \tau_e} = J_e , \tag{20}$$

and

$$\frac{1}{\sqrt{3}}\frac{\partial J_e}{\partial \tau_e} = -J_e + I_e^+ , \tag{21}$$

respectively, where I_e^+ is the radiation intensity in the upward direction which, in the diffusion approximation, is

$$I_e^+ = S_e + \frac{1}{\sqrt{3}}\frac{1}{\kappa_e \rho_e}\frac{\partial S_e}{\partial z} . \tag{22}$$

We estimate the extra flux entering equation (16) using the following relation

$$F_{\text{ex}}^{(e)} = \begin{cases} F_{\text{Sun}} - F_R^{(e)} & z < 0 , \\ 0 & z \geq 0 , \end{cases} \tag{23}$$

where $z = 0$ corresponds to $\tau_{5000} = 1$ and $F_{\text{Sun}} = -6.284 \times 10^{10}$ erg cm^{-2} s^{-1} (Allen 1963). The reason for invoking an additional energy flux, $F_{\text{ex}}^{(e)}$ in the photosphere, is to fit the temperature in these layers, as closely as possible, with the semiempirical VAL model atmosphere. For $z < 0$, the convective flux is zero, and, therefore, the equation for energy equilibrium becomes

$$\frac{\partial F_{\text{ex}}^{(e)}}{\partial \tau} + \frac{\partial F_R^{(e)}}{\partial \tau} = 0 .$$

Using equations (17), (18), and (23), we find

$$Q_{\text{ex}}^{(e)} = \frac{\partial F_{\text{ex}}^{(e)}}{\partial \tau_e} = 4\pi(J_e - S_e) .$$

Thus $Q_{\text{ex}}^{(e)}$ is a measure of the extent to which the external medium deviates from radiative equilibrium. This deviation could possibly reflect the error in the gray and angle approximations and may not, in fact, have anything to do with actual energy input (the author is thankful to the referee for pointing this out).

In the convection zone ($z \geq 0$), the condition given by equation (16) will, in general, not be satisfied using the guess stratification. To determine the stratification, which satisfies equation (16), we linearize F_c with respect to T, p, and $\nabla = d \ln T/d \ln p$. Thus,

$$F_c = F_c^{(0)} + \left(\frac{\partial F_c}{\partial p}\right)^{(0)} \delta p + \left(\frac{\partial F_c}{\partial T}\right)^{(0)} \delta T + \frac{\partial F_c^{(0)}}{\partial \nabla}\frac{p}{T}\,\delta\left(\frac{dT}{dp}\right) \tag{24}$$

(Gustafsson 1971), where $F_c^{(0)}$ is the flux obtained from the previous iteration and the δ's denote the corrections.

It is convenient to linearize F_R with respect to J, so that

$$F_R = F_R^{(0)} + \frac{\partial F_R^{(0)}}{\partial J}\,\delta J . \tag{25}$$

Let us now consider the transfer equation (eq. [18]) and linearize S with respect to T, so that

$$S = S^{(0)} + \frac{\partial S^{(0)}}{\partial T}\,\delta T , \tag{26}$$

where $\partial S^{(0)}/\partial T = 4\sigma T^3/\pi$. Finally, we require a relationship involving δp, which is provided by the equation of hydrostatic equilibrium. When integrating the latter equation, we assume that the pressure at the top remains fixed at the value taken from the VAL atmosphere.

Thus, we have a system of linear equations in δJ, δT, and δP which can be solved to determine new values of J, T, and p. From the ideal gas law, ρ can be determined. We determine the opacity by treating it as a function of ρ and T and interpolating from a table by Kurucz (1978). Corrections are now calculated to the new variables (keeping $F_{\text{ex},\,e}$ fixed), and the process is repeated until the maximum change in temperature, during the iteration process, becomes less than a degree. After each iteration, the ionization of hydrogen is self-consistently determined using Saha's equations so that μ can be calculated. The thermodynamic quantities are also updated at each step, following Mihalas (1967).

We use a numerical procedure to solve the linear equations. Let us divide the integration region into a grid, with points located at z_k; $k = 0, 1, \ldots, N$. Using finite differences to express derivatives, the equations can be cast in the form

$$-A_k X_{k-1} + B_k X_k - C_k X_{k+1} = D_k , \qquad k = 1, 2, \ldots, N - 1 , \tag{27}$$

with $X_k = (\delta J_k, \delta T_k, \delta p_k)^T$, where A_k, B_k, C_k are matrices of order 3 × 3 and D_k is a column vector of order 3. Details of the explicit structure of A, B, C, and D are given in Appendix B. We complete the system of equations by using boundary conditions. Two boundary conditions are needed for δJ, since it is determined from a second-order differential equation. These are obtained from the linearized forms of equations (20)–(22). In addition we need to specify only one boundary condition in the equation of hydrostatic equilibrium (eq. [1]), since it involves a first-order spatial derivative. We specify the pressure p_0 and flux at $z = z_0$. The boundary conditions yield

$$B_0 X_0 - C_0 X_1 = 0 , \tag{28}$$

and

$$-A_N X_{N-1} + B_N X_N = 0 . \tag{29}$$

Equations (27)–(29) can be combined to yield the following matrix equation:

$$\begin{bmatrix} B_0 & -C_0 & & & \\ -A_1 & B_1 & -C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -A_{N-1} & B_{N-1} & -C_{N-1} \\ & & & -A_N & B_N \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \\ X_N \end{bmatrix} = \begin{bmatrix} D_0 \\ D_1 \\ \vdots \\ D_{N-1} \\ D_N \end{bmatrix} \tag{30}$$

The coefficient matrix on the left-hand side of equation (30) has a block tridiagonal structure. We solve for the unknowns X_0, X_1, X_N using the Feautrier method, described in Mihalas (1978). This procedure is carried out numerically using a standard algorithm for inverting the 3 × 3 blocks.

b) *Flux Tube Atmosphere*

The atmosphere within the flux tube is determined iteratively, starting from the guess

$$T = T_e \tag{31}$$

at each height. Equations (1), (2), (6), and (31) imply that β, defined as $\beta = 8\pi p/B^2$, is constant with z (if the dependence of μ is neglected). The pressure and density are thus determined from

$$p = \frac{\beta}{\beta + 1} p_e , \tag{32}$$

$$\rho = \frac{\beta}{\beta + 1} \rho_e . \tag{33}$$

As before, the equations are linearized with respect to the same variables. The only difference is that we use the energy equation (15) and determine the mean radiation intensity using equation (9). Above the convection zone (i.e., where $\nabla < \nabla_a$), $F_c = 0$, which implies, on using equation (15), that

$$Q_{\rm ex} = 4\pi(J - S) . \quad 34)$$

Here $Q_{\rm ex}$ is a measure of the departure from radiative equilibrium within the tube. We choose

$$Q_{\rm ex} = Q_{\rm ex}^{(e)} . \tag{35}$$

Although the choice given by equation (35) may appear somewhat arbitrary, it is motivated primarily for mathematical reasons, so as to keep the influence of $Q_{\rm ex}$ on the stratification minimal. We estimate the effect of $Q_{\rm ex}$ by considering the case when it is zero [$Q_{\rm ex} = Q_{\rm ex}^{(e)} = 0$] and demonstrate in § IV*b* that $T_e - T$ is barely altered. Incidentally, equation (35) implies that $F_{\rm ex} < F_{\rm ex}^{(e)}$, because $d\tau/d\tau_e < 1$. The latter condition holds owing to the tube being rarer than its surroundings.

