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Exact solution of the incompressible Hall magnetohydrodynamics

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ABSTRACT

The Alfvén wave is known to be an exact solution of the ideal magnetohydrodynamics (MHD), and this has found use in modelling astrophysical turbulence. In this paper we show that the Hall MHD also submits itself to an exact solution in the incompressible limit. We compare the linear and the non-linear modes of the Hall MHD and comment on their probable role in describing turbulent fluctuations in different astrophysical situations.

Key words: MHD – turbulence – waves.

1 INTRODUCTION

The exact solution of the incompressible ideal magnetohydrodynamics (MHD), the non-linear Alfvén wave, has served as an essential reference point for all studies of Alfvénic turbulence in various space and astrophysical situations (Walen 1944a,b; Alfvén & Fälthammer 1963; Parker 1979; Shebalin, Matthaeus & Montgomery 1983). The essence of this arbitrary amplitude wave lies in the relation $\boldsymbol{b} = \pm\,\boldsymbol{v}$, the requirement that the velocity and the magnetic field fluctuations are either parallel or anti-parallel and have the same magnitude [the magnetic (velocity) field is normalized to the uniform ambient field B_0 (the Alfvén speed $V_{\mathrm{A}} = B_0/\sqrt{4\pi\rho}$, where ρ is the uniform mass density)]. When $\boldsymbol{b}$ and $\boldsymbol{v}$ are so related, the non-linear terms in the time-dependent MHD vanish; it is this effective linearization that yields the waves

$$\boldsymbol{b} = \pm\boldsymbol{v}, \tag{1}$$

$$\boldsymbol{b} = \hat{\boldsymbol{b}} \exp[\mathrm{i}\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp \pm \mathrm{i}(k_s s + k_s t)], \tag{2}$$

with an effective frequency $\omega = -(+)k_s$, propagating in a direction antiparallel (parallel) to the ambient field $\boldsymbol{B}_0 = \hat{\boldsymbol{e}}_s$. Notice that $k_s = \boldsymbol{k} \cdot \hat{\boldsymbol{e}}_s$ is the projection of the wave vector along the direction of the field line, and $\perp$ is perpendicular to $\hat{\boldsymbol{e}}_s$. In equation (2), time and space variables are, respectively, measured in units of the ion gyroperiod $\omega_{\mathrm{ci}}^{-1} = m_{\mathrm{i}}c/eB_0$ and the ion skin depth $\lambda_{\mathrm{i}} = c/\omega_{\mathrm{pi}}$, where $\omega_{\mathrm{pi}} = (4\pi e^2\, n/m_{\mathrm{i}})^{1/2}$ is the ion plasma frequency and n is the plasma density.

The importance of the non-linear Alfvénic state for MHD prompts one to speculate if a similar kind of an exact solution exists for Hall MHD (HMHD), a system which encompasses MHD, but can sustain a much richer spectrum of plasma states not accessible to MHD (Mahajan & Yoshida 1998; Wardle 1999; Mahajan et al. 2001; Ohsaki et al. 2001; Balbus & Terquem 2001; Ohsaki et al. 2002; Ohsaki & Mahajan 2004). In the framework of the HMHD, a plasma is treated as a two-fluid system. The electrons are assumed to be inertialess and the electric field is determined from the equation of motion of the electron fluid. This electric field is then substituted in the magnetic induction equation and the equation of motion of the ion fluid. As a result the magnetic field remains frozen to the electron fluid but not to the ion fluid. This is the essence of the HMHD which is an extension of the ideal MHD into the dispersive range. The Hall extension affects the transport of energy and magnetic induction without affecting the transport of ion momentum. Most of the effort in HMHD, so far, has been in what may properly be described as electron HMHD where the ion fluid has been assumed to be at rest. A tutorial on HMHD, along with a detailed list of references on applications in the structuring of plasma expansions, magnetic field transport and magnetic reconnection, can be found in Huba (2003). In this Letter we demonstrate that HMHD in the incompressible limit, indeed, admits an exact non-linear wave solution which in the long-wavelength limit ($k \ll 1$) reduces to the non-linear shear and compression waves (degenerate in the large plasma β limit) of MHD, while, for $k \gg 1$, one branch goes over to the ion cyclotron mode while the other branch becomes the whistler mode. The general solution spans the entire k range for which the HMHD equations are valid. Naturally the MHD relation between the velocity and the magnetic field perturbations is fundamentally transformed; it becomes a function of the wavevector $\boldsymbol{k}$.

