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EFFECT OF THE TIME SPENT BY THE PHOTON IN THE ABSORBED STATE ON THE TIME-DEPENDENT TRANSFER OF RADIATION

D. MOHAN RAO, K. E. RANGARAJAN, AND A. PERAIAH
Indian Institute of Astrophysics
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ABSTRACT

We derive the time-dependent transfer equation for a two-level atomic model taking into account bound-bound and bound-free transitions. The form of the transfer equation is similar in both the cases. We also propose a numerical scheme for solving the monochromatic time-dependent transfer equation when the time spent by the photon in the absorbed state is significant. The numerical method can be easily extended to solve the problem of time-dependent line formation or the bound-free continuum. We used this method to study three types of boundary conditions for the incident radiation on a scattering atmosphere. They are (1) an isotropic pulse, (2) a pulse of collimated radiation, and (3) a constant input of isotropic radiation. We present numerical results of the angular distribution of the time-dependent reflected and emergent intensities for various collisional de-excitation parameter values and also for different total optical depths T of the medium. The quantitative results show that the relaxation of the radiation field depends on the optical depth of the medium and also on the angle of emergence of the ray.

Subject headings: line formation — radiative transfer

I. INTRODUCTION

Two important quantities characterize the time dependence of a radiation field in its interaction with quantized atomic states. One quantity is the average time spent by the photon in the absorbed state, t_a (the upper state relaxation time), and the other is the average time spent by a photon between two successive scatterings, t_f. In most of the cases, one of these characteristic times is dominant and determines the temporal behavior of the radiation field. For a resonance line transition, normally the upper level is broadened only radiatively, and therefore $t_a = 1/A_{21}$, where A_{21} is the Einstein spontaneous emission coefficient; t_a is usually of the order 10^{-8} s. For a model atom consisting of a ground-level and continuum, the recombination time $t_a = 1/R_{c1}$, where R_{c1} is the radiative recombination rate. In the case of the Lyman continuum, $t_a \sim 5 \times 10^{10}(T)^{1/2}/N_e$ (Kneer 1976), where T and N_e are the temperature and electron density, respectively. Its value for $T = 6000$ K and photospheric densities of $N_e = 10^{13}$ cm^{-3} is of the order of 1 s or less. However, for the same temperature and chromospheric electron densities of the order of 10^{10} cm^{-3}, the recombination time has grown to the order of minutes. Due to the finite travel time of the photon, time spent by a photon between successive scatterings t_f is given by the relation $t_f = 1/(knc)$, where k is the absorption coefficient per particle, n is the number density of the particles, and c is the velocity of light. The quantity t_f connects the time derivative with other terms in the transfer equation. In a low-density medium like planetary nebula, for a resonance line (e.g., Lyα), we have $n \sim 1/\text{cm}^3$, $k = 10^{-12}$ cm^2, and $t_f \sim 10^2$ s.

Time dependence must be considered in a scattering atmosphere when the relaxation time of the radiation t_r is comparable to or longer than the characteristic time for changes in the impinging radiation field on the medium (Kunasz 1983). In an optically thick gas, where the photon is scattered many times before escape or destruction, t_r is comparable to the mean time spent by a photon in the gas, t_p, and is usually greater than t_a or t_f. The term t_p is related to t_a and t_f by the relation $t_p = \langle L\rangle/c + \langle N\rangle t_a$, where $\langle L\rangle$ and $\langle N\rangle$ are the mean path length and mean number of scatterings of the photon, respectively. In this paper we will study only the effects of t_a on radiative transfer and neglect the influence of t_f.

a) *Astrophysical Applications*

Time-dependent transfer effects will be important in the study of objects like active galactic nuclei, quasi-stellar objects, supernovae, and compact objects with accretion disks. They are also important when the source of energizing radiation is intrinsically constant but suddenly occulted or reinstated, as in planetary atmospheres or close binaries. The photon flight time t_f is important in low-density atmospheres, whereas t_a (usually the recombination time scale) is significant in high-density atmospheres for the study of time-dependent astrophysical phenomena.

Simultaneous observations using X-ray satellites and ground-based optical telescope have detected many pairs of coincident X-ray and optical bursts (Pederson *et al.* 1982; see also Hartmann and Woosley 1988). In all the cases, the optical flux is delayed with respect to X-rays by typically 1–4 s. Temporal burst profiles are similar in the optical and the X-ray band, but the optical emission is significantly wider. To explain these characteristics, Cominsky, London, and Klein (1987) performed numerical calculations of the reprocessing of high-energy X-radiation by solving the nonlinear diffusion equation. In the reprocessing scenario, intense bursts of high-energy X-rays are assumed to be emitted at irregular intervals from a neutron star in a close binary system with a low-mass companion star. In many cases, the radiative response of the illuminated atmosphere is very rapid, and the duration of the optical radiation depends on the duration of the impinging X-ray radiation. However, in certain cases, the reprocessed radiation is trapped in the medium by large opacity, and thus emerges on a much longer time scale. Though the models of Cominsky, London, and Klein (1987) explain the observations, the detailed time-dependent radiative transfer calculations of the bound-free continuum may help further in understanding the physical conditions of these objects. Future observational studies may show more evidence of the effects of the time-dependent radiative transfer due to the advances in astronomical detectors which will give us higher time resolution.

b) Laboratory Applications

In an optically thin gas, the time decay of the radiation field from the medium is $\sim e^{-A_{21}t}$. In an optically thick medium, where the photon is scattered several times, the decay will be much slower. Knowledge of the radiative decay rate, which is defined as the rate at which the energy in the excited state decays by means of radiative transfer, has important applications in laboratory physics. One practical application is in the design of fluorescent lamps. In these lamps, it is necessary to calculate the radiative decay rates of the Hg resonance radiation at the $\lambda 2537$ and $\lambda 1849$ caused by radiation trapping process (see Post 1986 for a detailed discussion). The study of time-dependent transfer is also useful in the measurements of transition probabilities.

Laboratory experiments which involve optically thick vapors are influenced by radiative trapping and diffusion. Huennekkens and Gallagher (1982) have measured the time-dependent fluorescence of sodium D lines following pulsed excitation of one D line, in the presence of radiation trapping with optical depths of ~ 10–2000. Measurements of the fundamental decay rate for the Hg resonance in cylindrical geometry have been reported (Post, Van deWeiger, and Cremers 1986). Recently, the spatial and temporal evolution of excited atoms in an optically thick magnesium vapor after an initial pulsed laser excitation has been studied using a delayed pulsed probe (Streater, Molander, and Cooper 1988). From the approximate treatment of the transfer equation with $t_a \gg t_f$, they found a solution for the time-dependent transfer equation with partial frequency redistribution that showed better agreement with the experimental results than the complete redistribution calculations.

c) Past Work

Sobolev (1963) obtained the reflection function through probabilistic arguments by considering the time spent by a photon in the absorbed state for semi-infinite media. Combining the time-dependent principle of invariance and the inverse method of Bellman, Kalaba, and Locke (1986), Matsumoto (1974) obtained the solution for semi-infinite homogeneous media by taking into account both t_a and t_f. However, for the techniques using the Laplace transforms, the numerical inversion process is ill-conditioned and often leads to unreliable results. Later, Matsumoto (1976) obtained a Von Neumann series solution for the Laplace transform of the equation. The series solution is separable into a time like factor and an angle factor. The angular factors are identical to those developed by Uesugi and Irvine (1970) in the theory of order of scatterings for the stationary case. Ganapol and Matsumoto (1986) extended the analysis to obtain a numerical evaluation for the reflected intensity from an anisotropically scattering atmosphere. Though the above methods are useful for studying the simple physical phenomena, and the results can be used as benchmarks for other numerical methods employing discretizations, they have certain disadvantages. They deal only with certain idealized cases like semi-infinite media. It is difficult to extend these techniques to time-dependent line transfer process where the source function contains the contribution from all the frequencies in the line. The numerical technique based on finite differences can tackle efficiently these problems.

In Appendix A, we derive the time-dependent line transfer equation under the assumption of complete redistribution. We obtain Milne's (1926) form of the transfer equation if we assume a square absorption profile. In this equation, the radiation field in one frequency does not depend on the radiation field in other frequencies; we call this case monochromatic radiative transfer. One of the earliest attempts to solve this equation was made by Chandrasekhar (1950), who obtained a series solution. Milne's theory may not be valid for an optically thick medium, where the escape of the radiation takes place through the line wings. Then we have to consider the detailed line transfer equation.

In Appendix B, we derive the time-dependent transfer equation for a bound-free transition. Our atomic model consists of only one bound and one free state. We can derive the monochromatic transfer equation by noting that for $h\nu/kT \gg 1$, the frequency variation of J_ν and B_ν show so rapid a drop with increasing ν that most of the contribution to the photoionization and recombination rate integrals come from $\nu \approx \nu_0$ (Mihalas 1978). The form of the transfer equation is similar to the line transfer equation derived in Appendix A. Hence the numerical technique which we have proposed in this paper can also be employed to study the problem of time-dependent bound-free continuum.

In § II, we shall propose a numerical scheme based on the discrete space theory of radiative transfer (Grant and Hunt 1969*a*, *b*) to solve the time-dependent radiative transfer equation for a square absorption profile. In § III, we present the results for a square absorption profile. We consider the cases where a slab is illuminated by (1) an isotropic pulse, (2) a pulse of collimated radiation, and (3) a constant input of isotropic radiation. Emergent and reflected intensity distributions are studied for various values of the collisional de-excitation parameter ϵ. We also evaluate the radiation field for different optical depths of the medium. Extension of the method to line transfer problems with the complete and partial frequency redistribution scattering processes is straightforward, and the results will be published in forthcoming papers.

II. METHOD OF SOLUTION

In this section, we shall present a numerical method for solving the time-dependent transfer equation when the profile function is a square profile. As we can see from Appendix A, the transfer equation for this case is (we change the sign of the right side of eq. [A21], due to our convention of measuring μ in the direction in which the optical depth is increasing):

$$\mu\left(\frac{dI_\nu}{d\tau}\right) = (1-\epsilon)\int_0^t e^{-(t-t')/t_a} J_\nu \frac{dt'}{t_a} + \epsilon B_\nu (1 - e^{-t/t_a}) - I_\nu \,. \tag{2}$$

Due to our assumption that the number density of the excited atoms n_2 at time $t = 0$ is 0 (see eq. [A4]), the thermal term in equation (1) vanishes at $t = 0$. We divide equation (1) into two equations, one for radiation traveling in the direction, $\mu(0 < \mu \leq 1)$, and the other equation for the radiation traveling in the opposite direction, $-\mu$. We will omit the subscript ν from the equations and write $I = I(t, \tau, \mu)$ and $\epsilon = \epsilon(\tau)$. For simplicity's sake, we shall omit the dependence of t_a on τ.

Sometimes it is convenient to distinguish between the attenuated incident radiation which penetrates to a depth τ at time t and the diffuse radiation $I(t, \tau, \mu)$ that results as a consequence of multiple scattering processes. Then the transfer equation for the diffuse radiation is given by

$$\pm \mu \frac{\partial I(t, \tau, \pm\mu)}{\partial \tau} = \frac{[1-\epsilon(\tau)]}{2} \int_0^t e^{-(t-t')/t_a} \frac{dt'}{t_a} \int_0^1 [I(t', \tau, \mu') + I(t', \tau, -\mu')]d\mu' + K(t, \tau, \mu) - I(t, \tau, \pm\mu) , \quad (2)$$

where

$$K(t, \tau, \mu) = h(t, \tau, \mu) + \epsilon B(\tau)(1 - e^{-t/t_a}) .$$

The function h is the source term due to the directly transmitted radiation and is independent of μ for isotropic scattering problems. The form of h will be discussed later in the text for each individual case.