Deep in the flux tube, where the atmosphere is optically thick, $J - S \to 0$, and equation (11) integrates to

$$F_c = cF_{\rm Sun}(\text{constant}) , \tag{36}$$

where c is a constant.

In practice, since the location of the level where convection begins, can vary during each iteration, we use the linearized versions of equations (8) and (15) along with equation (1) to self-consistently determine J, T, and p. At the top of the flux tube, we specify the pressure and mean intensity as follows:

$$p_0 = \frac{\beta_0}{\beta_0 + 1} p_0^{(e)} , \qquad J_0 = J_0^{(e)} ,$$

where β_0 is a fixed parameter and the subscript zero refers to values at the top of the tube located at $z = z_0$. Equations (2) and (5) allow B and a to be determined, for a fixed choice of β_0 and a_0. No boundary conditions are necessary on the energy equation (15), since we consider z_0 to be above the convection zone.

At the lower boundary, we use the energy equation (36). Details of the equations can be found in Appendix B. The quantities ρ, B, and a are calculated from equations (6), (2), and (5), respectively. We also recalculate μ, Q, κ, ∇_a, and C_p after each iteration, assuming them to be functions of ρ and T.

IV. RESULTS

In the calculations, the following choice of parameters was used: $\Lambda = 2\lambda_p$, $\alpha = 0.2$, $c = 0.2$, and $\beta = 1.5$. Upper and lower boundaries were taken at $z = -450$ km and $z = 1000$ km, respectively. The top boundary corresponds to a level in the atmosphere which is just below the temperature minimum and the lower boundary to a level in the convection zone. A uniform grid spacing of 10 km, corresponding to 145 mesh points, was found to be adequate to resolve the steepest of gradients. The numerical procedure required ~ 12 iterations for satisfactory convergence.

a) Variation of T_e and $F_{\rm ex}^{(e)}$ with z

Let us first consider Figure 1, which depicts the spatial variation in the external medium of the temperature T (*solid lines*) and the extra flux $F_{\rm ex}^{(e)}$ (*dashed lines*). Curve 1 corresponds to choosing $F_{\rm ex}^{(e)} = 0$, whereas curve 2 corresponds to choosing $F_{\rm ex}^{(e)}$ to fit the temperature to the VAL atmosphere. In the former case, the temperature is lower than the latter, apart from a small region close to the upper boundary. The magnitude of the extra flux never exceeds $\sim 0.2 F_{\rm Sun}$.

b) Variation of T, ΔT, and F with z

Figures 2*a* and 2*b* depict the spatial dependence of T, $\Delta T = T_e - T$, and the vertical flux $F = F_{R,z} + F_{c,z}$ (in units of $F_{\rm Sun}$), where $F_{R,z}$ is the Eddington flux, for finite and zero values of $F_{\rm ex}$, respectively. The corresponding scale values of the optical depth in the flux tube (assuming $\tau_0 = 0$) are also shown on the upper horizontal scale. Let us first focus our attention on Figure 2*a*. For $z \leq -100$ km, $\Delta T \approx 0$ because in these layers τ_a is very small and horizontal exchange of heat is very effective. On the other hand, in the layers just below $z = 0$, vertical radiative transport dominates and ΔT increases because $F_R < F_{R,e}$. In the deeper layers, for which $z \gtrsim 150$, the opacity becomes very large and the tube temperature is determined solely by convective energy transport. The temperature difference ΔT in these layers exists mainly due to the reduced efficiency of convective energy transport by the strong magnetic field. Let us consider the behavior of F (the sum of the vertical radiative and convective fluxes). Owing to lateral heat transport by radiation as well as owing to the extra heating in the photosphere, F is not constant with z. However, close to the top of the tube, $J \approx J_e$, so that

$$F = F_R \approx \left(\frac{\kappa_e \rho_e}{\kappa \rho}\right)[F_{\rm Sun} - F_{\rm ex}^{(e)}] , \tag{39}$$

where we have used equations (7), (19), and (23). The first term on the right-hand side is approximately constant, with the rough value $(\beta + 1)/\beta$. In the second term, $F_{\rm ex}^{(e)}$ remains fairly constant, with a value of about $-0.2 F_{\rm Sun}$, apart from $z \approx 0$, where it becomes negligibly small. Therefore, F varies also weakly in the upper layers of the tube. For $z \geq 0$, the radiative flux drops off sharply with depth owing to a rapid increase of opacity. The total flux in the deeper layers ($z \geq 200$ km) consists only of the convective flux, which remains constant.

It is instructive to plot the same quantities as in Figure 2*a* for $F_{\rm ex} = F_{\rm ex}^{(e)} = 0$. These are shown in Figure 2*b*. Although the

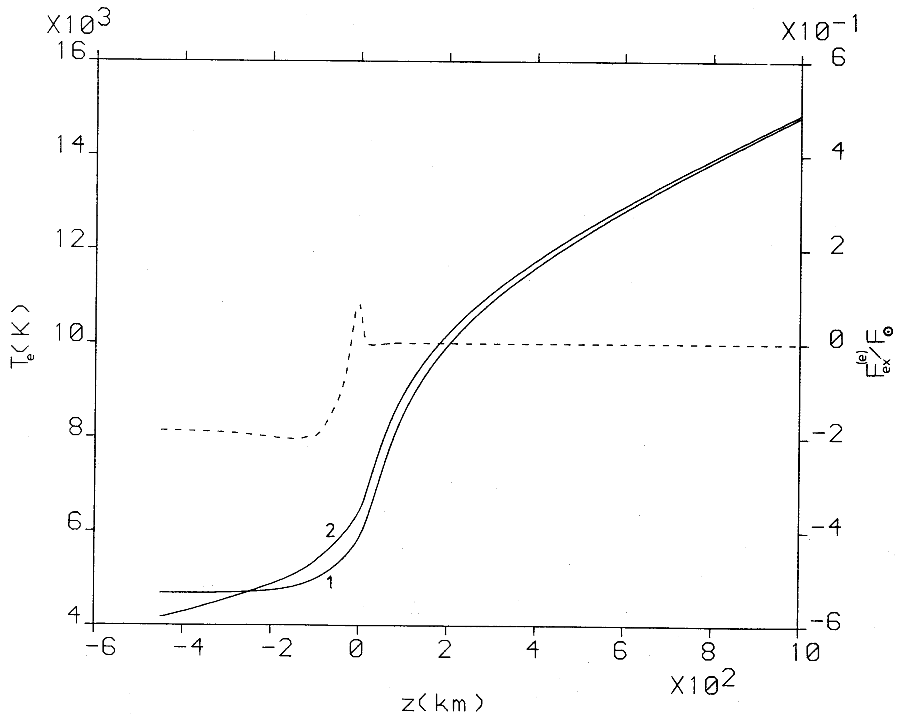

FIG. 1.—Variation of T_e (*solid lines*) and $F_{\rm ex}^{(e)}$ (*dashed line*) with z. Curves 1 and 2 correspond to $F_{\rm ex}^{(e)} = 0$ and choosing $F_{\rm ex}^{(e)}$ to fit the semiempirical VAL model atmosphere.

