

One-third spectrum in large scale Hall-magnetohydrodynamic turbulence

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In a recent study [H. Branover, A. Eidelman, and E. Golbraikh, "On the universality of large-scale turbulence," *Phys. Fluids* **16**, 845 (2004)] the properties of the large scale turbulence have been investigated theoretically and experimentally concluding that the kinetic energy spectrum goes as $k^{1/3}$ at large spatial scales and citing a few examples for the existence of such a spectrum in natural systems [J. C. Kaimal, R. A. Eversole, D. H. Lenschow, B. B. Stankov, P. H. Kahn, and J. A. Businger, "Spectral characteristics of the convective boundary layer over uneven terrain," *J. Atmos. Sci.* **39**, 1098 (1982); I. Gulpepe and D. O'C. Starr, "Dynamical structure and turbulence in cirrus clouds: Aircraft observations during FIRE," *ibid.* **52**, 4159 (1995)]. In this paper we show that the $1/3$ spectrum for the kinetic energy is a direct consequence of the magnetic helicity invariant of the Hall-magnetohydrodynamic turbulence. We present the simultaneous kinetic and magnetic energy spectra and propose the verification of the latter in the laboratory and natural systems. © 2004 American Institute of Physics. [DOI: 10.1063/1.1794756]

The phenomenology of the Kolmogorov¹ spectrum gets significantly extended when one ventures into the realm of the magnetohydrodynamic (MHD) turbulence. The MHD turbulence has been studied within the ideal MHD (Ref. 2) as well as beyond it, including nonideal effects such as the Hall effect in the two-fluid picture.³ The Kolmogorov phenomenology of the Hall-MHD (HMHD) has been investigated and applied to the observed spectra of the solar atmosphere and the solar wind⁴ successfully. Recently, the universality of the large scale turbulence with the kinetic energy spectrum going as $k^{1/3}$ has been concluded from the laboratory experiments on MHD turbulence.⁵ Evidence in favor of such a spectrum has also been seen in convective atmospheric boundary layer^{6,7} along with the usual Kolmogorov spectrum $k^{-5/3}$. Cirrus clouds⁸ have too been observed to support $k^{1/3}$ spectrum. In this paper, we present the spectral distributions of the kinetic and the magnetic energy densities, within the Hall-MHD, first derived for and applied to the solar atmospheric⁴ turbulence. While some of the spectral predictions of the Hall-MHD found ratification in the solar turbulence, the $1/3$ kinetic energy spectrum could not be tested due to lack of observations on those scales. Although the $1/3$ spectrum has been theoretically derived invoking the conservation of the total energy flux⁵ in a given volume, we show that the Hall-MHD with its new features offers a better alternative.

The HMHD system consists of³

$$\frac{\partial \mathbf{\Omega}_j}{\partial t} - \nabla \times [(\mathbf{U}_j - \mu_j \mathbf{\Omega}_j) \times \mathbf{\Omega}_j] = 0, \quad \text{where } j = 1, 2,$$

$$\mathbf{\Omega}_1 = \mathbf{B}, \quad \mathbf{\Omega}_2 = \mathbf{B} + \nabla \times \mathbf{V},$$

$$\mathbf{U}_1 = \mathbf{V} - \nabla \times \mathbf{B}, \quad \mathbf{U}_2 = \mathbf{V}, \quad (1)$$

where μ 's are scale parameters. The magnetic field \mathbf{B} , the velocity \mathbf{V} , the space and time scales are, respectively, normalized to an arbitrary field B_0 , the Alfvén speed $V_A = B_0 / \sqrt{4\pi Mn}$, the ion-skin depth $\lambda_i = c / (4\pi me^2 / M)^{1/2}$, and the cyclotron time $\tau_c = Mc / eB_0$.

In this model, inertialess electrons are frozen to the magnetic field lines; the ions, on the other hand, are not frozen in due to their additional vortical motion. Note that in Eq. (1), obtained by taking the curl of the ion force balance equation, the pressure gradient term $\nabla P / n$ has disappeared because it has been assumed to be a perfect gradient [by invoking an equation of state $P = P(n)$, for example]; the pressure has not been neglected.