For a slab atmosphere, with a given incident radiation input on the lower boundary of the atmosphere and with zero incident intensity on the other boundary, the boundary conditions for equation (1) are

$$I(t, \tau = T, -\mu) = f(t, \mu) ,$$
$$I(t, \tau = 0, \mu) = 0 , \quad (3)$$

where $f(t, \mu)$ is a given function. The diffuse intensity satisfies

$$I(t, \tau = T, -\mu) = 0 ,$$
$$I(t, \tau = 0, \mu) = 0 . \quad (4)$$

We shall approximate the angular integral in equation (2) as

$$\int_0^1 I(t, \tau, \mu')d\mu' = \sum_{j=1}^{J} I(t, \tau, \mu_j)c_j , \quad (5)$$

where the coefficient c_j and angle cosines μ_j are determined by the Gauss-Legendre quadrature of order J. We consider for the time variable a set of equally spaced points, $t_1 = 0$, $t_2 = t_1 + \Delta t, \ldots, t_i = t_1 + (i-1)\Delta t, \ldots t_I$, where t_I is the last time point up to which we find the radiation field. We shall approximate the time integral in equation (2) by the trapezoidal quadrature formula and write

$$\int_0^{t_i} e^{-(t_i-t')/t_a} I(t', \tau, \pm\mu_j) \frac{dt'}{t_a} = \sum_{k=1}^{i-1} a_k e^{-(t_i-t_k)/t_a} I(t_k, \tau, \pm\mu_j) \frac{\Delta t}{t_a} + a_i I(t_i, \tau, \pm\mu_j) \frac{\Delta t}{t_a} , \quad \text{for } i = 2, \ldots, I , \quad (6)$$

where coefficients $a_1, a_2, \ldots a_i$ are the trapezoidal weights and are given by $a_1 = 0.5$, $a_k = 1.0$ for $k = 2, \ldots, i-1$ and $a_i = 0.5$ for $i = 2, \ldots I$. We solve now a sequence of equations starting from time $t = t_1$ up to time $t = t_I$. Since the scattering integral vanishes from equation (2) for time $t = t_1 = 0$, we have

$$\pm \mu_j \frac{\partial I(t_1, \tau, \pm\mu_j)}{\partial \tau} = K(t_1, \tau, \mu_j) - I(t_1, \tau, \pm\mu_j) , \quad \text{for } j = 1, 2, \ldots, J . \quad (7)$$

We can solve equation (7) either in closed form, since $K(t_1, \tau, \mu)$ is a known function, or one can use a simple numerical approximation. Having obtained the radiation field $I(t_1 = 0, \tau, \pm\mu)$ throughout the medium, we will solve equation (2) for time $t = t_2$ by using the approximations (5) and (6).

Now the transfer equation for time $t = t_2$ is

$$\pm\mu_j \frac{\partial I(t_2, \tau, \pm\mu_j)}{\partial \tau} = \frac{1-\epsilon(\tau)}{2} \frac{\Delta t}{t_a} \times \sum_{j=1}^{J} [I(t_2, \tau, \mu_j) + I(t_2, \tau, -\mu_j)]a_2 c_j + K^{(1)}(t_2, \tau, \mu_j) - I(t_2, \tau, \pm\mu_j) , \quad (8)$$

where

$$K^{(1)}(t_2, \tau, \mu_j) = h(t_2, \tau, \mu_j) + \epsilon(\tau)B(\tau) \times (1 - e^{-t_2/t_a}) + \frac{1-\epsilon(\tau)}{2} e^{-(t_2-t_1)/t_a} \frac{\Delta t}{t_a} \times \sum_{j=1}^{J} [I(t_1, \tau, \mu_j) + I(t_1, \tau, -\mu_j)]a_1 c_j .$$

One can note that $K^{(1)}$ is a known term since it depends on the directly transmitted radiation, thermal sources, and radiation field at time t_1. Also, equation (8) has a similar form to the time-independent transfer equation

$$\pm\mu_j \frac{\partial I(\tau, \pm\mu_j)}{\partial \tau} = \frac{1-\epsilon(\tau)}{2} \sum_{j=1}^{J} [I(\tau, \mu_j) + I(\tau, -\mu_j)]c_j + K(\tau, \mu_j) - I(\tau, \pm\mu_j) ,$$

which is the discretized (angular) equivalent of the equation

$$\pm\mu \frac{\partial I(\tau, \pm\mu)}{\partial \tau} = \frac{1-\epsilon(\tau)}{2} \int_{-1}^{1} I(\tau, \mu')d\mu' + K(\tau, \mu) - I(\tau, \pm\mu) , \quad (9)$$

where

$$K(\tau, \mu) = h(\tau, \mu) + \epsilon(\tau)B(\tau) .$$

Hence one can solve equation (8) for $I(t_2, \tau, \pm\mu)$ by using the techniques which were employed for equation (9).

We can continue this procedure for obtaining $I(t_i, \tau, \mu)$ by using the intensity values $I(t_1, \tau, \mu), \ldots, I(t_{i-1}, \tau, \mu)$. We can write the transfer equation for $I(t_i, \tau, \mu)$ by combining equations (2), (5), and (6):

$$\pm \mu_j \frac{\partial I(t_i, \tau, \pm\mu_j)}{\partial \tau} = \frac{1-\epsilon(\tau)}{2} \times \sum_{j=1}^{J} [I(t_i, \tau, \mu_j) + I(t_i, \tau, -\mu_j)]d_i c_j + K^{(1)}(t_i, \tau, \mu_j) - I(t_i, \tau, \pm\mu_j) , \quad \text{for } i = 2, \ldots, I , \quad \text{for } j = 1, \ldots, J , \quad (10)$$

where

$$K^{(1)}(t_i, \tau, \pm\mu_j) = h(t_i, \tau, \mu_j) + \epsilon(\tau)B(\tau)(1 - e^{-t_i/t_a}) + \sum_{k=1}^{i-1} \sum_{j=1}^{J} I(t_k, \tau, \pm\mu_j)d_k c_j ,$$

and

$$d_k = a_k e^{-(t_i-t_k)/t_a} \frac{\Delta t}{t_a} , \quad \text{for } k = 1, \ldots, i . \quad (11)$$

a) Integration of the Time-Dependent Transfer Equation (10) over a Shell

We use the discrete space theory technique to solve the above equations. We describe briefly the procedure here. For details of this method, see Grant and Hunt (1969*a*, *b*). Let the medium be divided into N shells. Now we shall integrate equation (10) over a shell bounded by layers n and $n+1$. First, we recast the system of equations (10) into a matrix form by defining the following matrices:

$$\boldsymbol{M} = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_J \end{pmatrix}_{J \times J} ,$$

$$\boldsymbol{C} = \begin{pmatrix} c_1 & c_2 & \dots & c_J \\ c_1 & c_2 & \dots & c_J \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \dots & c_J \end{pmatrix}_{J \times J} , \quad (12)$$

and the intensity vector and source vector are

$$\boldsymbol{U}_i^{\pm}(\tau) = \begin{pmatrix} I^{\pm}(t_i, \tau, \mu_1) \\ \vdots \\ I^{\pm}(t_i, \tau, \mu_J) \end{pmatrix}_{J \times 1}$$

$$\boldsymbol{K}_i^{(1)\pm}(\tau) = \begin{pmatrix} K^{(1)\pm}(t_i, \tau, \mu_1) \\ \vdots \\ K^{(1)\pm}(t_i, \tau, \mu_J) \end{pmatrix}_{J \times 1} , \quad \text{for } i = 2, \dots, I . \quad (13)$$

Here subscript i refers to the ith time point. We shall rewrite equation (10) for all angles in matrix form using equations (12) and (13). This is given for a postively directed beam as

$$\boldsymbol{M} \frac{\partial \boldsymbol{U}_i^{+}(\tau)}{\partial \tau} + \boldsymbol{U}_i^{+}(\tau) = \frac{[1 - \epsilon(\tau)] d_i}{2} \times (\boldsymbol{C}\boldsymbol{U}_i^{+} + \boldsymbol{C}\boldsymbol{U}_i^{-}) + \boldsymbol{K}_i^{(1)+}(\tau) , \quad (14)$$

and for a negatively directed beam,

$$-\boldsymbol{M} \frac{\partial \boldsymbol{U}_i^{-}(\tau)}{\partial \tau} + \boldsymbol{U}_i^{-}(\tau) = \frac{[1 - \epsilon(\tau)] d_i}{2} \times (\boldsymbol{C}\boldsymbol{U}_i^{+} + \boldsymbol{C}\boldsymbol{U}_i^{-}) + \boldsymbol{K}_i^{(1)-}(\tau) . \quad (15)$$

Now we integrate equations (14) and (15) over the optical depth coordinate from τ_n to τ_{n+1}. Then we obtain

$$\boldsymbol{M}[\boldsymbol{U}_{i,n+1}^{+} - \boldsymbol{U}_{i,n}^{+}] + \tau_{n+1/2} \boldsymbol{U}_{i,n+1/2}^{+} = \tau_{n+1/2} \left[\boldsymbol{K}_{i,n+1/2}^{(1)+} + \frac{(1-\epsilon)_{n+1/2} d_i}{2} (\boldsymbol{C}\boldsymbol{U}_{i,n+1/2}^{+} + \boldsymbol{C}\boldsymbol{U}_{i,n+1/2}^{-}) \right] , \quad (16)$$

$$\boldsymbol{M}(\boldsymbol{U}_{i,n}^{-} - \boldsymbol{U}_{i,n+1}^{-}) + \tau_{n+1/2} \boldsymbol{U}_{i,n+1/2}^{-} = \tau_{n+1/2} \left[\boldsymbol{K}_{i,n+1/2}^{(1)-} + \frac{(1-\epsilon)_{n+1/2} d_i}{2} \times \boldsymbol{C}\boldsymbol{U}_{i,n+1/2}^{+} + \boldsymbol{C}\boldsymbol{U}_{i,n+1/2}^{-}) \right] \quad \text{for } i = 2, \dots, I , \quad (17)$$

where $\boldsymbol{U}_{i,n}^{+} = \boldsymbol{U}_i^{\pm}(\tau_n)$, and the variables subscripted with $n + 1/2$ such as $\boldsymbol{U}_{i,n+1/2}^{\pm}$, $\tau_{n+1/2}$ and $\epsilon_{n+1/2}$ are the averages over the shell whose boundaries are τ_n and τ_{n+1}.