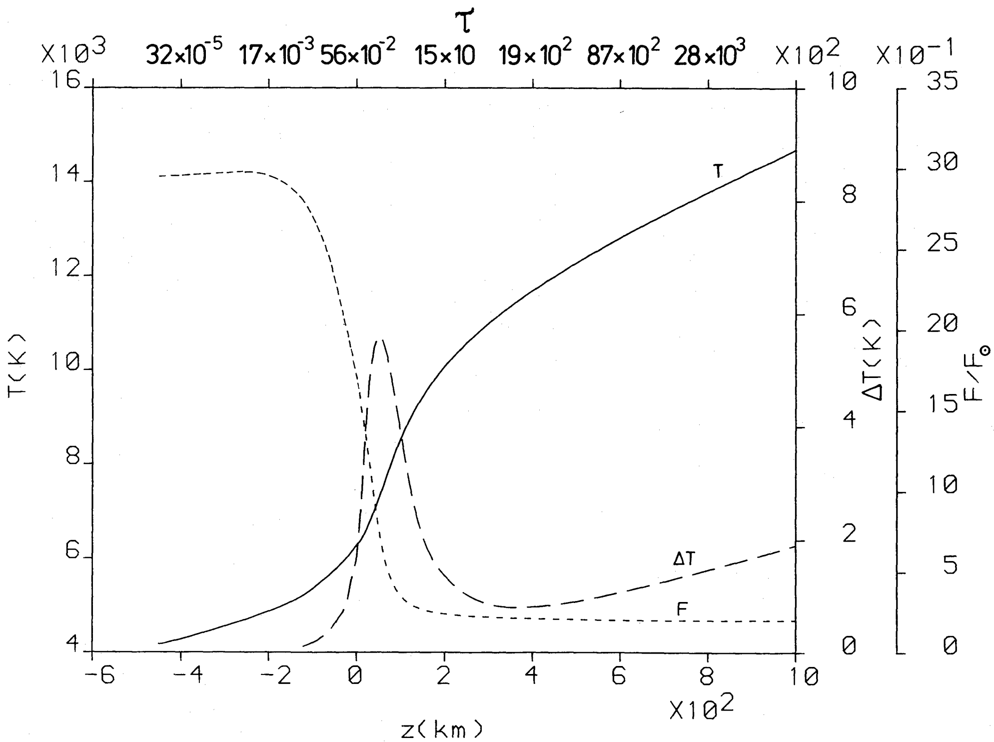

FIG. 2a

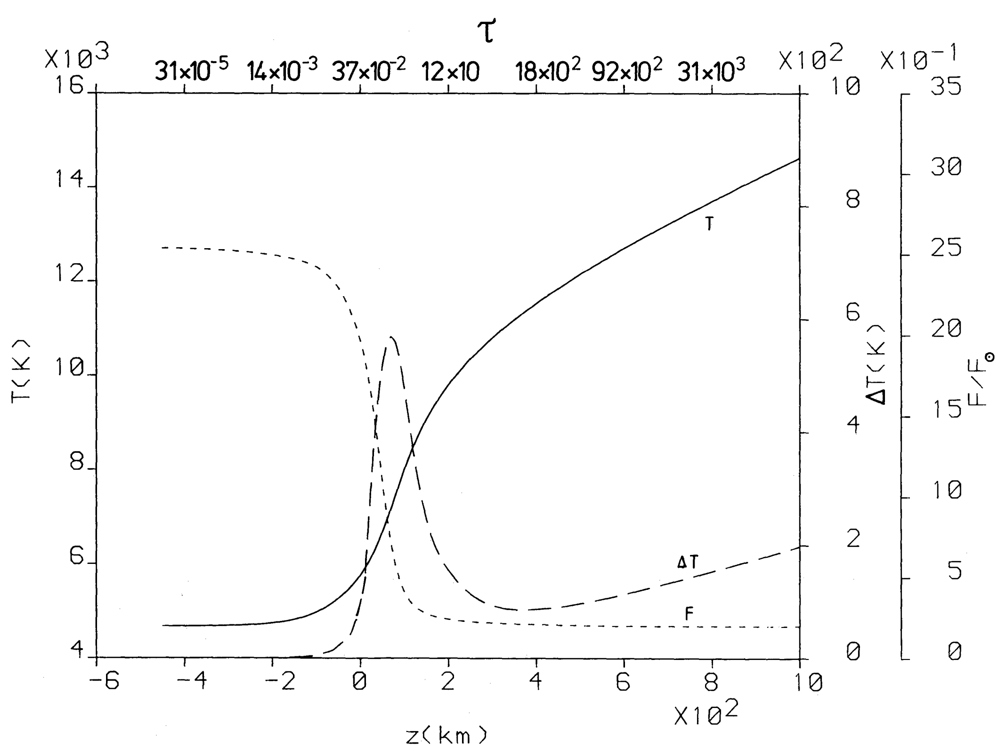

FIG. 2b

FIG. 2.—Variation of T (*solid line*), ΔT (*long dashes*), and F (*short dashes*) with z and τ (in the tube) for (*a*) finite F_{ex} and (*b*) $F_{ex} = 0$, assuming $\beta_0 = 1.5$ and $\alpha = 0.2$

temperature profile is slightly different than before, ΔT has roughly the same behavior. This is because ΔT is appreciably different from zero only in layers for which $z \geq 0$. The flux F has roughly the same variation as before, except that its absolute magnitude is lower because $F_{ex}^{(e)} = 0$.

In the ensuing discussion, we present results only for $F_{ex} \neq 0$. Since this extra flux is always small, it does not seem worthwhile to treat the possibility $F_{ex} = 0$ because the equilibrium values are only slightly different.

c) *Variation of β with z*

Figure 3 shows the dependence of β on z, for $\beta_0 = 1.5$, 3, and 5 (corresponding to the curves a, b, and c, respectively). For $z < 0$, β is practically constant with z, since $T_e/\mu_e \approx T/\mu$. However, for $z \geq 0$, ΔT is positive so that β increases with z, the slope increasing with β_0. In order to understand this behavior, we use equation (2) and the definition of β to find $d\beta/dz \propto \beta^2(T_e/\mu_e - T/\mu)$. This explains why $|d\beta/dz|$ increases with β and with ΔT.

d) *Variation of T and B with τ*

It is also of some interest to examine how various quantities change with τ. Observationally, it is the optical depth, which is more significant. In Figures 4*a* and 4*b*, the τ dependence of T and B for $\beta_0 = 1.5$, 3, and 5 are presented. For purposes of comparison, the variation with optical depth in the external atmosphere of T_e is also shown in Figure 4*a*. It should be borne in mind when comparing temperatures, corresponding to a fixed value of τ, that these are not at the same geometric depth. The temperature for any τ is in general higher in the tube compared to the external medium, with the difference increasing with decreasing β. This is because a reduction in β leads to a more rarefied tube, which implies that any optical depth level, say $\tau = 1$, occurs deeper in the tube where T is higher.