The three quadratic invariants of the Hall-MHD system are found to be

$$\text{total energy } E = \frac{1}{2} \int (V^2 + B^2) d^3x = \frac{1}{2} \sum_k |V_k|^2 + |B_k|^2, \quad (2)$$

$$\begin{aligned} \text{magnetic helicity } H_M &= \frac{1}{2} \int \mathbf{A} \cdot \mathbf{B} d^3x \\ &= \frac{1}{2} \sum_k \frac{i}{k^2} (\mathbf{k} \times \mathbf{B}_k) \cdot \mathbf{B}_{-k}, \end{aligned} \quad (3)$$

$$\begin{aligned} \text{generalized helicity } H_G &= \frac{1}{2} \int \mathbf{A} + \mathbf{V} \cdot (\mathbf{B} + \nabla \times \mathbf{V}) d^3x \\ &= \frac{1}{2} \sum_k \left[\frac{i\mathbf{k} \times \mathbf{B}_k}{k^2} + \mathbf{V}_k \right] \\ &\quad \cdot [\mathbf{B}_{-k} - i\mathbf{k} \times \mathbf{V}_{-k}], \end{aligned} \quad (4)$$

where \mathbf{A} is the vector potential. Note that $H_G - H_M$ is a combination of the kinetic and the cross helicities.

We split the fields⁹ into their ambient and the fluctuating parts (there is no ambient flow),

$$\mathbf{B} = \hat{e}_s + \mathbf{b}; \quad \mathbf{V} = \mathbf{v}, \quad (5)$$

and substitute in Eq. (1) to get

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times [(\mathbf{v} - \nabla \times \mathbf{b}) \times \hat{e}_s + (\mathbf{v} - \nabla \times \mathbf{b}) \times \mathbf{b}], \quad (6)$$

$$\frac{\partial}{\partial t}(\mathbf{b} + \nabla \times \mathbf{v}) = \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v} + \mathbf{b}) + \mathbf{v} \times \hat{e}_s]. \quad (7)$$

The nonlinear problem represented by Eqs. (6) and (7) is converted to a set of linear problems (the time honored method for solving nonlinear equations) by imposing the following conditions:

$$\mathbf{v} - \nabla \times \mathbf{b} = \alpha \mathbf{b}, \quad (8)$$

$$\mathbf{b} + \nabla \times \mathbf{v} = \beta \mathbf{v}, \quad (9)$$

where α and β are like the separation constants. With the nonlinearities so taken care of, we are left with the remaining time-dependent linear equations

$$\frac{\partial \mathbf{b}}{\partial t} = \alpha \nabla \times [\mathbf{b} \times \hat{e}_s], \quad (10)$$

$$\frac{\partial}{\partial t}(\mathbf{v}) = (1/\beta) \nabla \times [\mathbf{v} \times \hat{e}_s]. \quad (11)$$

Apparently we have traded a close nonlinear system (six equations for six variables) for an overdetermined linear system (8)–(11) with 12 equations in six variables. Acceptable solutions, therefore, will be possible only under some particular conditions that will remove the overdetermination. To seek them, we first note that (10) and (11) admit

$$\mathbf{b} = \mathbf{b}_k \exp [i\mathbf{k} \cdot \mathbf{x} + i\alpha(\hat{e}_s \cdot \mathbf{k})t], \quad (12)$$

$$\mathbf{v} = \mathbf{v}_k \exp \left(i\mathbf{k} \cdot \mathbf{x} + i\frac{1}{\beta}(\hat{e}_s \cdot \mathbf{k})t \right). \quad (13)$$

If the exponential solutions (12) and (13) are to satisfy the linear equations (8) and (9), we must require $\beta = 1/\alpha$. In addition, substituting Eqs. (12) and (13) into Eqs. (8) and (9) leads to

$$\mathbf{v}_k - i\mathbf{k} \times \mathbf{b}_k = \alpha \mathbf{b}_k, \quad (14)$$

$$\mathbf{v}_k + i\mathbf{k} \times \mathbf{v}_k = \frac{1}{\alpha} \mathbf{v}_k \quad (15)$$

which, after simple manipulation, yield

$$\mathbf{v}_k - \alpha \mathbf{b}_k = i\alpha \mathbf{k} \times \mathbf{v}_k, \quad (16)$$

$$\mathbf{v}_k - \alpha \mathbf{b}_k = i(\mathbf{k} \times \mathbf{b}_k), \quad (17)$$

with the consequences

$$\mathbf{b}_k = \alpha \mathbf{v}_k \quad (18)$$

relating \mathbf{b}_k and \mathbf{v}_k , and

$$\mathbf{k} \times \mathbf{v}_k = -i\frac{1 - \alpha^2}{\alpha} \mathbf{v}_k = -i\lambda \mathbf{v}_k. \quad (19)$$

The first of these establishes the HMHD equivalent of the Alfvénic condition for MHD ($\mathbf{b} = \pm \mathbf{v}$), and the second, the Fourier transform of a Beltrami equation ($\nabla \times \mathbf{G} = \lambda \mathbf{G}$), has to be solved to complete the story; the solvability constraint will end up relating α with k giving the “dispersion relation” $\alpha = \alpha(k)$, and α is determined by

$$\alpha_{\pm} = \left[-\frac{k}{2} \pm \left(\frac{k^2}{4} + 1 \right)^{1/2} \right] \quad (20)$$

and

$$\lambda = \pm k. \quad (21)$$

Thus the fluctuations are more in the nature of nonlinear waves as discussed in detail in Ref. 9. In order to derive the spectral energy distributions we resort to the Kolmogorov hypotheses according to which the spectral cascades proceed at constant rates ε_E , ε_H , and ε_G , respectively, for the energy, the magnetic helicity, and the generalized helicity governed by the eddy turnover time $(k\mathbf{v}_k)^{-1}$.