We shall approximate the quantities $\boldsymbol{U}_{i,n+1/2}^{+}$ and $\boldsymbol{U}_{i,n+1/2}^{-}$ by

$$\boldsymbol{U}_{i,n+1/2}^{+} = \tfrac{1}{2}(\boldsymbol{U}_{i,n+1}^{+} + \boldsymbol{U}_{i,n}^{+}) ,$$

and

$$\boldsymbol{U}_{i,n+1/2}^{-} = \tfrac{1}{2}(\boldsymbol{U}_{i,n+1}^{-} + \boldsymbol{U}_{i,n}^{-}) , \quad (18)$$

which is called the "diamond scheme." Substituting equation (18) into equations (16) and (17) and rearranging the terms, we get

$$\begin{aligned} &\{\boldsymbol{M} + 0.5\tau_{n+1/2}[\boldsymbol{I} - 0.5(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}]\} \\ &\quad \times \boldsymbol{U}_{i,n+1}^{+} - 0.25\tau_{n+1/2}(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}\boldsymbol{U}_{i,n}^{-} \\ &\quad = 0.25\tau_{n+1/2}(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}\boldsymbol{U}_{i,n+1}^{-} \\ &\qquad + \{\boldsymbol{M} - 0.5\tau_{n+1/2}[\boldsymbol{I} - 0.5(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}]\} \boldsymbol{U}_{i,n}^{+} \\ &\qquad + \boldsymbol{K}_{i,n+1/2}^{(1)+} \tau_{n+1/2} , \end{aligned} \quad (19)$$

$$\begin{aligned} &\{\boldsymbol{M} + 0.5\tau_{n+1/2}[\boldsymbol{I} - 0.5(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}]\} \\ &\quad \times \boldsymbol{U}_{i,n}^{-} - 0.25\tau_{n+1/2}(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}\boldsymbol{U}_{i,n+1}^{+} \\ &\quad = 0.25\tau_{n+1/2}(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}\boldsymbol{U}_{i,n}^{+} \\ &\qquad + \{\boldsymbol{M} - 0.5\tau_{n+1/2}[\boldsymbol{I} - 0.5(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C}]\} \boldsymbol{U}_{i,n+1}^{-} \\ &\qquad + \boldsymbol{K}_{i,n+1/2}^{(1)-} \tau_{n+1/2} . \end{aligned} \quad (20)$$

Here $\boldsymbol{I}$ is the identity matrix of appropriate dimension. It is now straightforward to put these equations in the canonical form

$$\begin{pmatrix} \boldsymbol{U}_{i,n+1}^{+} \\ \boldsymbol{U}_{i,n}^{-} \end{pmatrix} = \begin{bmatrix} \boldsymbol{t}_i(n+1, n) & \boldsymbol{r}_i(n, n+1) \\ \boldsymbol{r}_i(n+1, n) & \boldsymbol{t}_i(n, n+1) \end{bmatrix} \times \begin{pmatrix} \boldsymbol{U}_{i,n}^{+} \\ \boldsymbol{U}_{i,n+1}^{-} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Sigma}_{i,n+1/2}^{+} \\ \boldsymbol{\Sigma}_{i,n+1/2}^{-} \end{pmatrix} . \quad (21)$$

Now we shall express the r (reflection) and t (transmission) matrices in terms of the following auxiliary matrices:

$$\begin{aligned} \boldsymbol{Q}_{n+1/2} &= 0.5(1 - \epsilon_{n+1/2}) d_i \boldsymbol{C} , \\ \boldsymbol{S}^{+} &= \boldsymbol{M} - 0.5\tau_{n+1/2}(\boldsymbol{I} - \boldsymbol{Q}_{n+1/2}) , \\ \tilde{\boldsymbol{S}} &= 0.5\tau_{n+1/2} \boldsymbol{Q}_{n+1/2} , \\ \boldsymbol{\Delta} &= [\boldsymbol{M} + 0.5\tau_{n+1/2}(\boldsymbol{I} - \boldsymbol{Q}_{n+1/2})]^{-1} , \\ \tilde{\boldsymbol{r}} &= \boldsymbol{\Delta}\tilde{\boldsymbol{S}} , \quad \boldsymbol{t}^{+} = (\boldsymbol{I} - \tilde{\boldsymbol{r}}\tilde{\boldsymbol{r}})^{-1} . \end{aligned}$$

Then

$$\boldsymbol{t}_i(n+1, n) = \boldsymbol{t}^{+}(\boldsymbol{\Delta}\boldsymbol{S}^{+} + \tilde{\boldsymbol{r}}\tilde{\boldsymbol{r}}) = \boldsymbol{t}_i(n, n+1) ,$$

$$\boldsymbol{r}_i(n+1, n) = \boldsymbol{t}^{+}\tilde{\boldsymbol{r}}(\boldsymbol{I} + \boldsymbol{\Delta}\boldsymbol{S}^{+}) = \boldsymbol{r}_i(n, n+1)$$

and

$$\begin{aligned} \boldsymbol{\Sigma}_{n+1/2}^{+} &= \tau_{n+1/2} \boldsymbol{t}^{+}(\boldsymbol{\Delta}\boldsymbol{K}_{i,n+1/2}^{(1)+} + \tilde{\boldsymbol{r}}\boldsymbol{\Delta}\boldsymbol{K}_{i,n+1/2}^{(1)+}) , \\ \boldsymbol{\Sigma}_{n+1/2}^{-} &= \tau_{n+1/2} \boldsymbol{t}^{+}(\boldsymbol{\Delta}\boldsymbol{K}_{i,n+1/2}^{(1)-} + \tilde{\boldsymbol{r}}\boldsymbol{\Delta}\boldsymbol{K}_{i,n+1/2}^{(1)-}) , \\ &\text{for } i = 2, \dots, I . \end{aligned} \quad (22)$$

Once we calculate the r and t operators for each shell, the total internal radiation field can be calculated using the algorithm given by Grant and Hunt (1969*a*, *b*).

b) Stability Considerations

From physical considerations, we know that the reflection and transmission operators r and t must be nonnegative. For this condition to be satisfied, we need to have $\Delta \geq 0$, $S^{+} \geq 0$.

This condition can be achieved only if

$$\tau_{n+1/2} \leq \tau_{\text{crit}} = \min_{i,j} \left| \frac{\mu_j}{0.5[1 - 0.5(1 - \epsilon_{n+1/2})d_i c_j]} \right| . \quad (23)$$

From equation (11), we can see that the maximum value of d_i is $\Delta t/t_a$ and hence we choose τ_{crit} as $2\mu_1$. To study the radiation field with finer time resolution, and also to keep the error term in the trapezoidal integration formula for the time integral as low as possible, we choose $\Delta t/t_a \simeq 0.2$. But if the radiation field varies in a time scale comparable to t_a, we can choose $\Delta t/t_a \simeq 1$.

It may be pointed out that Peraiah and Grant (1973) derived the condition for the r and t matrices to be nonnegative for a medium without time-dependent transfer effects as

$$\tau_{n+1/2} \leq \tau_{\text{crit}} = \min_{j} \left| \frac{\mu_j}{0.5[1 - 0.5(1 - \epsilon_{n+1/2})c_j]} \right| . \quad (24)$$

III. DISCUSSION OF THE SPECIFIC CASES AND THE RESULTS

In this section, we shall discuss in detail the various cases considered and the transfer equation associated with those cases. The results of calculation are shown in graphical form. In all the cases, we assume that no radiation is incident on the upper boundary of the medium, i.e.,

$$I(t, \tau = 0, \mu) = 0 . \quad (25)$$

We set $t_a = 1$ throughout our calculations and time t is measured in units of t_a.

a) *Effect of an Isotropic Pulsed Input*

We consider the cases when an isotropic pulse impinges on the lower boundary of a homogeneous medium. The boundary condition for this case is given by

$$I(t, \tau = T, -\mu) = I_0 t_a \delta(t) , \quad (26)$$

where $\delta(t)$ is the Dirac delta function. Now the directly transmitted radiation I^{dir} in the slab is given by equation

$$I^{\text{dir}}(t, \tau, -\mu) = I_0 t_a \delta(t) e^{-(T-\tau)/\mu} .$$

The scattering integral of I^{dir} is

$$h(t, \tau) = 0.5(1 - \epsilon)I_0 \int_0^t \int_0^1 e^{-(t-t')/t_a}\delta(t')e^{-(T-\tau)/\mu'} d\mu' dt'$$

$$= 0.5(1 - \epsilon)I_0 e^{-t/t_a} \int_0^1 e^{-(T-\tau)/\mu'} d\mu'$$

$$= 0.5(1 - \epsilon)I_0 e^{-t/t_a}[e^{-(T-\tau)} - (T - \tau)E_1(T - \tau)] , \quad (27)$$

where E_1 is the well-known exponential integral of order 1. Hence the transfer equation for the diffuse intensity is

$$\pm \mu \frac{\partial I(t, \tau, \pm\mu)}{\partial \tau} + I(t, \tau, \pm\mu) = 0.5(1 - \epsilon) \int_0^t \int_0^1 e^{-(t-t')/t_a}[I(t, \tau, \mu') + I(t, \tau, -\mu')]d\mu' \frac{dt'}{t_a} + \epsilon B(1 - e^{-t/t_a}) + 0.5(1 - \epsilon)I_0 e^{-t/t_a} \int_0^1 e^{-(T-\tau)/\mu'} d\mu' , \quad (28)$$

with

$$I(t, \tau = T, -\mu) = 0 ,$$
$$I(t, \tau = 0, \mu) = 0 .$$

i) *Results for a Conservatively Scattering Medium*

The computed solutions of equation (28) are shown in Figures 1–10 for different values of the collisional de-excitation parameter ϵ. The time-dependent intensities reflected in the directions $\mu = 0.21$ and $\mu = 0.78$ are displayed in Figures 1–6 when the total optical depth of the medium is 1, 2, and 15. First we shall study the results for $\epsilon = 0$, and the effects of thermal sources will be examined later.

For a conservatively scattering atmosphere (i.e., $\epsilon = 0$), we can see from Figures 1 and 2 that the reflected radiation for $\mu = 0.21$ decays more rapidly with time compared to that for $\mu = 0.78$. The same behavior can be seen from Figures 3 and 4 for $T = 2$ and from Figures 5 and 6 for $T = 15$. In Table 1, we

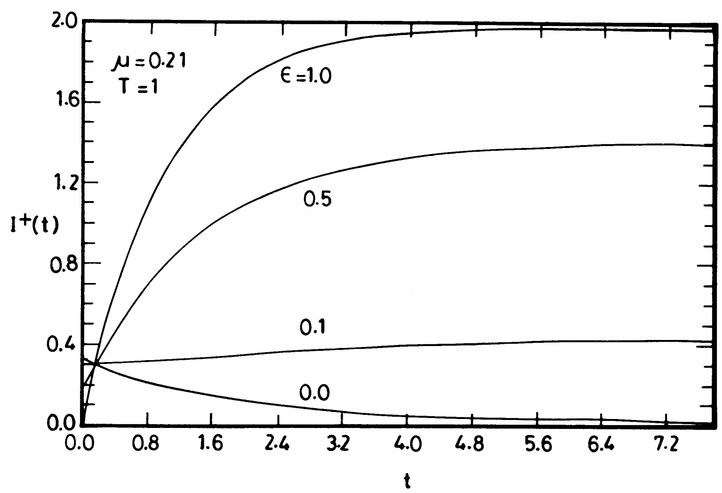

FIG. 1.—Reflected intensity as a function of time t, at $\mu = 0.21$, for different collisional de-excitation parameters ϵ, when an isotropic pulse is incident on the medium with total optical depth $T = 1.0$.