Turning our attention to the τ dependence of B, we find from Figure 4*b* that the field increases monotonically with τ. The field at any optical depth is inversely proportional to β_0, which is straightforward to understand.

e) *Effect of Varying α*

In the calculations, presented so far, we made the ad hoc assumption that α, parameterizing the reduction of convective efficiency by a strong magnetic field, was a constant with a value 0.2. We now consider the sensitivity of our results on α. Figure 5 depicts the z variation of T for two values of α, 0.8 and 0.2, assuming $\beta = 1.5$. It is clear that the temperature profiles are almost identical. This is to be expected in the upper layers of the tube where energy transport occurs mainly through radiation. In the deeper layers, where convection dominates, the temperature is slightly higher for $\alpha = 0.8$ compared to $\alpha = 0.2$. We have, therefore, demonstrated that a large change in F_c has a comparatively small effect on the temperature profile. This is possible because, although F_c varies linearly with α, it depends very sensitively on T (see eqs. [12]–[14]). Therefore, a large variation in α requires a comparatively small change in T, for a given value of F_c.

Let us briefly discuss the dependence of the temperature stratification on the magnetic field strength. In the top layers of the tube, the temperature variation is fairly insensitive to β, because τ_a is small and lateral heat exchange is efficient. On the other hand, in the convection zone τ is rather large, so that the tube is hotter than the outside at equal optical depths for reasons already explained.

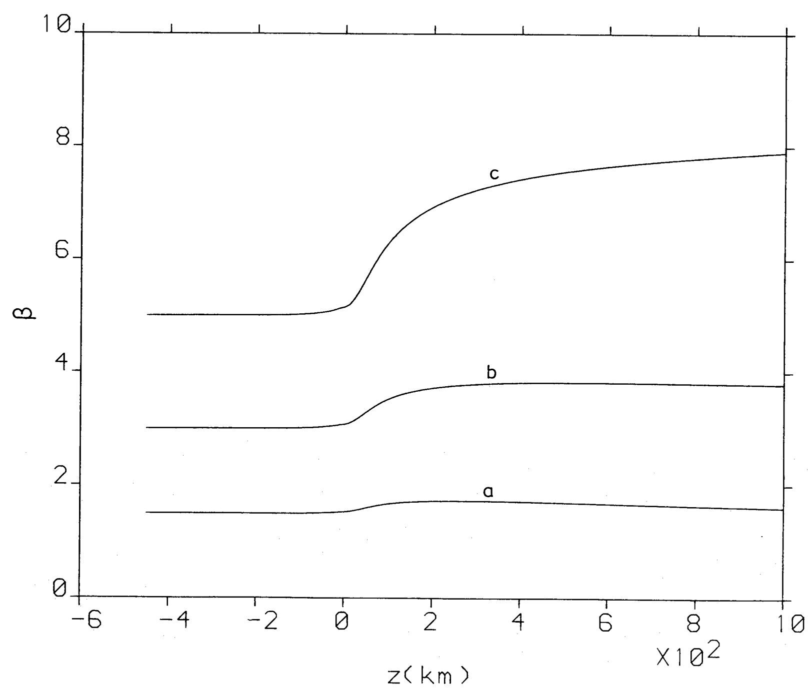

FIG. 3.—Spatial dependence of β for $\beta_0 = 1.5$, 3, and 5 corresponding to curves *a*, *b*, and *c*, respectively, assuming $\alpha = 0.2$

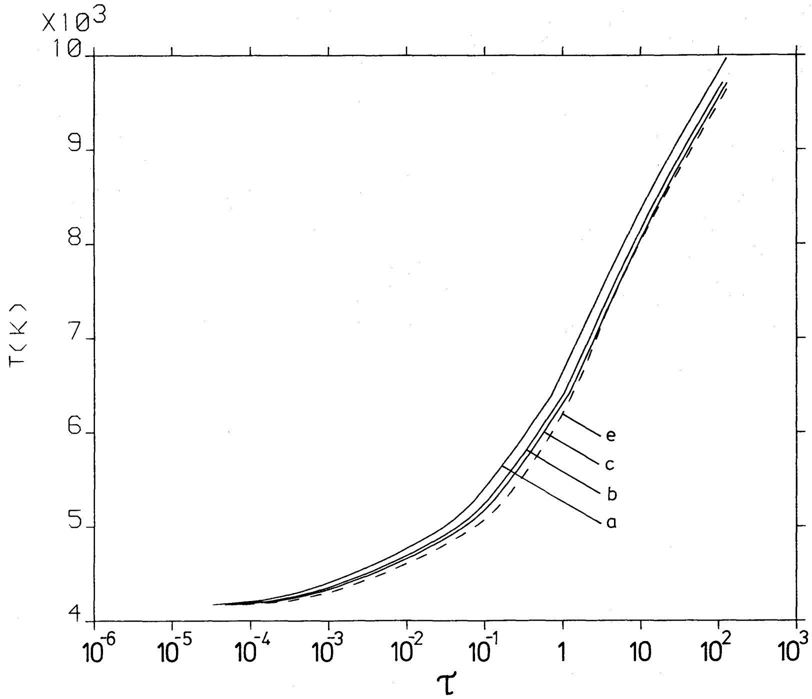

FIG. 4*a*

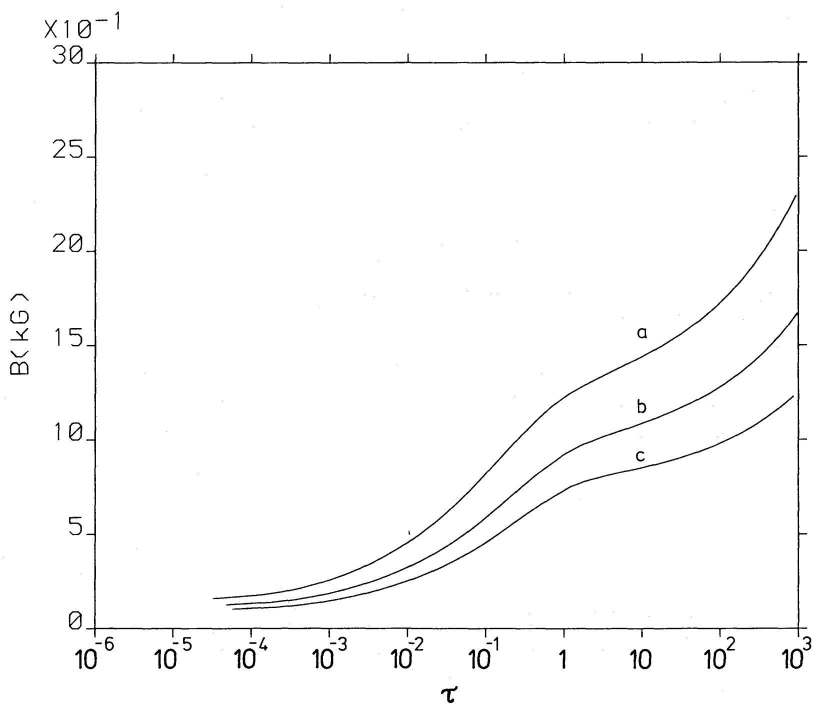

FIG. 4*b*

FIG. 4.—Variation with τ of (*a*) T and (*b*) B for different β_0, assuming $\alpha = 0.2$. Curves *a*, *b*, and *c* denote the values 1.5, 3, and 5 for β_0 and *e* denotes T_e.