2 NON-LINEAR HALL MHD

In the Alfvénic units defined above, the dimensionless equations

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times [(\boldsymbol{V} - \nabla \times \boldsymbol{B}) \times \boldsymbol{B}] \tag{3}$$

and

$$\frac{\partial(\boldsymbol{B} + \nabla \times \boldsymbol{V})}{\partial t} = \nabla \times [\boldsymbol{V} \times (\boldsymbol{B} + \nabla \times \boldsymbol{V})] \tag{4}$$

constitute the HMHD in the incompressible limit ($\nabla \cdot \boldsymbol{V} = 0$) along with the condition for divergence-free magnetic field ($\nabla \cdot \boldsymbol{B} = 0$).

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Equation (4) is obtained by adding the induction equation and curl of the equation of motion of the ion fluid. To look for wave-like solutions we split the fields into their ambient and fluctuating parts (there is no ambient flow),

$$\boldsymbol{B} = \hat{\boldsymbol{e}}_s + \boldsymbol{b}; \qquad \boldsymbol{V} = \boldsymbol{v}, \tag{5}$$

and substitute in (3)–(4),

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times [(\boldsymbol{v} - \nabla \times \boldsymbol{b}) \times \hat{\boldsymbol{e}}_s + (\boldsymbol{v} - \nabla \times \boldsymbol{b}) \times \boldsymbol{b}], \tag{6}$$

$$\frac{\partial}{\partial t}(\boldsymbol{b} + \nabla \times \boldsymbol{v}) = \nabla \times [\boldsymbol{v} \times (\nabla \times \boldsymbol{v} + \boldsymbol{b}) + \boldsymbol{v} \times \hat{\boldsymbol{e}}_s]. \tag{7}$$

The non-linear problem represented by (6)–(7) is converted to a set of linear problems (the time-honoured method for solving non-linear equations) by imposing the conditions

$$\boldsymbol{v} - \nabla \times \boldsymbol{b} = \alpha \boldsymbol{b} \tag{8}$$

and

$$\boldsymbol{b} + \nabla \times \boldsymbol{v} = \beta \boldsymbol{v} \tag{9}$$

to eliminate the non-linear terms. Here α and β are like the separation constants. With the non-linearities so taken care of, we are left with the remaining time-dependent linear equations

$$\frac{\partial \boldsymbol{b}}{\partial t} = \alpha \nabla \times [\boldsymbol{b} \times \hat{\boldsymbol{e}}_s] \tag{10}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} = (1/\beta) \nabla \times [\boldsymbol{v} \times \hat{\boldsymbol{e}}_s]. \tag{11}$$

Apparently we have traded a close non-linear system (six equations for six variables) for an over-determined linear system (8)–(11) with 12 equations in six variables. Acceptable solutions, therefore, will be possible only under some particular conditions that will remove the over-determination. To seek them, we first notice that (10) and (11) admit

$$\boldsymbol{b} = \boldsymbol{b}_k \exp[\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\alpha(\hat{\boldsymbol{e}}_s \cdot \boldsymbol{k})t] \tag{12}$$

and

$$\boldsymbol{v} = \boldsymbol{v}_k \exp[\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\frac{1}{\beta}(\hat{\boldsymbol{e}}_s \cdot \boldsymbol{k})t]. \tag{13}$$