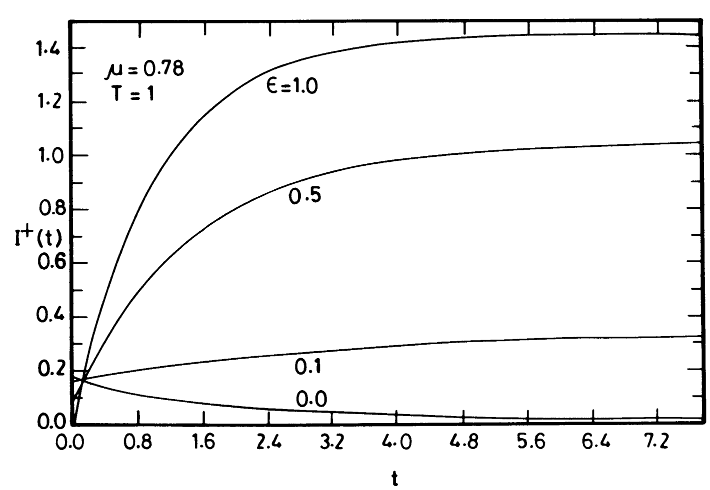

FIG. 2.—Same as Fig. 1 for $\mu = 0.78$

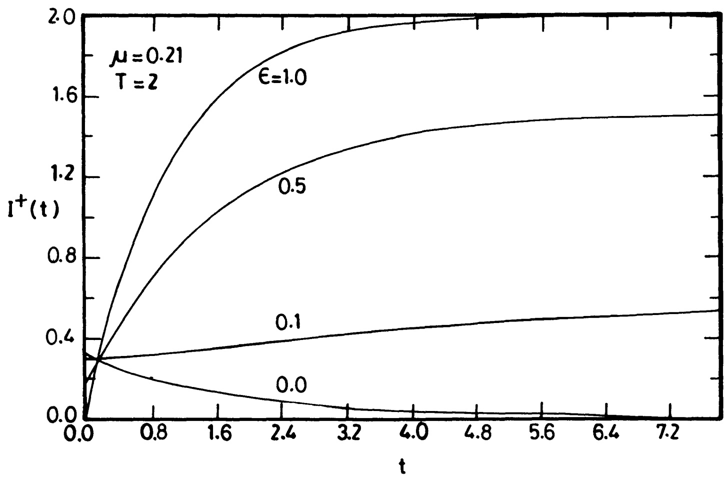

FIG. 3.—Same as Fig. 1 for $T = 2.0$

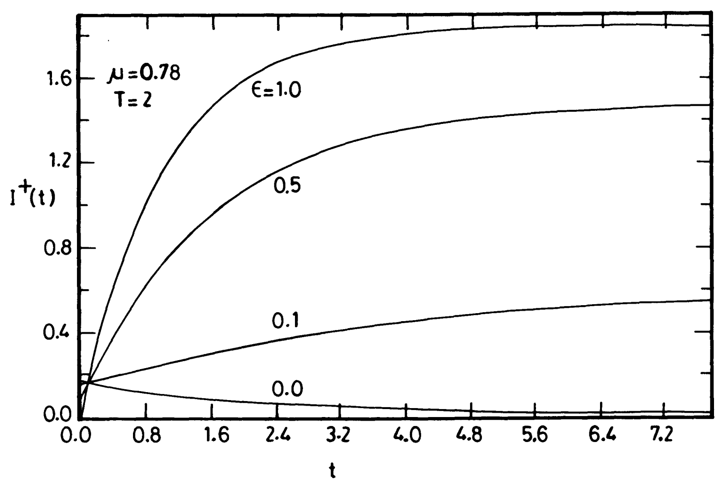

FIG. 4.—Same as Fig. 2 for $T = 2.0$

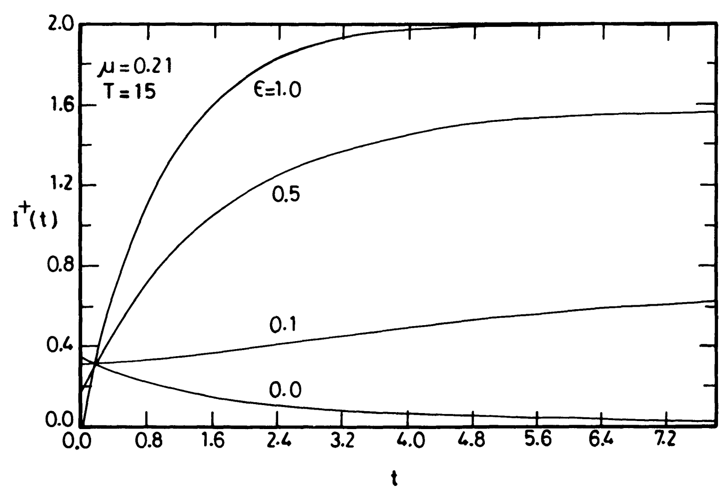

FIG. 5.—Same as Fig. 1 with $T = 15.0$

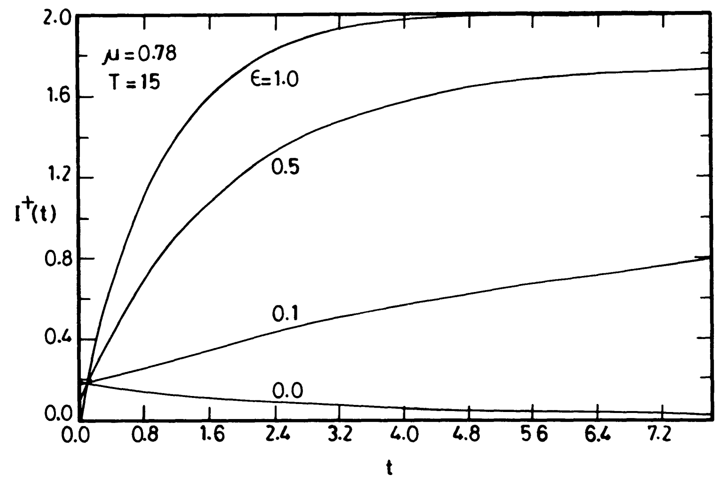

FIG. 6.—Same as Fig. 2 with $T = 15.0$

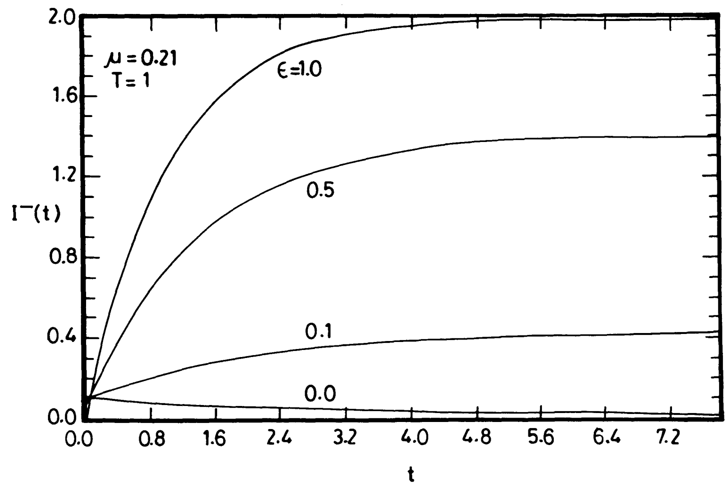

FIG. 7.—Emergent intensity as a function of time t, at $\mu = 0.21$, for different collisional de-excitation parameters ϵ, when an isotropic pulse is incident on the medium with $T = 1.0$.

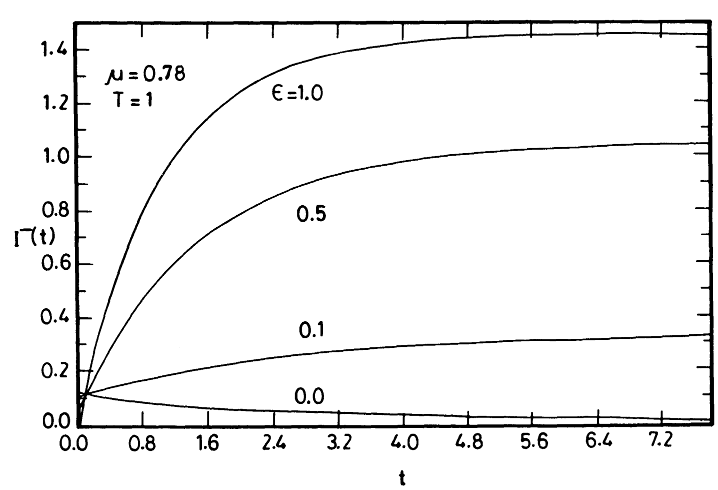

FIG. 8.—Same as Fig. 7 for $\mu = 0.78$

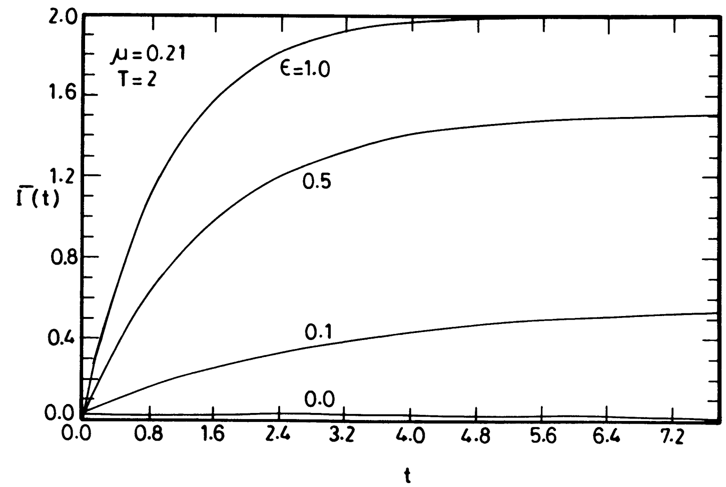

Fig. 9.—Same as Fig. 7 with $T = 2.0$

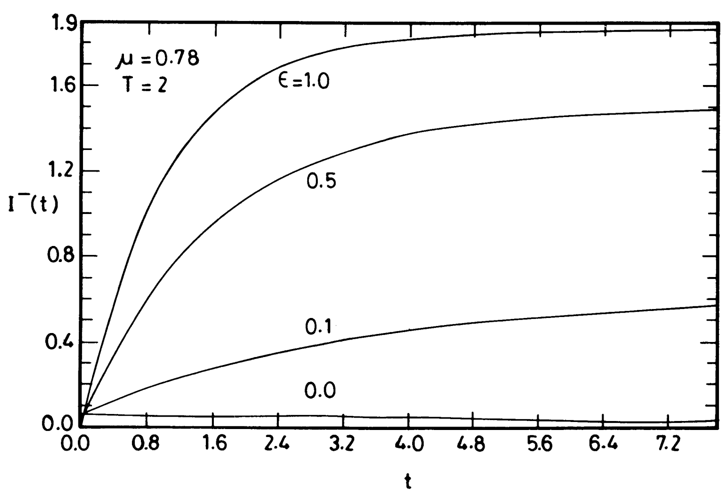

Fig. 10.—Same as Fig. 8 with $T = 2.0$

TABLE 1

Decay Time for the Reflected Radiation[a]

Total Optical Depth	Time Taken for the Radiation Field to Decay by 50%		Time Taken for the Radiation Field to Decay by 75%	
	$\mu = 0.21$	$\mu = 0.78$	$\mu = 0.21$	$\mu = 0.78$
1.........	1.2	1.4	2.6	3.1
2.........	1.3	1.8	2.8	4.0
15.........	1.3	1.8	2.9	4.4
With a Searchlight Boundary Condition				
1.........	1.4	1.8	3.2	3.6
2.........	1.6	2.2	3.6	4.8
15.........	1.6	2.5	3.6	6.0
For the Emergent Radiation Field with Searchlight Boundary Condition				
1.........	2.1	1.7	4.1	3.8
2.........	5.2	4.0	9.0	7.7

Note.—Time is measured in units of t_a.