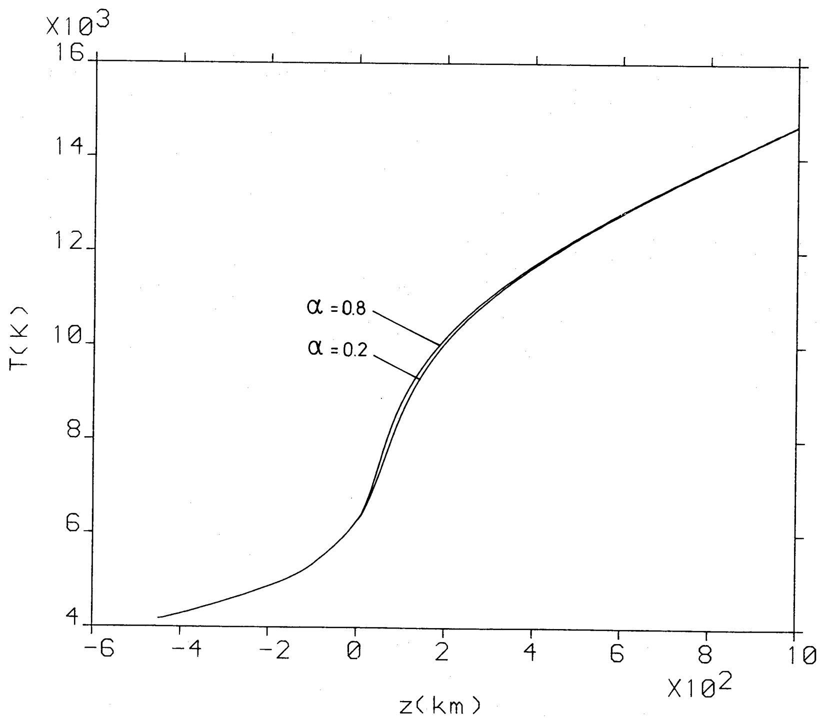

FIG. 5.—Variation of T with z for $\alpha = 0.2$ and $\alpha = 0.8$, assuming $\beta_0 = 1.5$

V. DISCUSSION

a) General Comments

We have used the linearization technique of Auer and Mihalas (1968), as modified by Gustafsson (1971), to solve the equilibrium equations for a flux tube. The numerical method, although computationally fairly robust, is not entirely free of pitfalls. A major problem can occur if at any stage during the iterations, the temperature gradient becomes subadiabatic in the convection zone. This leads to an enormous increase in temperature in the optically thick regions, since the required flux can only be transported by radiation. In the following iteration, the convective flux becomes extremely large, leading to a virtual breakdown of the numerical scheme. Gustafsson (1971) has suggested a method to overcome this difficulty, although for the flux tube atmosphere it did not always work. Fortunately, this limitation did not prove unduly restrictive on the parameter range that we could treat.

Our treatment of convective transport deserves some mention. In addition to the uncertainties of treating convection using a mixing length theory, there is the further question of how the strong magnetic field will modify energy transport in an IFT? We chose to parameterize this effect by introducing an efficiency parameter $\alpha(B^2)$ in the standard expression for F_{cz}, the convective flux in the vertical direction. It was assumed that the magnetic field suppresses horizontal energy transport. As regards the functional form of α, our choice of a constant value deserves comment. Let us first consider the possibility $\alpha = 0$. In this case, energy transport can only occur through radiation. However, in the deeper layers of the convection zone, the supply of energy into the flux tube will be virtually blocked off. Table 1 shows the values of a and τ_a at different z. The convection zone begins roughly at $z = 0$. Below this level, the magnetic field strength increases and the tube narrows. However, the decrease in a is offset by the sharp rise in opacity, so that τ_a increases with z. For instance at a depth of 400 km, $\tau_a = 940$. Thus, a more likely alternative is that although the field reduces convective transport, it does not completely stop it. We arbitrarily selected $\alpha = 0.2$. However, our results indicate that the thermodynamic state of the flux tube atmosphere is comparatively insensitive to α.

In our calculations we assumed that the atmosphere in an IFT is in hydrostatic equilibrium. We justified this in the Introduction, on the basis, that our aim was to provide an "average" atmospheric model. We appealed to observations and numerical calculations, which indicate that time-averaged flows may be small. This does not necessarily mean that flows are unimportant. Although

TABLE 1

TUBE RADIUS a AND τ_a IN A FLUX TUBE

z (km)	a (km)	τ_a
−200.........	106	2.4×10^{-2}
0.........	74	1.1
200.........	59	140
400.........	48	940

systematic flows may be absent in flux tubes (Stenflo and Harvey 1984), oscillatory motions may be present and may contribute to asymmetries in V line profiles. This topic will form the basis of a subsequent paper in this series.

b) *Observational Implications*

Let us consider the level in the flux tube where the continuum optical depth $\tau_{5000} = 1$. The geometric displacement of this level relative to the level in the external medium, where the same condition holds, defines the Wilson depression z_W. Roughly speaking, we consider $z = z_W$ as the "observable level" in an IFT with reference to an observation at disk center. We have already found that the temperature in an IFT is greater than the ambient medium for the same optical depth because of its reduced density. In addition, there is also the possibility that the extra heating term Q_{ex} could contribute to this difference. However, as shown in § IV*b*, the effect is small for $z \lesssim 0$. Furthermore, at $z = z_W$, $Q_{ex} = 0$.

Thus, observationally, IFTs would appear brighter than their surroundings, which indeed appears to be the case. In Table 2, the values of various quantities are presented at $= z_W$, corresponding to different β_0. As an illustration, consider $\beta_0 = 1.5$ for which $B(z_W) = 1228$ G. This value is within the observed range. The Wilson depression z_W is 25 km and $T(z_W) = 6678$ K, which corresponds to a temperature excess of ~ 500 K over the surrounding atmosphere. Furthermore, the vertical flux at $z = z_W$, which is almost exclusively carried by radiation, is about a factor 1.3 larger than the ambient value, due to lateral inflow of heat. We also find that the diameter of a tube at the "observable" level is in the neighborhood of 150 km, which is well below the resolution limit of ground-based instruments. However, with the advent of space observations in the near future, it will be possible to resolve single tubes. Nevertheless, ground observations, utilizing techniques independent of spatial resolution, can provide useful information, averaged over an ensemble of flux tubes (e.g., Stenflo and Harvey 1984). The values given in Table 1 are broadly compatible with observations, bearing in mind that dynamical effects have been neglected, apart from $\beta = 0.1$, which appears to be too hot compared to semiempirical models (e.g., Chapman 1979; Solanki 1986).

c) *Comparison with Spruit (1976, 1977)*

Spruit uses a two-dimensional analysis, which has the advantage of providing more quantitative results on the horizontal structure. On the other hand, he invokes the diffusion approximation to model radiative transport, which, as already pointed out, is totally unrealistic in the photosphere and in the overlying layers. A further difference between our treatments is in the treatment of convective transport. Spruit (1977) assumes $F_c \propto \nabla$ (in a subsequent chapter he also considers $F_c \propto \nabla - \nabla_a$), instead of $F_c \propto \nabla - \nabla'$. In spite of these, and a few additional differences, there are some broad similarities in our results. It is interesting, for example, to compare the total vertical flux along the tube axis, as shown in Figure 5 of Spruit (1976) for $r_{00} = 84$ km. The rapid increase in F as z decreases, for $z \lesssim 0$, the attainment of a maximum, and the subsequent decrease with z are common features. Even though a comparison of the precise values is not useful, owing to differences in the treatment of radiative transfer, the trend is fairly similar. Both sets of calculations also demonstrate the importance of radiative heat exchange in the photosphere. The temperature profiles in these layers (see Fig. 3*b* for $r_0 = 84$ km; Spruit 1976) are not greatly different. However, in the deeper layers of the convection zone, the temperature difference ΔT (relative to the ambient medium) found in this paper is not so large compared to Spruit (1976), who finds $\Delta T_{max} \approx 2000$ K. Since the atmosphere is highly opaque in these layers, the discrepancy is possibly due to the fact that we use different expressions to calculate F_c. It seems unlikely from the results presented in § IV*d* that a reduction in the vertical flux entering the base of a flux tube will lead to such large values for ΔT.