If the exponential solutions (12) and (13) are to satisfy the linear equations (8) and (9), we must require $\beta = 1/\alpha$. In addition, substituting (12) and (13) into (8) and (9) leads to

$$\boldsymbol{v}_k - \mathrm{i}\boldsymbol{k} \times \boldsymbol{b}_k = \alpha \boldsymbol{b}_k \tag{14}$$

and

$$\boldsymbol{b}_k + \mathrm{i}\boldsymbol{k} \times \boldsymbol{v}_k = \frac{1}{\alpha}\boldsymbol{v}_k \tag{15}$$

which, after simple manipulation, yield

$$\boldsymbol{v}_k - \alpha \boldsymbol{b}_k = \mathrm{i}\alpha \boldsymbol{k} \times \boldsymbol{v}_k,$$

$$\boldsymbol{v}_k - \alpha \boldsymbol{b}_k = \mathrm{i}(\boldsymbol{k} \times \boldsymbol{b}_k).$$

Two consequences immediately follow:

$$\boldsymbol{b}_k = \alpha \boldsymbol{v}_k, \tag{16}$$

relating $\boldsymbol{b}_k$ and $\boldsymbol{v}_k$, and

$$\boldsymbol{k} \times \boldsymbol{v}_k = -\mathrm{i}\frac{1-\alpha^2}{\alpha}\boldsymbol{v}_k = -\mathrm{i}\lambda \boldsymbol{v}_k. \tag{17}$$

The first of these establishes the HMHD equivalent of the Alfvénic condition for MHD, and the second shows that the Fourier transform of a Beltrami equation ($\nabla \times \boldsymbol{G} = \lambda \boldsymbol{G}$) has to be solved to complete the story; the solvability constraint will end up relating α and k, giving the dispersion relation $\omega = -\alpha(k)(\hat{\boldsymbol{e}}_s \cdot \boldsymbol{k})$.

The solutions of (17) are well-known and we could just quote them. For completeness, however, we recapitulate a few steps in the process. Suppressing the indices for a simplified notation, we derive from (17) that (i) dotting with $\boldsymbol{v}$ yields $\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{v}_\mathrm{r}^2 - \boldsymbol{v}_\mathrm{i}^2 + 2\mathrm{i}\boldsymbol{v}_\mathrm{r} \cdot \boldsymbol{v}_\mathrm{i} = 0$ implying $v_\mathrm{i} = \pm v_\mathrm{r}$ and $\boldsymbol{v}_\mathrm{r} \cdot \boldsymbol{v}_\mathrm{i} = 0$, and (ii) and dotting with $\boldsymbol{k}$ gives $\boldsymbol{k} \cdot \boldsymbol{v} = 0 \Rightarrow \boldsymbol{k} \cdot \boldsymbol{v}_\mathrm{r} = 0 = \boldsymbol{k} \cdot \boldsymbol{v}_\mathrm{i}$. Clearly the subscript r(i) denotes the real(imaginary) part. Crossing (17) with $\boldsymbol{k}$ and using $\boldsymbol{k} \cdot \boldsymbol{v} = 0$, we obtain (remembering that λ is a function of α)

$$\lambda = \pm k. \tag{18}$$

Keeping track of the $\pm$ may be notationally complicated. Since the physics is the same, we will investigate the option $v_\mathrm{i} = v_\mathrm{r}$ and $\lambda = k$. For this choice, it is straightforward to show that $\hat{\boldsymbol{v}}_\mathrm{r}$, $\hat{\boldsymbol{v}}_\mathrm{i}$ and $\hat{\boldsymbol{k}}$ form a right-handed orthogonal triad of unit vectors.

Let us first choose $\hat{\boldsymbol{v}}_\mathrm{r}$, $\hat{\boldsymbol{v}}_\mathrm{i}$ and $\hat{\boldsymbol{k}}$ to be $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$ and $\hat{\boldsymbol{e}}_z$ respectively. This choice dictates the following expressions for the velocity and the magnetic fields ($\boldsymbol{k} = k\hat{\boldsymbol{e}}_z$, and A_0 is a constant amplitude):

$$\boldsymbol{b} = \alpha \boldsymbol{v}, \tag{19}$$

$$\boldsymbol{v} = A_0[\hat{\boldsymbol{e}}_x + \mathrm{i}\hat{\boldsymbol{e}}_y] \exp[\mathrm{i}kz + \mathrm{i}\alpha k(\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_s)t], \tag{20}$$

with α determined by

$$k = \lambda = \frac{1-\alpha^2}{\alpha}, \tag{21}$$

$$\alpha_\pm = \left[-\frac{k}{2} \pm \left(\frac{k^2}{4} + 1\right)^{1/2}\right]. \tag{22}$$