[a] For the case of conservative scattering atmosphere, with the isotropic pulse as input.

show the time taken for the initial reflected radiation to decay by 50% and by 75% for different optical depths of the medium. The values remain the same for the radiation reflected in the direction $\mu = 0.21$ for all optical depths of the medium. But for the radiation reflected in the near normal direction $\mu = 0.78$, the time spent by the photon to decay by 50% and 75% increases with the optical depth of the medium.

Van de Hulst and Irvine (1963) pointed out that the time-dependent problem for $t_a \neq 0$ and $t_f = 0$ is equivalent to the problem of finding the distribution of photons over the number of scatterings. All our results can be interpreted using this fact. A photon reflected at the grazing angle $\mu = 0.21$ experiences few scatterings and spends little time before it appears on the surface. On the other hand, a photon reflected at an angle nearer to the normal ($\mu = 0.78$) undergoes a large number of scatterings and spends a long time before its reappearance. By similar reasoning, an increase in the optical depth of the medium has more effect on the photons reflected in the normal direction than the photons reflected at the grazing angle.

The emergent intensities are plotted in Figures 7 and 8 for $T = 1$ and in Figures 9 and 10 for $T = 2$. Now the photons emerging along the direction $\mu = 0.78$ experiences few scatterings, which results in the rapid dropping of the emergent intensity distribution compared to the photons coming in the direction $\mu = 0.21$. The amount of time taken for the radiation field to decay by 50% is 2.2 and 1.8 for $\mu = 0.21$ and $\mu = 0.78$, respectively, when $T = 1$. The values are 7.0 and 4.4 for $\mu = 0.21$ and $\mu = 0.78$, respectively, when $T = 2$. So we can see that the effect of optical depth is more pronounced for the time-dependent emergent radiation field than on the reflected radiation field. The reason may be due to our chosen boundary condition. The source due to the pulse on the lower boundary of the medium has maximum value in the lower layers of the medium. This radiation has to undergo many scatterings before emerging from the top of the medium. But the maximum amount of reflected radiation from the lower boundary comes from the shallow layers of the medium, undergoing only a few scatterings. Consequently, optical depth effects are less on the reflected radiation and more on the emergent time-dependent radiation.

ii) *Effects of Thermal Sources*

We can see from Figures 1–6 the evolution of a radiation field to steady state for $\epsilon = 0$. As ϵ increases from 0 to 1, the contribution of the thermal sources to the radiation field is enhanced, and the time-dependent intensity reaches a higher steady state value. We can see from equation (28), that for a given ϵ, the source term due to the pulse (the last term on the right-hand side of the equation) decreases and the thermal source (the second term on the right-hand side) saturates to the value ϵB as time $t \to \infty$. This is due to our assumption of a constant thermal source in the medium.

When we increase ϵ, the photons become thermalized within fewer scatterings. Hence the radiation reaches steady state at a faster rate. For example, from Figure 4, we see that the radiation reaches 75% of its steady state value at time point $t \simeq 1.2$ for $\epsilon = 1.0$ and at $t \simeq 2.0$ for $\epsilon = 0.5$. This behavior is common for both the angles. When $\epsilon \neq 0$, we have thermal sources situated throughout the medium, and hence the emergent intensity does not have characteristics different from those of the reflected radiation, contrary to the case of conservatively scattering medium ($\epsilon = 0$). This can be seen from Figures 1–10.

b) Effects of the Searchlight-Beam Boundary Condition on the Time-dependent Radiation Field

A slab is subject to a searchlight beam with Dirac-delta time distribution. The incident radiation can be written as

$$I(t, \tau = T, -\mu) = I_0 t_a \delta(t)\delta(\mu - \mu_0) ,$$
$$I(t, \tau = 0, \mu) = 0 , \qquad (29)$$

where μ_0 is the cosine of the angle at which the beam falls on the medium.

i) *Results for a Conservatively Scattering Medium*

The numerical solutions for this case are plotted in Figures 11 to 20 for the same set of values of the de-excitation parameter $\epsilon(\epsilon = 0, 0.1, 0.5, \text{and } 1)$ as in the previous case. First we consider the case of a conservatively scattering medium ($\epsilon = 0$), and the effect of thermal sources will be studied later. It can be derived as in the previous case that the source term due to directly transmitted radiation is

$$h(t, \tau) = 0.5(1 - \epsilon)I_0 e^{-t/t_a} e^{-(T-\tau)/\mu_0} .$$

We choose I_0 and $\mu_0 = 1$.

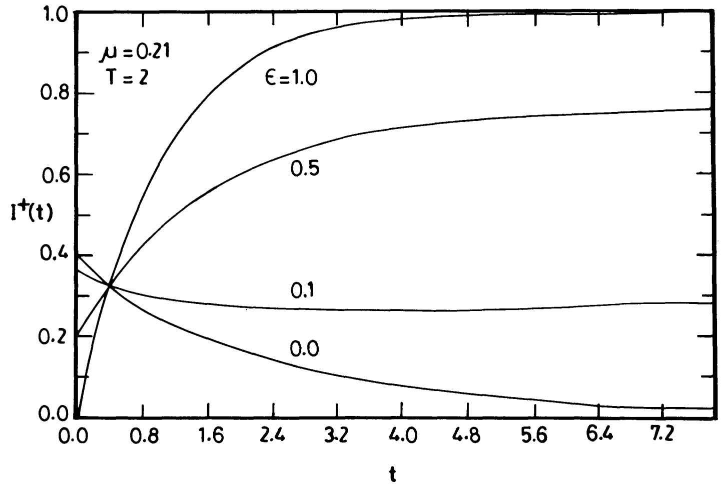

FIG. 13.—Same as Fig. 11 with $T = 2.0$

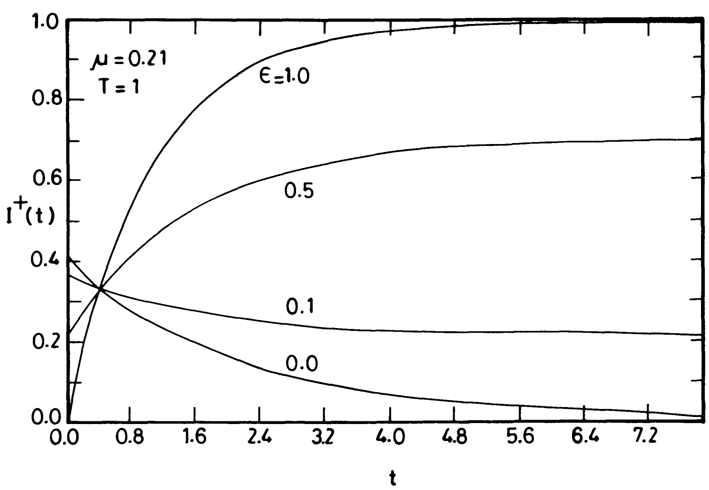

FIG. 11.—Reflected intensity as a function of time t, at $\mu = 0.21$, for different colisional de-excitation parameters ϵ, when a searchlight beam is incident on the medium with $T = 1.0$.