Another difference in the two approaches is that, unlike Spruit, we do not *a priori* specify the Wilson depression z_W. Instead we parameterize different flux tube equilibria in terms of β_0 and determine z_W self-consistently. The latter approach has the advantage that observations provide a better guide to the value of magnetic field (and hence β_0) than to the value of z_W.

d) *Comparison with Ferrari et al. (1985) and Kalkofen et al. (1986)*

Ferrari *et al.* (1985) also use the thin flux tube approximation to construct model equilibrium flux tube atmospheres, parameterized by β_0. There are, however, two essential differences between their approach and ours. These are essentially related to how energy transport within the flux tube is modeled. The first difference concerns the treatment of radiative transfer, and the second regards the effect of a strong magnetic field on convection. Let us first discuss the question of radiative transfer. Ferrari *et al.* treat the tube as a plane-parallel atmosphere, thus neglecting horizontal variations which permit lateral heat exchange. In a subsequent paper, Kalkofen *et al.* (1986) attempt to estimate this effect by integrating the transfer equation, within the framework of a two-stream approximation, along a single ray at an angle

$$\theta = \cos^{-1} \frac{1}{\sqrt{3}}$$

TABLE 2

VARIOUS QUANTITIES AT $\tau_{5000}^{\rm int}$ IN A FLUX TUBE

β_0	z_W (km)	T (K)	p (dyn cm^{-2})	ρ (g cm^{-3})	B (G)	F ($F_{\rm Sun}$)	a (km)
0.1	105	8445	2.20×10^4	4.10×10^{-8}	2300	2.14	63
1.5	25	6678	9.34×10^4	2.11×10^{-7}	1228	1.26	70
3.0	10	6410	1.06×10^5	2.57×10^{-7}	926	1.09	72
5.0	5	6364	1.13×10^5	2.78×10^{-7}	741	1.02	73

NOTE.—The radius of the tube at the upper boundary is taken as 200 km.

to the radial direction. Within the tube, the specific intensity I is assumed to be constant along the ray. For this assumption to be valid, the following condition should hold:

$$\left|\frac{I}{I}\frac{dI}{ds}\right|^{-1} = \left|\frac{4}{T}\frac{dT}{ds}\right|^{-1} \gg a \sec\theta ,$$

or

$$\left|\frac{1}{T}\frac{dT}{dz}\right|^{-1} \gg \frac{a}{2\sqrt{2}} .$$

In the vicinity of $z = 0$, this is unlikely to be true in view of the steep temperature gradient. A better approximation could be to use the generalized Eddington approximation of Unno and Spiegel (1966) applied to a thin flux tube. This has the advantage that it permits both vertical and horizontal energy radiative transport to be considered. Even though we have used the latter approach, it should be borne in mind that we are attempting to treat a transfer problem which is inherently two-dimensional, within a one-dimensional framework. As a next step, one could solve the transfer equation along a few rays at different angles. It is hoped to attempt this in a forthcoming publication.

The second main difference with the approach of Ferrari *et al.*, is that the present investigation does not assume that convective energy transport within the tube is totally suppressed. Ferrari *et al.* consider a shallower tube, extending some 140 km below $\tau_e = 1$, compared to 1000 km in this paper. We find that when $z \lesssim 140$ km, $F_R/F_c \gtrsim 10$ for $\beta = 0.1$. Thus, any assumption regarding convective inhibition is irrelevant in determining the equilibrium structure in these layers. On the other hand, in the deeper layers the opacity becomes extremely large, so that convection appear to be the only effective way of transporting energy in such regions.

Despite the difference in treatment it is interesting to compare our results. For $\beta_0 = 0.1$, Ferrari *et al.* find $T_i - T_e \sim 2900$ K at $\tau = 1$ compared to ~ 2200 K (see Table 2). The main reason for this discrepancy could be that, unlike them, we permit horizontal exchange of heat, which has the effect of reducing the temperature excess. Nevertheless, as already stated, even when lateral transport is considered, $\beta_0 = 0.1$ yields a temperature structure which appears to be too high. Ferrari *et al.* also consider a second choice, $\beta_0 = 0.25$, which too yields a rather large temperature excess. A more realistic possibility would have been to select β_0 in the range 1.5–2.0. However, this would probably have limited their treatment to even a shallower tube, extending no deeper than some 50 km below $\tau_e = 1$.

Finally, let us compare field strengths. At $z = 0$, they find $B \sim 1500$ G, whereas our calculations yield $B \sim 1700$ G (for $\beta_0 = 0.1$). The total flux, density, and pressure cannot be compared, since they have not presented any values for these quantities.

VI. SUMMARY AND CONCLUSIONS

The aim of the present contribution was to present model calculations to self-consistently determine the equilibrium atmosphere in an IFT. We used the generalization of the Eddington approximation to three dimensions, to develop a zeroth-order transfer equation valid for an axisymmetric thin flux tube. Such a treatment permitted radiative transport to occur in both the vertical and horizontal directions. We also included convection, using a mixing length formalism and using a parameter α, to incorporate its inhibition by a magnetic field. We solved the equilibrium equations for a thin flux tube. It was found convenient to parameterize the different equilibria in terms of β_0. The results suggest that the temperature on the axis of a flux tube is lower than the ambient medium at the same geometric level. However, at equal optical depths, the temperature in the tube is higher. At $\tau = 1$, the temperature difference could typically be some 500 K. We also found that for $z < 0$, the height variation of β is weak. For $z \geq 0$, β shows a stronger z dependence, which increases with β_0. Horizontal heat exchange plays an important role in determining the thermodynamic state of the gas in the flux tube. Lastly, the equilibrium stratification does not appear to be strongly influenced by the precise value of α.

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APPENDIX A

TRANSFER EQUATION FOR A THIN FLUX TUBE

Let us expand J about $r = 0$, as follows:

$$J(r, z) = J_0(z) + \epsilon R J_1(z) + \epsilon^2 R^2 J_2(z) + \cdots , \tag{A1}$$

where $r = \epsilon R$. We assume that J has no azimuthal dependence. It is easily seen that J_1 must vanish because div F_R must remain finite at $r = 0$, In general, it turns out that only even powers enter the expansion in equation (A1). From equations (7)–(8), we find, to zeroth order in ϵ,

$$\frac{1}{3\kappa\rho}\left[\frac{\partial}{\partial z}\left(\frac{1}{\kappa\rho}\frac{\partial J_0}{\partial z}\right) + 4J_2\right] = J - S . \tag{A2}$$

We estimate J_2 in an approximate fashion by using the relation

$$J(a, z) = J_e(z) \; ; \tag{A3}$$

where J_e is the mean intensity in the external medium. Using equations (A1) and (A3), we find

$$J_2 = (J_e - J_0)/a^2 \, . \tag{A4}$$

Substituting equation (A4) in equation (A2), we finally arrive at

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} + \frac{4}{3}\left(\frac{J_e - J}{\tau_a^2}\right) = J - S \, , \tag{A5}$$

where $d\tau = \kappa\rho \, dz$ and $\tau_a = \kappa\rho a$ and we have dropped the subscript zero.