From (20) and (22), we extract the effective frequency of the circularly polarized wave,

$$\omega_\pm = -k\left[-\frac{k}{2} \pm \left(\frac{k^2}{4} + 1\right)^{1/2}\right](\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_s), \tag{23}$$

a result which is valid over a wide range of k from $k \ll 1$ at the MHD end to the $k \gg 1$ in the Hall-dominated regime.The k dependence of the separation constant, implying a k-dependent relationship between $\boldsymbol{b}$ and $\boldsymbol{v}$, is one of the defining and distinguishing characteristics of the new broad-band fully non-linear wave.

Let us examine the two extreme limits of the general result, equation (23). For $k \ll 1$,

$$\alpha_\pm \to \pm 1, \quad \omega_\pm \to \mp k(\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_s), \tag{24}$$

reproducing the k-independent MHD Alfvénic relationship for both the co- and the counter-propagating waves. In the $k \gg 1$ regime, however,

$$\alpha_+ \to 1/k, \quad \alpha_- \to -k, \tag{25}$$

with

$$\omega_\mathrm{N}+ \to -\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_s, \quad \omega_\mathrm{N}- \to (\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_s)k^2. \tag{26}$$

In order to make contact with the familiar, we recall the linear dispersion relation for HMHD (Ohsaki & Mahajan 2004) for $\boldsymbol{k} = k\hat{\boldsymbol{e}}_z$, $\boldsymbol{B}_0 = \hat{\boldsymbol{e}}_z$ and the normalizing length-scale λ_i so that $\epsilon = 1$:

$$(\omega^2 - k^2)[\omega^4 - (1+\beta)k^2\omega^2 + \beta k^4] = k^4\omega^2(\omega^2 - \beta k^2). \tag{27}$$

In the $\beta \to \infty$ limit which corresponds to the non-linear incompressible HMHD solution considered here, we find the linear modes for $k \gg 1$, $\omega_{\rm L}+ = \pm\, k_z^2$ and $\omega_{\rm L}- = \pm 1$.

It is easy to recognize that the $\omega_{\rm N}+$ mode for parallel and antiparallel propagation is the magnetosonic–cyclotron branch $\omega_{\rm L}-$, while the $\omega_{\rm N}-$ represents the shear–whistler mode $\omega_{\rm L}+$. In this limit the magnitudes of the velocity and magnetic fields can vastly differ (they still remain aligned). The respective relationships are

$$\boldsymbol{v} \to k\boldsymbol{b} \tag{28}$$

for the $\omega_{\rm N}+$ branch, and

$$\boldsymbol{b} \to k\boldsymbol{v} \tag{29}$$

for the $\omega_{\rm N}-$ branch; the compressional–whistler mode has an abundance of turbulent magnetic energy over the turbulent kinetic energy, while in the shear–cyclotron mode $\omega_{\rm N}+$, the kinetic energy dominates. The non-linearly correct relationship of (28) strongly strengthens the results of Krishan & Mahajan (2004) where exactly these relationships were invoked to model the observed solar granulation spectrum.