When Hall current and the fluid vorticity effects are dominant, the three spectra for the velocity and magnetic field fluctuations⁹ reduce to, in the $k \gg 1$ regime $\alpha_{\pm} \rightarrow 1/k$,

$$W_E(k) = (2\varepsilon_E)^{2/3} [1 + k^{-2}]^{-2/3} k^{-5/3}, \quad M_E(k) = k^{-2} W_E(k), \quad (22)$$

$$W_H(k) = (2\varepsilon_H)^{2/3} k^{1/3}, \quad M_H(k) = k^{-2} W_H(k), \quad (23)$$

$$W_G(k) = (2\varepsilon_G)^{2/3} [1 + k^2]^{-4/3} k^{1/3}, \quad M_G(k) = k^{-2} W_G(k). \quad (24)$$

If the turbulence is, in addition, dominated by velocity field fluctuations ($\mathbf{v}_k^2 \gg \mathbf{b}_k^2$), the spectral expressions under the joint dominance of the Hall term and the velocity fluctuations ($k \gg 1$) further simplify to

$$W_{E_{1a}}(k) = (2\varepsilon_E)^{2/3} k^{-5/3}, \quad M_{E_{1a}}(k) = (2\varepsilon_E)^{2/3} k^{-11/3}, \quad (25)$$

$$W_{H_{1a}}(k) = (2\varepsilon_H)^{2/3} k^{1/3}, \quad M_{H_{1a}}(k) = (2\varepsilon_H)^{2/3} k^{-5/3}, \quad (26)$$

$$W_{G_{1a}}(k) = (2\varepsilon_G)^{2/3} k^{-7/3}, \quad M_{G_{1a}}(k) = (2\varepsilon_G)^{2/3} k^{-13/3}. \quad (27)$$

For the second root of $\alpha \approx k$, $k \gg 1$, we find the following spectra:

$$W_{E_{1b}}(k) = (2\varepsilon_E)^{2/3} k^{-3}, \quad M_{E_{1b}}(k) = (2\varepsilon_E)^{2/3} k^{-1}, \quad (28)$$

$$W_{H_{1b}}(k) = (2\varepsilon_H)^{2/3} k^{-7/3}, \quad M_{H_{1b}}(k) = (2\varepsilon_H)^{2/3} k^{-1/3}, \quad (29)$$

$$W_{G_{1b}}(k) = (2\varepsilon_G)^{2/3} k^{-7/3}, \quad M_{G_{1b}}(k) = (2\varepsilon_G)^{2/3} k^{-1/3}. \quad (30)$$

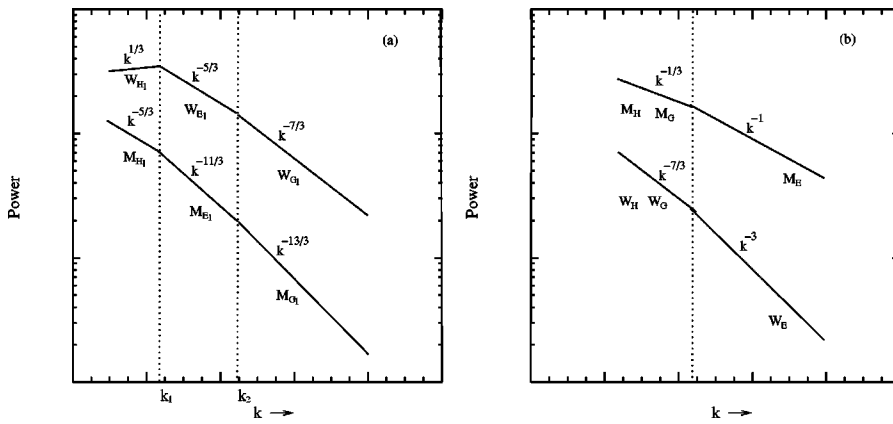


FIG. 1. (a) Schematic magnetic (M) and kinetic (W) spectra in the Hall state for $\alpha_+ \rightarrow 1/k$ ($k \geq 1$). (b) Schematic M and W spectra in the Hall state for $\alpha_- \rightarrow -k$ ($k \geq 1$).