APPENDIX B

STRUCTURE OF MATRICES A, B, C, AND D

I. EXTERNAL ATMOSPHERE

We evaluate the fluxes at levels $\tau_{k-1/2} = (\tau_k + \tau_{k-1})/2$, where $k = 1, 2, \ldots, N$. Using equation (24), we find that the convective flux can be expressed as

$$F_{c,k-1/2} = F^{(0)}_{c,k-1/2} + \left(\frac{\partial F_c^{(0)}}{\partial p}\right)^{T,\nabla}_{k-1/2} \delta p_{k-1/2}$$
$$+ \left(\frac{\partial F_c^{(0)}}{\partial T}\right)^{p,\nabla}_{k-1/2} \delta T_{k-1/2} + \left(\frac{\partial F_c^{(0)}}{\partial \nabla}\right)^{p,T}_{k-1/2} \frac{p_{k-1/2}}{T_{k-1/2}}\left[\frac{\delta T_k - \delta T_{k-1}}{p_k - p_{k-1}} - \frac{(T_k - T_{k-1})}{(p_k - p_{k-1})^2}(\delta p_k - \delta p_{k-1})\right], \tag{B1}$$

where we have dropped the index e, which we use only where there is an ambiguity.

In a similar manner, we write down an equivalent expression for the radiative flux. Using equations (17), (19), and (25), we have

$$F_{R,k-1/2} = F^{(0)}_{R,k-1/2} - \frac{4\pi}{3}\frac{\delta J_k - \delta J_{k-1}}{\Delta\tau_{k-1/2}} \, , \tag{B2}$$

where

$$\Delta\tau_{k-1/2} = \tfrac{1}{2}(\kappa_k \rho_k - \kappa_{k-1}\rho_{k-1})(z_k - z_{k-1}) \, .$$

Henceforth, the superscript 0, denoting values from the previous iteration, will be dropped and in the subsequent expressions will denote the grid level. Substituting equations (B1) and (B2) in equation (16), we obtain.

$$- a_{11}^{(k)}\delta J_{k-1} - a_{12}^{(k)}\delta T_{k-1} - a_{13}^{(k)}\delta p_{k-1} + b_{11}^{(k)}\delta J_k + b_{12}^{(k)}\delta T_k + b_{13}^{(k)}\delta p_k = d_1^{(k)} \, , \qquad k = 1, 2, \ldots, N \, , \tag{B3}$$

with

$$a_{11}^{(k)} = e_k \, , \qquad a_{12}^{(k)} = -f_k + h_k \, , \qquad a_{13}^{(k)} = -q_k - s_k \, ,$$
$$b_{11}^{(k)} = e_k \; ; \qquad b_{12}^{(k)} = f_k + h_k \, , \qquad b_{13} = q_k - s_k \, ,$$
$$d_1^{(k)} = F_{\rm Sun} + a_{11}^{(k)} J_{k-1} - b_{21}^{(k)} J_k - F_{c,k-1/2} - F_{{\rm ex},k-1/2} \, ,$$

where

$$e_k = -4\pi/(3\,\Delta\tau_{k-1/2}) \, , \qquad f_k = +\frac{1}{2}\left(\frac{\partial F_c}{\partial T}\right)_{k-1/2} , \qquad h_k = \frac{p_{k-/2}}{T_{k-1/2}}\frac{1}{(p_k - p_{k-1})}\left(\frac{\partial F_c}{\partial \nabla}\right)_{k-1/2} ,$$
$$q_k = \frac{1}{2}\left(\frac{\partial F_c^{(0)}}{\partial p}\right)_{k-1/2} , \qquad s_k = g_k(T_k - T_{k-1})/(p_k - p_{k-1}) \, .$$

We assume that the top boundary is above the convection zone, so that $F_{c,0} = 0$. Using equations (16) and (20), we find

$$-\frac{4\pi}{3}\left(\frac{\partial J}{\partial \tau}\right)_0 = F_{\rm Sun} - F_{{\rm ex},0} \, . \tag{B4}$$

Expanding J, in a Taylor series about J_0 and using equation (18), we obtain

$$b_{11}^{(0)}\delta J_0 + b_{12}^{(0)}\delta T_0 - c_{11}^{(0)}\delta J_1 - c_{12}^{(0)}\delta T_1 = d_1^{(0)} \, , \tag{B5}$$

with

$$b_{11}^{(0)} = 1 + \frac{1}{2}\chi_1 , \quad b_{12}^{(0)} = -\frac{1}{2}\chi_1^2 \left(\frac{\partial S}{\partial T}\right)_0 ,$$
$$c_{11}^{(0)} = 1 , \quad c_{12}^{(0)} = 0 ,$$
$$\alpha_1^{(0)} = \frac{1}{2}\chi_1^2 S_0 - b_{11}^{(0)} J_0 + c_{11}^{(0)} J_1 + \frac{\sqrt{3}}{4\pi}\chi_1(F_{\rm Sun} - F_{{\rm ex},0}) ,$$

where

$$\chi_k = \sqrt{3}\,\Delta\tau_{k-1/2} \quad \text{and} \quad S_k = \sigma T_k^4/\pi .$$

Next we use the transfer equations to relate δJ_k with δT_k. Using equation (18), we have

$$-a_{21}^{(k)}\,\delta J_{k-1} - a_{22}^{(k)}\,\delta T_{k-1} + b_{21}^{(k)}\,\delta J_k + b_{22}^{(k)}\,\delta T_k - c_{21}^{(k)}\,\delta J_{k+1} - c_{22}^{(k)}\,\delta T_{k+1} = d_2^{(k)} , \quad k = 1, 2, N-1 , \tag{B6}$$

where

$$a_{21}^{(k)} = 1/(3\,\Delta\tau_{k-1/2}\,\Delta\tau_k) , \quad a_{22}^{(k)} = 0 ,$$
$$b_{21}^{(k)} = 1 + \frac{1}{3\,\Delta\tau_k}\left(\frac{1}{\Delta\tau_{k-1/2}} + \frac{1}{\Delta\tau_{k+1/2}}\right) , \quad b_{22}^{(k)} = -\left(\frac{\partial S}{\partial T}\right)_k ,$$
$$c_{21}^{(k)} = 1/(3\,\Delta\tau_{k+1/2}\,\Delta\tau_k) , \quad c_{22}^{(k)} = 0 ,$$
$$d_2^{(k)} = S_k + a_{21}^{(k)} J_{k-1} - b_{21}^{(k)} J_k + c_{21}^{(k)} J_{k+1} .$$