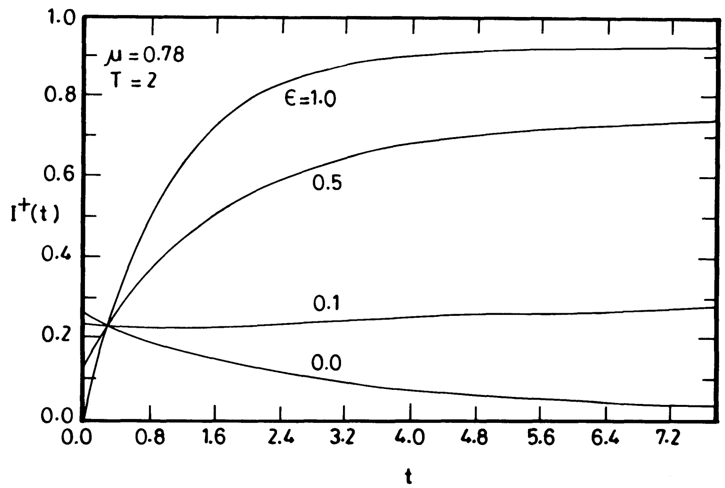

FIG. 14.—Same as Fig. 12 with $T = 2.0$

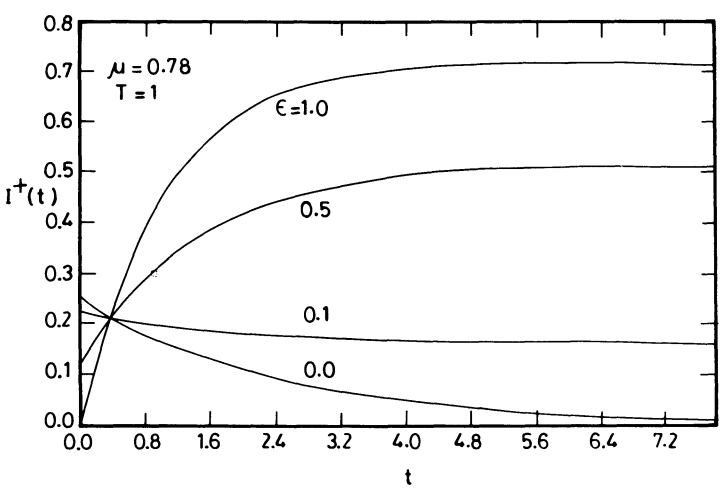

FIG. 12.—Same as Fig. 11 for $\mu = 0.78$

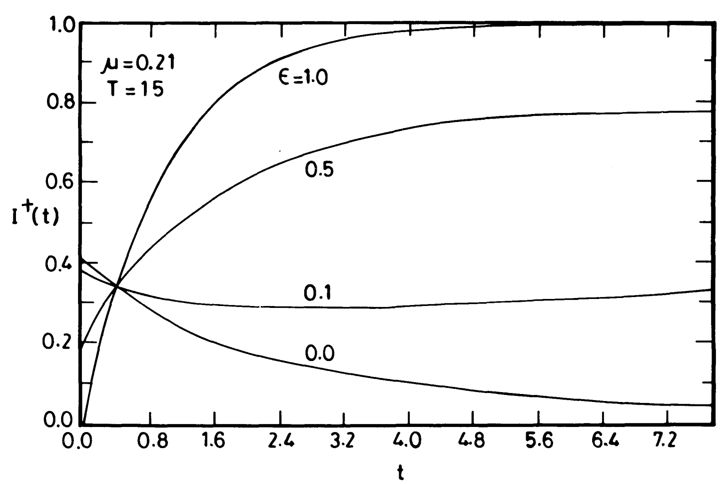

FIG. 15.—Same as Fig. 11 with $T = 15.0$

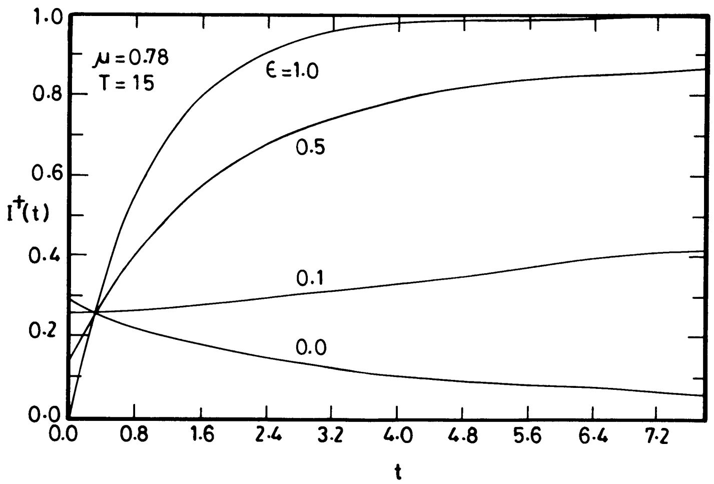

FIG. 16.—Same as Fig. 12 with $T = 15.0$

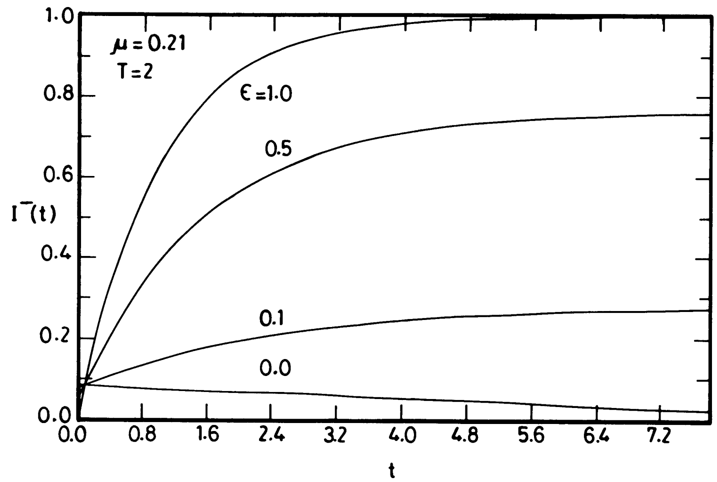

FIG. 19.—Same as Fig. 17 with $T = 2.0$

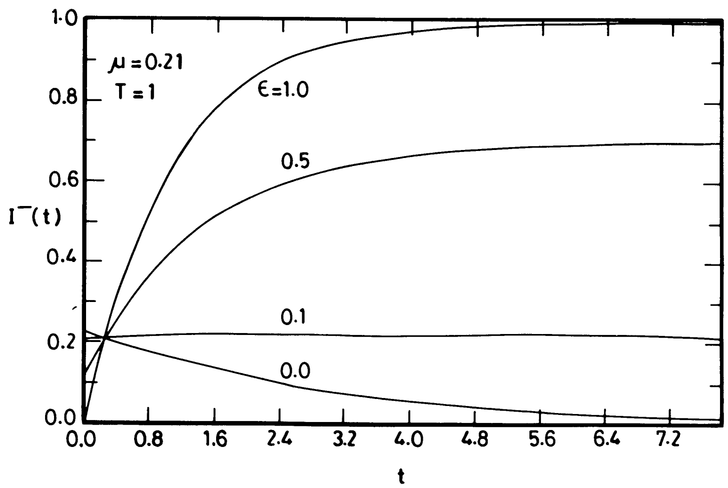

FIG. 17.—Emergent intensity as a function of time t, at $\mu = 0.21$, for different collisional de-excitation parameters ϵ, when a searchlight beam is incident on the medium with $T = 1.0$.

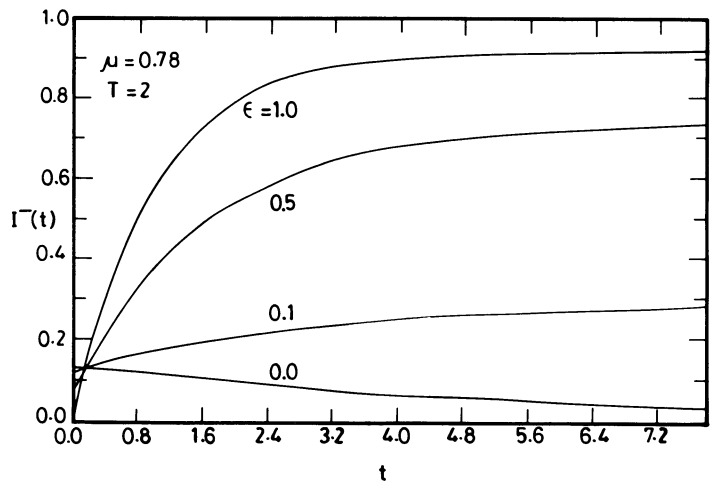

FIG. 20.—Same as Fig. 18 with $T = 2.0$

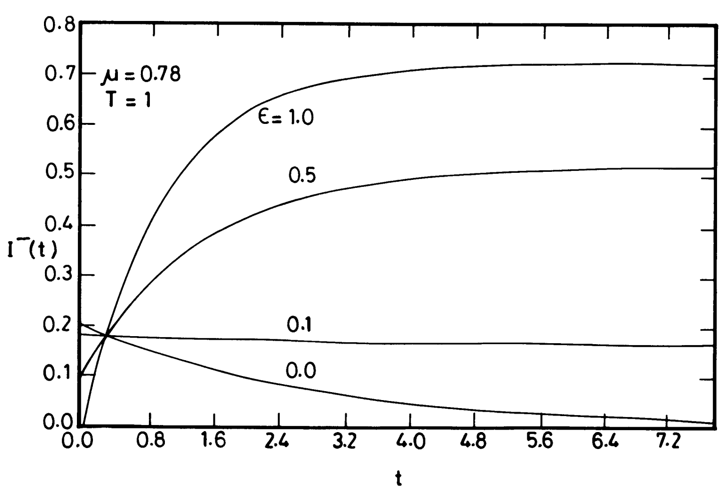

FIG. 18.—Same as Fig. 17 for $\mu = 0.78$

The source term due to the searchlight beam is larger than the term for the isotropic pulse (compare eq. [27] and the above equation):

$$0.5e^{-t/t_a}e^{-(T-\tau)} > 0.5e^{-t/t_a}\int_0^1 e^{-(T-\tau)\mu'}\,d\mu' . \qquad (30)$$

Hence the reflected intensity values at earlier time points are higher than for the case of an isotropic pulse. This can be seen from the figures for the time-dependent reflected intensities (Figs. 11 and 12) for $\mu = 0.21$ and $\mu = 0.78$ ($T = 1$). The same can be seen from Figures 13 and 14 for $T = 2$, and from Figures 15 and 16 for $T = 15$. In Table 1, we tabulated the values of time taken by the reflected radiation field to decay by 50% and 75%.

From Table 1, we can see that the values are slightly higher than the quantities for the isotropic case. Also, when $\mu = 0.21$, the values are nearly same for the medium with $T = 2$ and $T = 15$. We see that the reflected radiation in the direction $\mu = 0.78$ takes longer time to decay than the radiation reflected in the grazing angle, as it undergoes a larger number of scatterings and spends more time in the medium before it escapes.

The diffuse emergent intensities are plotted in Figures 17 and 18 when $T = 1$, and in Figures 19 and 20 when $T = 2$. The intensities start with higher values than in the case of the isotropic pulse. In Table 1, the values of time taken by the emergent radiation field to decay by 50% and 75% are tabulated. We can see from the above table that the values are smaller compared to the case of the isotropic pulse.

We checked the time-dependent reflected intensities for $T = 15$ with the Ganapol and Matsumoto (1986) values for the semi-infinite atmosphere. We found that the results for the semi-infinite atmosphere, using their code, agreed within 5% to our calculations of reflected intensities from a medium with $T = 15.0$.

ii) *Effect of Constant Thermal Source*

We see from Figures 11–16 that the radiation field evolves to the steady state for $\epsilon = 0$ as in the case when an isotropic pulse impinges on the medium. We see that the reflected radiation increases with time and reaches the steady state. When $\epsilon = 0.1$, we see that the radiation field is a slowly varying function of time.

The effect of thermal sources on the transmitted intensities for $\mu = 0.1$ and $\mu = 0.78$ can be seen in Figures 17 and 18 when $T = 1$ and in Figures 19 and 20 when $T = 2$. They all reach steady state as $t \to \infty$ for $\epsilon \neq 0$ as in the case of the reflected radiation.

c) *Effect of Constant Input of Radiation*

A slab is subjected to a constant input of radiation. The boundary condition is given by

$$I(t, \tau = T, -\mu) = H(t) , \tag{31}$$

where $H(t)$ is the Heaviside unit step function defined by

$$\begin{aligned} H(t) &= 0 \quad \text{for } t < 0 , \\ &= 1 \quad \text{for } t \geq 0 . \end{aligned} \tag{32}$$

Now the incident distribution is nonsingular, and so we solve the transfer equation directly without taking recourse to the steps of splitting the radiation field into directly transmitted and diffuse radiation field.

The reflected intensities for $\mu = 0.21$ and $\mu = 0.78$ are plotted in Figures 21 and 22 for different collisional de-excitation parameter values ($\epsilon = 0$, 0.5, and 1.0) for $T = 2$. For a conservatively scattering medium, if we consider the time $\tilde{t}$ at which the ratio $I(\tilde{t}, T, \mu)/\hat{I}(T, \mu)$ is 0.5, [$\hat{I}(T, \mu)$ is the steady state value], $\tilde{t}$ is 1.5 for $\mu = 0.21$, and 1.9 for $\mu = 0.78$. Hence the photons reflected in the direction $\mu = 0.21$ approach the steady state faster than the photons reflected in the normal direction. As explained previously, the photons reflected in the grazing direction experience few scatterings and hence spend less time in the medium. Evolution of the radiation field to the steady state is faster, since thermal sources increase in the medium. The time $\tilde{t}$ at which the radiation field builds to 50% of the steady state is nearly same for the radiation reflected in both the directions. This is because the thermal sources dominate the picture and they are isotropic. The $\tilde{t}$ is 1.1 and 0.7 for $\epsilon = 0.5$ and 1.0, respectively. The reflected intensity distribution when $T = 15.0$ is plotted in Figures 25 and 26. The $\tilde{t}$ is 2.0 and 3.0 for a conservatively scattering atmosphere. When thermal sources are present, $\tilde{t}$ is nearly the same for both the angles and is equal to 1.1 for $\epsilon = 0.5$ and 0.7 for $\epsilon = 1.0$.

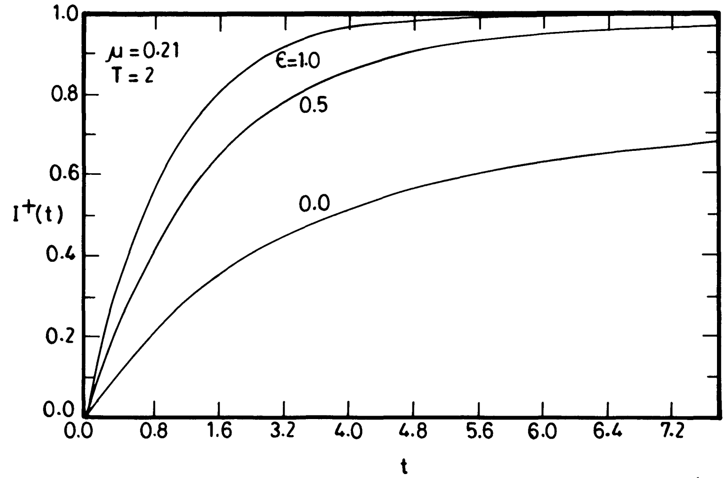

FIG. 21.—Reflected intensity as a function of time t, at $\mu = 0.21$, for different collisional de-excitation parameters ϵ, when a constant isotropic radiation is incident on the medium with $T = 2.0$.