To construct the three-dimensional solution, we note that the triad $\hat{\boldsymbol{v}}_{\rm r} = \hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{v}}_{\rm i} = \hat{\boldsymbol{e}}_y$ and $\hat{\boldsymbol{k}} = \hat{\boldsymbol{e}}_z$ could be replaced by two other independent cyclic combinations. Since (17) is linear, the solutions can be added. The most general three-dimensional solution, therefore, may be written as

$$\begin{aligned} \boldsymbol{v} = {} & B(\hat{\boldsymbol{e}}_x + {\rm i}\hat{\boldsymbol{e}}_y)\exp[{\rm i}kz + {\rm i}\alpha k(\hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_s)t] \\ & + C(\hat{\boldsymbol{e}}_y + {\rm i}\hat{\boldsymbol{e}}_z)\exp[{\rm i}kx + {\rm i}\alpha k(\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_s)t] \\ & + A(\hat{\boldsymbol{e}}_z + {\rm i}\hat{\boldsymbol{e}}_x)\exp[{\rm i}ky + {\rm i}\alpha k(\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_s)t]. \end{aligned} \tag{30}$$

It is straightforward to verify that (30) exactly solves the original system (10), (11), (8) and (9) with $\beta = 1/\alpha$,

$$\frac{\partial \boldsymbol{b}}{\partial t} = \alpha\nabla\times[\boldsymbol{b}\times\hat{\boldsymbol{e}}_s], \tag{31}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = \alpha\nabla\times(\boldsymbol{v}\times\hat{\boldsymbol{e}}_s), \tag{32}$$

$$\boldsymbol{b} = \alpha\boldsymbol{v} \tag{33}$$

and $[\lambda = (1-\alpha^2)/\alpha]$

$$\nabla\times\boldsymbol{v} = \lambda\boldsymbol{v}. \tag{34}$$

Since the defining equations consist of only real variables, either $\mathrm{Im}\,\boldsymbol{v}$ or $\mathrm{Re}\,\boldsymbol{v}$ could be a solution. Let us write down the imaginary part,

$$\begin{aligned} \boldsymbol{v} = {} & \hat{\boldsymbol{e}}_x[A\cos(ky + \alpha k\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_s t) \\ & + B\sin(kz + \alpha k\hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_s t)] \\ & + \hat{\boldsymbol{e}}_y[B\cos(kz + \alpha k\hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_s t) \\ & + C\sin(kx + \alpha k\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_s t)] \\ & + \hat{\boldsymbol{e}}_z[C\cos(kx + \alpha k\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_s t) \\ & + A\sin(ky + \alpha k\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_s t)]. \end{aligned} \tag{35}$$

This is the time-dependent generalization of the famous *ABC* solution of $\nabla\times\boldsymbol{G} = k\,\boldsymbol{G}$. This comes as no surprise because the original system (10), (11), (8) and (9) can be cast as a single Beltrami equation with a new ∇ defined in terms of the time-translated ‘coordinates’ $X_i = x_i + \alpha k\hat{\boldsymbol{e}}_{x_i}\cdot\hat{\boldsymbol{e}}_s$.

3 CONCLUSION

Exact time-dependent three-dimensional solutions to interacting field theories are quite rare. To the best of our knowledge, this is the only time-dependent, three-dimensional exact and fully non-linear wave solution to a physical system of as great an interest and complication as the Hall MHD. It is to be noted that the HMHD does not submit to an exact solution in the presence of density fluctuations ($\nabla\cdot\boldsymbol{V}\neq 0$). However, pressure fluctuations through temperature fluctuations are admissible within the system described here. One recalls that the pressure force has not been neglected in the equation of motion of the ion fluid. It does not appear explicitly in equation (4) because it is the curl of the equation of motion of the ion fluid. The importance of knowing an exact non-linear solution can never be overestimated. It is expected to provide a reference solution for a variety of investigations, analytical as well as numerical, relating to plasma turbulence, in particular, for the enquiries that probe into the short-scale domain of the turbulent spectrum. More specifically, our preliminary work on the turbulence on the solar atmosphere (Krishan & Mahajan 2004) shows that a scale-dependent relationship between the velocity and the magnetic field is absolutely essential in order to account for the observed spectra. The exact solutions provide a basis for building quantitative models of turbulence, in the identification of the nature of fluctuations and their interactions. The usual and valid objection to the identification of turbulent fluctuations with linear modes can now be laid to rest. The exact solutions offer correct representations of turbulent fluctuations with the added advantage of the possibility of their identification in the familiar terminology.

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