Using the boundary conditions given by equations (20) and (22), we arrive at the following equations:

$$b_{21}^{(0)}\,\delta J_0 + b_{22}^{(0)}\,\delta T_0 - c_{21}^{(0)}\,\delta J_1 - c_{12}^{(0)}\,\delta T_1 = d_2^{(0)} , \quad -a_{21}^{(N)}\,\delta J_{N-1} - a_{22}^{(N)}\,\delta T_{N-1} + b_{21}^{(N)}\,\delta J_N + b_{22}^{(N)}\,\delta T_N = d_2^{(N)} \tag{B7}$$

where

$$a_{21}^{(N)} = 1 , \quad a_{22}^{(N)} = 0 ,$$
$$b_{21}^{(0)} = 1 + \chi_1 + \frac{1}{2}\chi_1^2 , \quad b_{21}^{(N)} = 1 + \chi_N + \frac{1}{2}\chi_N^2 , \quad b_{22}^{(0)} = -\frac{1}{2}\chi_1^2\left(\frac{\partial S}{\partial T}\right)_0 , \quad b_{22}^{(N)} = -\left(\chi_N + \frac{1}{2}\chi_N^2\right)\left(\frac{\partial S}{\partial T}\right)_N ,$$
$$d_2^{(0)} = \frac{1}{2}\chi_1^2 S_0 - b_{21}^{(0)} J_0 + c_{21}^{(0)} J_1 , \quad d_2^{(N)} = \left(\chi_N + \frac{1}{2}\chi_N^2\right) S_N + a_{21}^{(N)} J_{N-1} - b_{21}^{(N)} J_N .$$

Finally, we relate δp with δT by using the condition for hydrostatic equilibrium given by equation (1). This yields the following equations:

$$-a_{32}^{(k)}\,\delta T_{k-1} - a_{33}\,\delta p_{k-1} + b_{32}\,\delta T_k + b_{33}\,\delta p_k = d_3^{(k)} , \quad k = 1, 2, \ldots, N , \tag{B8}$$

where

$$a_{32}^{(k)} = -C_k\,p_{k-1/2}/(2T_{k-1/2}^2) , \quad a_{33}^{(k)} = 1 + C_k/(2T_{k-1/2}) ,$$
$$b_{32}^{(k)} = -a_{32}^{(k)} , \quad b_{33}^{(k)} = 2 - a_{33}^{(k)} ,$$
$$d_3^{(k)} = a_{33}^{(k)} p_{k-1} - b_3^{(k)} p_k ,$$

with

$$C_k = \mu_k\, g\, \Delta z .$$

At $k = 0$, we assume that the pressure is known. Let this value be denoted by $P_{\rm top}$. Thus, we have

$$b_{33}\,\delta p_0 = d_3^{(0)} ,$$

where

$$b_{33}^{(0)} = 1 \quad \text{and} \quad d_3^{(0)} = P_{\rm top} - P_0 .$$

II. FLUX TUBE ATMOSPHERE

The analysis for the flux tube atmosphere proceeds like the previous case with the difference that we use the energy equation (15) and equation (9) for the radiation intensity. The former yields the following set of equations

$$-\tilde{a}_{12}^{(k)}\,\delta T_{k-1} - \tilde{a}_{13}^{(k)}\,\delta p_{k-1} + \tilde{b}_{11}^{(k)}\,\delta J_k + \tilde{b}_{12}^{(k)}\,\delta T_k + \tilde{b}_{13}^{(k)}\,\delta p_k - \tilde{c}_{12}^{(k)}\delta T_{k+1} - \tilde{c}_{13}^{(k)}\,\delta p_{k+1} = \tilde{d}_1^{(k)} , \quad k = 1, 2, \ldots, N-1 , \tag{B9}$$

with

$$\tilde{a}_{12}^{(k)} = a_{12}^{(k)} , \qquad \tilde{a}_{13}^{(k)} = a_{13}^{(k)} ,$$

$$\tilde{b}_{11}^{(k)} = 4\pi , \qquad \tilde{b}_{12}^{(k)} = b_{12}^{(k+1)} + a_{12}^{(k+1)} - 4\pi \Delta\tau_k \left(\frac{\partial S}{\partial T}\right)_k , \qquad b_{13}^{(k)} = b_{13}^{(k)} + a_{13}^{(k+1)} ,$$

$$\tilde{c}_{12}^{(k)} = f_k + h_k , \qquad \tilde{c}_{13}^{(k)} = q_k - s_k ;$$

$$\tilde{d}_1^{(k)} = 4\pi \Delta\tau_k(S_k - J_k) + F_{c,k+1/2} - F_{c,k-1/2} + Q_{\text{ex},k}\Delta\tau_k .$$

It should be noted that in the present context the matrix elements $a_{ij}^{(k)}$ and $b_{ij}^{(k)}$, although defined in the previous section for the external atmosphere, now refer to quantities inside the flux tube.

At the upper boundary, we have

$$\tilde{b}_{11}^{(0)} \delta J_0 + \tilde{b}_{12}^{(0)} \delta T_0 = \tilde{d}_1^{(0)} , \tag{B10}$$

where

$$\tilde{b}_{11} = 1 , \qquad \tilde{b}_{12}^{(0)} = -\left(\frac{\partial S}{\partial T}\right)_0 , \qquad \tilde{d}_1^{(0)} = S_0 - J_0 .$$

For the lower boundary, we use the boundary condition that the total vertical flux $F_z = F_{\text{bot}}$. This yields

$$-\tilde{a}_{11}^{(N)} \delta J_{N-1} - \tilde{a}_{12}^{(N)} \delta T_{N-1} - \tilde{a}_{13}^{(N)} \delta p_{N-1} + \tilde{b}_{11}^{(N)} \delta J_N + \tilde{b}_{12}^{(N)} \delta T_N + \tilde{b}_{13}^{(N)} \delta p_N = \tilde{d}_1^{(N)} , \tag{B11}$$

where

$$\tilde{a}_j^{(N)} = a_j^{(N)} , \qquad \tilde{b}_j^{(N)} = b_j^{(N)} , \qquad j = 1, \ldots, 3 ;$$

$$\tilde{\alpha}_1^{(N)} = F_{\text{bot}} + \tilde{a}_{11}^{(N)} J_{N-1} - \tilde{b}_{21}^{(N)} J_N - F_{c,N-1/2} .$$

The relation which determines J (eq. [9]) yields the following set of equations:

$$-\tilde{a}_{21}^{(k)} \delta J_{k-1} - \tilde{a}_{22}^{(k)} \delta T_{k-1} + \tilde{b}_{21}^{(k)} \delta T_k + \tilde{b}_{22}^{(k)} \delta J_k - \tilde{c}_{21}^{(k)} \delta J_{k+1} - \tilde{c}_{22}^{(k)} \delta T_{k+1} = d_2^{(k)} , \qquad k = 1, \ldots, N-1 , \tag{B12}$$

where

$$\tilde{a}_{2i}^{(k)} = a_{2i}^{(k)} , \qquad \tilde{c}_{2i}^{(k)} = c_{2i}^{(k)} , \qquad i = 1, 2 ;$$

$$\tilde{b}_{21}^{(k)} = b_{21}^{(k)} + \frac{4}{3\tau_a^2} , \qquad \tilde{b}_{22}^{(k)} = b_{22}^{(k)} , \qquad \tilde{d}_2^{(k)} = d_2^{(k)} + \frac{4}{3\tau_a^2} [J_e^{(k)} - J^{(k)}] .$$

At the upper boundary, we use the boundary condition $J_0 = J_{0,e}$, which yields the following equations

$$\tilde{b}_{21}^{(0)} \delta J_0 = \tilde{d}_2^{(0)} ,$$

with

$$\tilde{b}_{21}^{(0)} = 1 , \qquad \tilde{d}_2 = J_{0,e} - J_0 .$$

For $z = z_N$, we use equation (B7).

The last set of equations, which stem from the condition for hydrostatic equilibrium are identical to equation (B8).

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