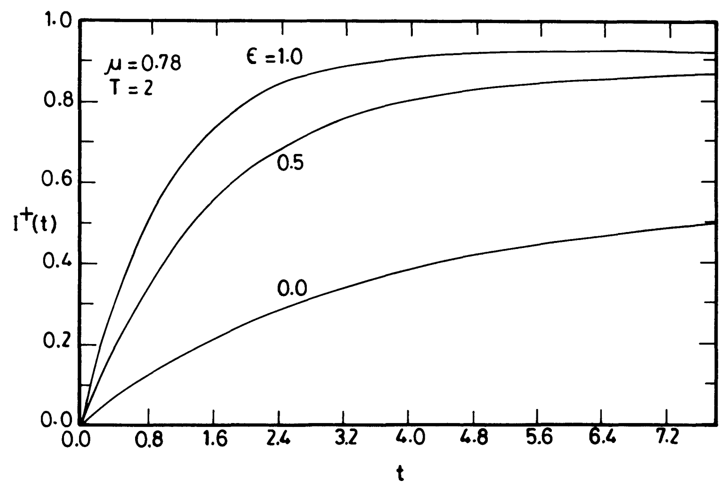

FIG. 22.—Same as Fig. 21 for $\mu = 0.78$

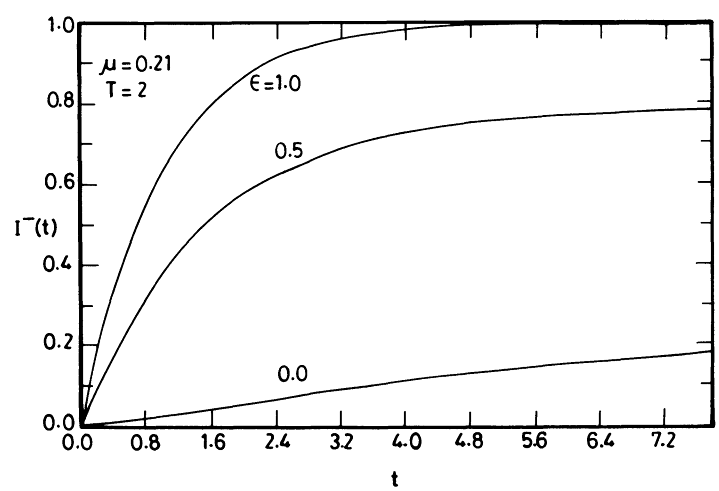

FIG. 23.—Emergent intensity as a function of time t, at $\mu = 0.21$, for different collisional de-excitation parameters ϵ, when a constant isotropic radiation is incident on the medium with total optical depth $T = 2.0$.

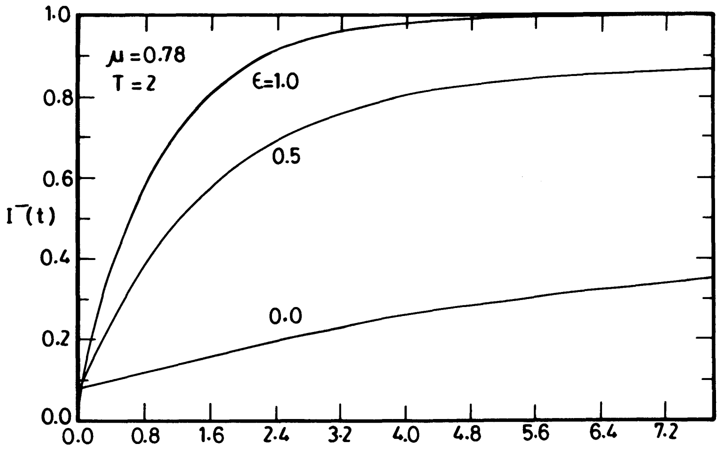

FIG. 24.—Same as Fig. 23 for $\mu = 0.78$

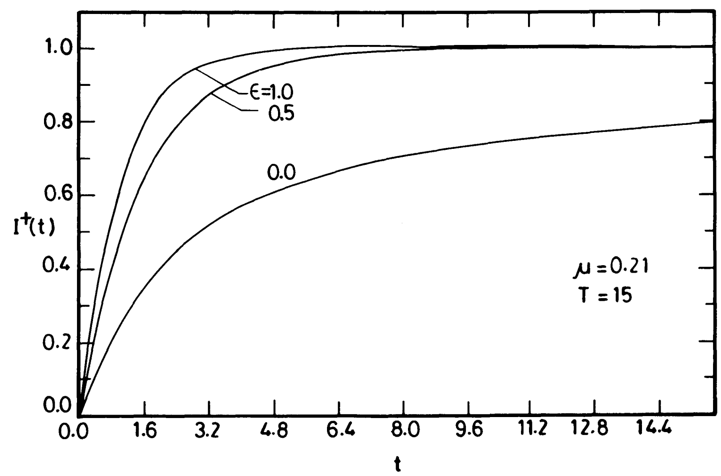

FIG. 25.—Same as Fig. 21 with $T = 15.0$

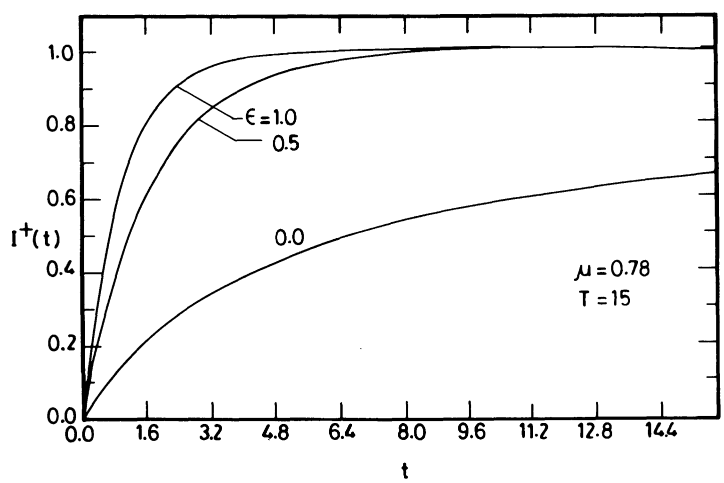

FIG. 26.—Same as Fig. 22 with $T = 15.0$

The time-dependent emergent intensities are plotted in Figures 23 and 24 when $T = 2.0$. The value of $\tilde{t}$ to which we referred earlier is 3.2 and 1.8 for $\mu = 0.21$ and $\mu = 0.78$, respectively, when $\epsilon = 0.0$. From this, we can see that the emergent photons coming in the directions $\mu = 0.78$ reach the steady state faster. The $\tilde{t}$ is 1.1 and 0.7 for both the angles when $\epsilon = 0.5$ and 1.0, respectively.

d) Discussion of the Numerical Checks Employed to Verify the Results

To verify whether the time-dependent radiation field is areaching the correct steady state value, we solved separately the time-independent radiative transfer equation using the method due to Peraiah and Grant (1973). We found that the steady state values attained by the time-dependent radiation field as $t \to \infty$ tallied with the solutions of the steady state equation within 1%.

We should note that the steady state value, which is attained by the time-dependent radiation field when a constant source of radiation impinges on the medium, is independent of the value of t_a. We should also note that the numerical value of the time-dependent radiation field can reach a correct steady state value even if an error is committed in the calculation of the radiation field at initial time points. As an additional check for the radiation at earlier times, we compared the time-dependent reflected intensity distribution for a conservatively scattering atmosphere with $T = 15.0$ with those of the Ganapol and Matsumoto (1986) results for the semi-infinite atmosphere.

IV. CONCLUSIONS

We have showed quantitatively that the time-dependent problem for $t_a \gg t_f$ is the same as finding the distribution of photons over the number of scatterings. When an isotropic pulsed beam of radiation impinges on a conservatively scattering medium, the reflected intensity distribution decays gradually with time as the reflected ray approaches the normal direction. For the same input of radiation, if the thermal sources are present in the medium ($\epsilon \neq 0$), the emergent and reflected intensity distributions reach to their respective steady state values, and as ϵ increases, they reach the steady state values at a faster rate, irrespective of the direction of the radiation field. If a conservatively scattering medium is subjected to an isotropic radiation of constant intensity, the reflected intensity distribution reaches the steady state faster as the angle of reflection approaches the grazing angle. The transmitted intensity distribution reaches the steady state faster when the transmitted ray approaches the normal direction.

One of us (D.M.R.) would like to thank B. D. Ganapol for supplying the Fortran code for calculating the time-dependent radiation field in semi-infinite atmosphere, and W. Kalkofen for some useful suggestions. We thank M. Sreenivasa Rao and B. A. Varghese for help in preparation of the manuscript. We are grateful to the unknown referee for improving the text.

APPENDIX A

DERIVATION OF THE TIME-DEPENDENT LINE TRANSFER EQUATION

The rate equation for a two-level atom is given by

$$n_1\left(B_{12}\int \phi_\nu J_\nu d\nu + C_{12}\right) - n_2\left(A_{21} + B_{21}\int \phi_\nu J_\nu d\nu + C_{21}\right) = \frac{\partial n_2}{\partial t}\,, \tag{A1}$$

where the symbols have their usual meanings.

For resonance lines, n_1 may be taken as a constant and independent of time. With this approximation, equation (A1) becomes

$$\frac{\partial}{\partial t}\left(\frac{n_2}{n_1}\right) + \left(\frac{n_2}{n_1}\right)\left(A_{21} + B_{21}\int \phi_\nu J_\nu d\nu + C_{21}\right) = \left(B_{12}\int \phi_\nu J_\nu d\nu + C_{12}\right). \tag{A2}$$

Solution of equation (A2) for ratio n_2/n_1 will contain terms which are non-linear in the radiation field. Then it is difficult to obtain a compact form for the time-dependent line source function if one includes the stimulated emission terms, unlike in the case of the steady state derivation. Therefore we will neglect stimulated emission and obtain a simpler relation in terms of the radiation field. This assumption is justified, since stimulated emission is important only in very few cases of common interest (for, e.g., lasers and masers). Now equation (A2) becomes

$$\frac{\partial}{\partial t}\left(\frac{n_2}{n_1}\right) + \left(\frac{n_2}{n_1}\right)(A_{21} + C_{21}) = \left(B_{12}\int \phi_\nu J_\nu d_\nu + C_{12}\right). \tag{A3}$$

We shall assume C_{12} and C_{21} are constant with respect to time. Solving equation (A3), we get

$$\left(\frac{n_2}{n_1}\right) = \int_0^t e^{-(t-t')/t_a}\left(B_{12}\int \phi_\nu J_\nu d_\nu + C_{21}\frac{g_2}{g_1}e^{-h\nu/kT_e}\right)dt'\,, \tag{A4}$$

where

$$t_a = \frac{1}{A_{21} + C_{21}}\,, \quad \text{and } n_2(t=0) = 0\,.$$

We will use the relation (A4) in the derivation of the line transfer equation. The transfer equation is given by

$$\frac{dI_\nu}{ds} = [n_2(A_{21} + B_{21}I_\nu) - n_1 B_{12} I_\nu]\phi_\nu(h\nu/4\pi)\,. \tag{A5}$$