In the opposite case wherein $k \ll 1$, one obtains the standard Alfvénic state with $v_k = b_k$, and the corresponding spectra are (suffixes 1a and 1b are used for the Hall-dominated and 2 for the Alfvén limit)

$$M(k) = W(k), \tag{31}$$

$$W_{E_2}(k) = (\varepsilon_E)^{2/3} k^{-5/3}, \tag{32}$$

$$W_{H_2}(k) = (2\varepsilon_H)^{2/3} k^{-1}, \tag{33}$$

$$W_{G_2}(k) = (\varepsilon_G)^{2/3} k^{-1}. \tag{34}$$

The rationale as well as the modality for stringing together different spectral branches originates in the hypothesis of selective dissipation. It was, first, invoked in studies of two-dimensional hydrodynamic turbulence.¹⁰ The idea is that in a given k range, the particular invariant which suffers the strongest dissipation, controls the spectral behavior (determined, in turn, by arguments of Kolmogorov). Thus if the k ranges associated with different invariants are distinct and separate, we have a straightforward recipe for constructing the entire k spectrum in the extended inertial range. In two-dimensional hydrodynamic turbulence, for instance, the enstrophy invariant, because of its stronger k dependence (and hence larger dissipation) compared to the energy invariant, dictates the large k spectral behavior. Therefore, the entire inertial range spectrum has two segments—the energy dominated low k , and the enstrophy dominated high k ($\propto k^{-3}$). The procedure amounts to placing the spectrum with the highest negative exponent at the highest k end, and the one with the lowest negative exponent of k at the lowest k end.

The magnetic spectrum $M(k)$ and the kinetic spectrum $W(k)$, constructed by following the procedure delineated above, is shown in Figs. 1(a) and 1(b), and (Fig. 2) for the Hall-dominated regime, Eqs. (26)–(31) [Alfvénic state, Eqs. (32)–(34)].

We thus observe that the spectral branch $k^{1/3}$ of the kinetic energy spectrum originates from the magnetic helicity invariant H_M under the dominance of the Hall effect ($k \geq 1$) as well as that of the kinetic energy ($v_k \geq b_k$) for $\alpha_+ \rightarrow 1/k$, and it should operate in large spatial scale regime as dictated by the hypothesis of selective dissipation. In the Alfvén regime ($k \ll 1$) the corresponding spectral branch $W_{H_2}(k)$

$\propto (k^{-1})$. There are two breaks in the spectra displayed in Figs. 1(a) and 1(b) and one break in Fig. 2. They are due to the change of the controlling invariant: at k_1 , the control is transferred from magnetic helicity H_M to the total energy E and at k_2 , from the total energy E to the generalized helicity H_G . Within the framework of the Kolmogorov hypothesis combined with the selective dissipation hypothesis, the positions of the spectral breaks indicate the scales of energy injection. The energy injected at k_2 , e.g., will cascade towards small k as $k^{-5/3}$ and towards large k as $k^{-7/3}$. Similarly, the energy injected at k_1 will cascade towards large k as $k^{-5/3}$ and towards small k as $k^{1/3}$. This is also in agreement with the conclusions of the papers 1 and 2. The $k^{1/3}$ spectrum in paper 1 is derived by invoking the constancy of the total energy flux in the entire volume (k^{-3}), i.e., dimensionally, $(kv_k) \times (v_k^2)(k^{-3}) = \text{const}$. In contrast, our derivation relies on the global invariant H_M . Although the two approaches are dimensionally identical, the underlying physics is very different. Additionally, our small spatial scale spectra also show, the steepened $k^{-7/3}$ branch comparable to the spectral branches A , B , and C of paper 1.

We propose that the measurement of the concomitant magnetic energy spectrum which should carry much less energy than the kinetic spectrum ($v_k \geq b_k$) may shed light on the type of the controlling invariant.

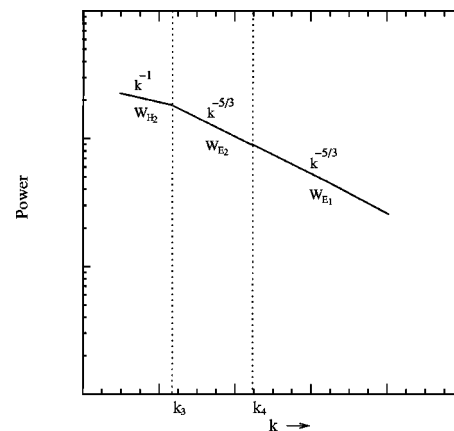


FIG. 2. Schematic magnetic (M) and kinetic ($W \equiv M$) spectra in the Alfvénic state ($k \leq 1$).

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