We define the optical depth scale in terms of the frequency integrated line opacity (which characterizes the average opacity in the line as a whole), $d\tau = -\chi_{12}\,dz$, where

$$\chi_{12} = (n_1 B_{12} - n_2 B_{21})(h\nu/4\pi)\,. \tag{A6}$$

We assume that the medium is stratified in plane-parallel layers whose normal is in the z-direction, and we denote the cosine of the angle made by the ray to the normal by μ. Then $dz = \mu\,ds$. Now the equation of transfer is given by

$$\mu(dI_\nu/d\tau) = \phi_\nu(I_\nu - S)\,, \tag{A7}$$

where

$$S = n_2 A_{21}/(n_1 B_{12} - n_2 B_{21})\,. \tag{A8}$$

Note that here the dimensions of $d\tau$ are s^{-1} (frequency units). Neglecting the stimulated emission term in equation (A8), the source function S reduces to

$$S = (n_2/n_1)(A_{21}/B_{12}) = (n_2/n_1)(g_1/g_2)(2h\nu^3/c^2)\,. \tag{A9}$$

Now substituting for the ratio (n_2/n_1) from equation (A4), we get the source function S as

$$S = (2h\nu^3/c^2)(g_1/g_2)\int_0^t e^{-(t-t')/t_a}\left(B_{12}\int \phi_\nu J_\nu d\nu + C_{21}\frac{g_2}{g_1}e^{-h\nu/kT_e}\right)dt'\,. \tag{A10}$$

Now defining the parameter ϵ',

$$\epsilon' = \frac{C_{21}(1 - e^{-h\nu/kT_e})}{A_{21}}\,, \tag{A11}$$

we have

$$\epsilon = \frac{\epsilon'}{1+\epsilon'} = \frac{C_{21}(1 - e^{-h\nu/kT_e})}{A_{21} + C_{21}(1 - e^{-h\nu/kT_e})}\,, \tag{A12}$$

When $h\nu/kT \gg 1$ (i.e., stimulated emission is negligible), equation (A12) becomes

$$\epsilon = \frac{C_{21}}{A_{21} + C_{21}} , \tag{A13}$$

and the Planck function can be written as

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{(e^{h\nu/kT_e} - 1)} = \frac{2h\nu^3}{c^2} e^{-h\nu/kT_e} . \tag{A14}$$

Using the relations (A12)–(A14), equation (A10) can be written as

$$S = (1 - \epsilon) \int_0^t e^{-(t-t')/t_a} \int \phi_\nu J_\nu d\nu \frac{dt'}{t_a} + \epsilon B(1 - e^{-t/t_a}) . \tag{A15}$$

It is convenient to work with the dimensionless frequency variable x, measured from the line center in units of Doppler widths [i.e., $x = (\nu_0 - \nu)/\Delta\nu_D$). In terms of this variable, we shall write the Doppler profile as

$$\phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} , \tag{A16}$$

and the Voigt profile as

$$\phi(x) = \frac{a}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(x - y)^2 + a^2} , \tag{A17}$$

which are normalized such that

$$\int_{-\infty}^{\infty} \phi(x) dx = 1 . \tag{A18}$$

Note that

$$\phi(\nu) = \frac{1}{\Delta\nu_D} \phi(x) . \tag{A19}$$

We will absorb a factor of $\Delta\nu_D$ into the definition of χ and write $d\tau = -\chi_1 dz$, where $\chi_1 = \chi_{12}/\Delta\nu_D$, χ_{12} being given by equation (A6). Now $d\tau$ is dimensionless. Then the line source function S is given by

$$S = (1 - \epsilon) \int_0^t e^{-(t-t')/t_a} \int_{-\infty}^{\infty} \phi(x) J(x) dx \frac{dt'}{t_a} + \epsilon B(1 - e^{-t/t_a}) , \tag{A20}$$

and the transfer equation becomes

$$\mu(dI_x/d\tau) = \phi_x(I_x - S) . \tag{A21}$$

Let us assume that the absorption through the line is uniform and has a width $\Delta\nu$ (Milne 1926). Then the profile function is given by

$$\phi(\nu) = 1/\Delta\nu . \tag{A22}$$

With this profile function, the expression for the source function reduces to

$$S = (1 - \epsilon) \int_0^t e^{-(t-t')/t_a} J_\nu \frac{dt'}{t_a} + \epsilon B_\nu(1 - e^{-t/t_a}) . \tag{A23}$$

APPENDIX B

DERIVATION OF THE TIME-DEPENDENT TRANSFER EQUATION FOR THE BOUND-FREE CONTINUUM

Let our model atom consist of a bound level i and the contiuum k. Then the time-dependent ionization equation can be written as

$$\frac{\partial n_k}{\partial t} = n_i(R_{ik} + C_{ik}) - n_k(R_{ki} + C_{ki}) , \tag{B1}$$

where n_k is the ion density and n_i is the number density of the particles in the bound state i. R_{ik} and C_{ik} are the photoionization and collisional rate coefficients from the level i to the continuum k, respectively. C_{ki} and R_{ki} are the collisional and spontaneous recombination rates, respectively. The quantities R_{ki} and R_{ik} are defined by the relations

$$R_{ki} = 4\pi\left(\frac{n_i}{n_k}\right)^* \int_{\nu 0}^{\infty} \alpha_{ik}(\nu)(h\nu)^{-1} B_\nu \, d\nu \,,$$

$$R_{ik} = 4\pi \int_{\nu 0}^{\infty} \alpha_{ik}(\nu)(h\nu)^{-1} J_\nu \, d\nu \,,$$

where $\alpha_{ik}(\nu)$ is the absorption cross section, and the superscript "*" denotes LTE values. For the definition of the quantities C_{ik} and C_{ki}, see Mihalas (1978). We shall neglect the stimulated recombination terms from the rate equations. We can make the assumption that $(n_k/n_i) \ll 1$ if the medium is sufficiently cool for the formation of an optically thick bound-free continuum. Dividing equation (1) by n_i, we have

$$\frac{\partial}{\partial t}\left(\frac{n_k}{n_i}\right) = (R_{ik} + C_{ik}) - \left(\frac{n_k}{n_i}\right)(R_{ki} + C_{ki}) \,. \tag{B2}$$

Then

$$\frac{d}{dt}\left[e^{(R_{ki}+C_{ki})t} \frac{n_k}{n_i}\right] = e^{(R_{ki}+C_{ki})t}(R_{ki} + C_{ki}) \,. \tag{B3}$$

Let $t_a = 1/(R_{ki} + C_{ki})$. Then

$$\frac{d}{dt}\left(e^{t/t_a} \frac{n_k}{n_i}\right) = e^{t/t_a}(R_{ik} + C_{ik}) \,. \tag{B4}$$

Integrating equation (B4) from 0 to t, and taking $n_k(t = 0) = 0$, we have

$$\frac{n_k(t)}{n_i} = \int_0^t e^{-(t-t')/t_a}(R_{ik} + C_{ik})dt' \,. \tag{B5}$$

By assuming that C_{ik} is independent of time t, and using the definition for R_{ik}, we can write equation (B5) as

$$\frac{n_k(t)}{n_i} = 4\pi t_a \int_0^t e^{-(t-t')/t_a} \int_{\nu 0}^{\infty} \frac{\alpha_{ik}(\nu)}{h\nu} J_\nu \, d\nu \frac{dt'}{t_a} + C_{ik}(1 - e^{-t/t_a})t_a$$

$$= 4\pi \frac{1}{(R_{ki} + C_{ki})} \int_0^t e^{-(t-t')/t_a} \int_{\nu 0}^{\infty} \left[\frac{\alpha_{ik}(\nu)}{h\nu} J_\nu \, d\nu\right] \frac{dt'}{t_a} + C_{ik}(1 - e^{-t/t_a})t_a \,. \tag{B6}$$

We will use the relation (B6) in the derivation of the transfer equation for the bound-free continuum. The transfer equation is given by

$$\mu \frac{dI_\nu}{dz} = \eta_\nu - \chi_\nu I_\nu \,, \tag{B7}$$

where the emission coefficient η_ν is given by

$$\eta_\nu = \frac{2h\nu^3}{c^2} n_k\left(\frac{n_i}{n_k}\right)^* \alpha_{ik}(\nu) e^{-h\nu/kT} \,,$$

and the absorption coefficient χ_ν is

$$\chi_\nu = n_i \alpha_{i\nu}(\nu) \,.$$

If we define the optical depth $d\tau_\nu = -n_i \alpha_{ik}(\nu) dz$, equation (B7) can be written as

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \frac{n_k}{n_i}\left(\frac{n_i}{n_k}\right)^* B_\nu$$

$$= I_\nu - 4\pi\left(\frac{n_i}{n_k}\right)^* \frac{B_\nu}{(R_{ki} + C_{ki})} \int_0^t e^{-(t-t')/t_a}\left[\int_{\nu 0}^{\infty} \frac{\alpha_{ik}(\nu)}{h\nu} J_\nu \, d\nu\right] \frac{dt'}{t_a} - C_{ik}(1 - e^{-t/t_a})t_a\left(\frac{n_i}{n_k}\right)^* B_\nu \,. \tag{B8}$$

Equation (B8) can be written as

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \gamma_\nu \int_0^t e^{-(t-t')/t_a}\left(\int_{\nu 0}^{\infty} \Phi_{i\nu} J_\nu \, d\nu\right) \frac{dt'}{t_a} - \epsilon_\nu B_\nu(1 - e^{-t/t_a}) \tag{B9}$$

where

$$\Phi_{i\nu} = 4\pi\alpha_{ik}/h\nu \,, \qquad \gamma_\nu = (n_i/n_k)^* B_\nu/(R_{ki} + C_{ki}) \,, \qquad \epsilon_\nu = (n_i/n_k)^* C_{ik}/(R_{ki} + C_{ki}) \,.$$

Equation (B9) is similar to the steady state transfer equation for bound-free continuum (see eqs. [7–131] in Mihalas 1978) except for the time-dependent terms.

We can derive the monochromatic transfer equation from equation (B9) by noting that, for $h\nu/kT \gg 1$, the frequency variation of J_ν and B_ν shows so rapid a drop with increasing ν that practically all of the contribution to the photoionization and recombination rate integrals comes from $\nu \simeq \nu_0$. Then we can approximate the integrals

$$\int_{\nu 0}^{\infty} \Phi_{i\nu} J_\nu = 4\pi w_0(\alpha_0/h_\nu)J_0 ,$$

$$\int_{\nu 0}^{\infty} \Phi_{i\nu} B_\nu = 4\pi w_0(\alpha_0/h\nu_0)B_0 ,$$

where W_0 is an approximate weight factor. Then

$$\epsilon = \frac{(n_i/n_k)^* C_{ik}}{(n_i/n_k)^*[4\pi\alpha_0 B_0 w_0(h\nu)^{-1} + C_{ik}]} = \frac{C_{ik}}{4\pi\alpha_0 B_0 w_0(h\nu)^{-1} + C_{ik}} ,$$

and similarly

$$\gamma = \frac{B_0}{4\pi\alpha_0 B_0 w_0(h\nu)^{-1} + C_{ik}} .$$

At $\nu = \nu_0$, equation (B9) can be written as

$$\mu \frac{dI}{d\tau} = I - (1 - \tilde{\epsilon}) \int_0^t e^{-(t-t')/t_a} J_0 \frac{dt'}{t_a} - \tilde{\epsilon} B_0(1 - e^{-t/t_a}) , \tag{B10}$$

where $\tilde{\epsilon} = C_{ik}/[4\pi w_0(\alpha_0/h\nu_0)B_0 + C_{ik}]$.

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A. PERAIAH, K. E. RANGARAJAN, and D. MOHAN RAO: Indian Institute of Astrophysics, Bangalore 560 034, India