J. Astrophys. Astr. (2000) **21**, 299-302

Models of Flux Tubes from Constrained Relaxation

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Abstract. We study the relaxation of a compressible plasma to an equilibrium with flow. The constraints of conservation of mass, energy, angular momentum, cross-helicity and relative magnetic helicity are imposed. Equilibria corresponding to the energy extrema while conserving these invariants for parallel flows yield three classes of solutions and one of them with an increasing radial density profile, relevant to solar flux tubes is presented.

Key words. Hydromagnetics—Sun: atmosphere, magnetic fields.

1. Invariants of the system

There is a remarkable concentration of the magnetic field of the Sun into isolated flux tubes at the visible surface, and also in the form of coronal loops, where the field strengths are of the order of 1500 G. It is our aim to seek equilibrium structures appropriate to these flux tubes, using principles of plasma relaxation. The subject of turbulent relaxation of a plasma has been examined by several workers with the aim of predicting a set of equilibria after the plasma has passed through a disruptive phase (e.g. Woltjer 1960; Taylor 1974; Rieman 1980). The results have been applied to fusion devices as well as to astrophysical plasmas (krishan 1996; Finn & Antonsen 1983). For a review of the utility of the helicity concept see Brown *et al.* (1999).

Taylor (1974) hypothesized the existence of a global helicity invariant while the system would resistively relax to a state of minimum energy which would be topologically inaccessible in perfect MHD. Berger & Field (1984) allowed for open field lines in the volume and constructed a relative helicity invariant which has the equivalent form (Kusano *et al.* 1985)

$$H_R = H(\mathbf{B}_c, \mathbf{B}_c) + 2H(\mathbf{P}, \mathbf{B}_c), \tag{1}$$

where $\mathbf{B}_c$ is the closed field in the volume under consideration and P represents the open field lines in the volume, whose self-helicity is taken to be zero. By taking a variation of this expression, we find, using $\delta H(\mathbf{B}) = 2\int d^3x \delta\mathbf{B} \cdot \mathbf{A}$, that

$$\begin{aligned}\delta H_R &= 2\int_v d^3x\delta\mathbf{B}_c \cdot \mathbf{A}_c + 2\int_v d^3x\delta\mathbf{B}_c \cdot \mathbf{A}_P \\ &= 2\int_v d^3x\delta\mathbf{B}_c \cdot \mathbf{A} = 2\int_v d^3x\delta\mathbf{B} \cdot \mathbf{A} = \delta H(\mathbf{B}),\end{aligned} \tag{2}$$

where we have employed the fact that $\delta A p = 0$ and $\delta P = 0$ as P is determined completely by the fixed boundary conditions (there is an alternate route in the Appendix A of Berger (1984)). We emphasize the result that the variation of the relative helicity is equal to the variation of the helicity itself.

Here, we investigate a three dimensional model of relaxed states, drawing on the framework of Finn & Antonsen (1983), who assume ideal MHD, but with large thermal conductivity. This has two consequences; the cross-helicity, $K = \int d^3x v \cdot B$, an invariant in ideal MHD remains conserved in the limit of large thermal conductivity and it turns out, under the same assumptions, that the entropy functional, $S = \int d^3x \rho \ln (p/\rho^\gamma)$ is also an invariant. Note however that adiabaticity is not assumed to allow for entropy change during the relaxation process.

We consider the usual system of MHD equations with the energy equation suitably modified to include large parallel thermal conductivity. It can be shown, in spite of parallel heat flux, due to the boundary condition $B \cdot \hat{n} = 0$, that the total energy, $E = \int d^3x(\frac{1}{2}\rho v^2 + \frac{1}{8\pi}\mathbf{B}^2 + \frac{p}{\gamma - 1})$, is conserved. In all H_R, K, S, and the total mass, $M = \int d^3xr$, and E are the' global invariants of the system. The entropy functional increases for finite thermal conductivity and is invariant, if it is infinite. The standard variational method of extremizing $E^* = E - ½ \mu H_R - \alpha K - nS - \delta M$ yields the following equations

$$p = nT = n(\gamma - 1)\eta m, \tag{3}$$

$$v = \frac{\alpha \mathbf{B}}{\rho}, \tag{4}$$

$$\nabla \times \left[\left(\frac{1}{4\pi} - \frac{\alpha^2}{\rho}\right)\mathbf{B}\right] = \mu \mathbf{B}, \tag{5}$$

$$\frac{m\alpha^2 \mathbf{B}^2}{2\rho^2} + T \ln (n/n_0) = 0, \tag{6}$$

where $\ln n_0 = m\delta/T + (\ln (T/m\gamma) - \gamma)/(\gamma - 1)$

2. Analysis of the parallel flow

We study the parallel flows by introducing the parameters, $\nu = n/n_0$, $\nu_1 = 4\pi\alpha^2/(mn_0)$, $\mathbf{b} = \mathbf{B}/B_{max}$, $\varepsilon_0 = \nu_1 B^2 max/(8\pi n_0 T)$ and $\mu_0 = \mu R$, where R is the length scale in the problem and B_{max} is the maximum field strength. The system then reduces to

$$s \times [(1 - \nu_1/\nu)b] = \mu_0 b, \tag{7}$$

$$\nu^2 \ln \nu = -\varepsilon_0 b^2. \tag{8}$$

There are two branches evident from the above equations: for $\varepsilon_0 > 1/(2e)$, there are no solutions and for $\varepsilon_0 < 1/(2e)$, two solutions exist. The solution with $\nu < 1/\sqrt{e}$ has ν monotonically decreasing with b^2 and vice-versa for $\nu > 1/\sqrt{e}$. The Taylor state, $s \times \mathbf{B} = \mu \mathbf{B}$ is found in the limit $\nu_1 \to 0$. For a fixed μo, the equilibria exist only in the restricted part of the $\varepsilon_0 - \nu_1$ plane, where solutions with $\nu < 1/\sqrt{e}$ called class I solutions and solutions with $\nu > 1/\sqrt{e}$ can be distinguished as class II and class III by the sign of dB^2/dp. So, there are three classes of solutions possible; class I with radially decreasing density and magnetic energy, class II with radially increasing magnetic energy but a decreasing density profile and finally, class III with increasing density and decreasing magnetic energy profile.

In cylindrical symmetry, the equations reduce to

$$b'_\phi = \frac{\mu_0}{\sigma} b_z - \frac{b_\phi}{r} - \frac{\sigma'}{\sigma} b_\phi, \tag{9}$$

$$b'_z = -\frac{\sigma'}{\sigma} b_z - \frac{\mu_0}{\sigma} b_\phi, \tag{10}$$

where $\sigma = 1 - v_1/v$. The solutions were obtained by solving the coupled non-linear equations (9-10) with the constraint, (8), and the boundary conditions, $b_\phi(0) = 0$ and $b^2 = 1$ at the appropriate boundary ($r/R = 0$ or 1) decided by the solution class. A point to note is that by forcing $b_\phi(1) = 0$, one can ensure that no net current, I_z, is carried by the flux tube and B_{max} can then be consistently determined. However, this solution can also be obtained by enforcing $b^2 = 1$ at 1 or 0 (depending on the solution class, whether it is increasing or decreasing) and truncating the solution at the point where $b_\phi = 0$, with the value of μ scaled appropriately. Zero net current flux tubes are qualitatively similar to the flux tube with carrying finite current.

3. Application to solar flux tubes

Class III solutions (see Fig. 1) with hollow profiles, holds for $v < 1/\sqrt{e}$, or if the condition $v \ln (ev^2) > v_1$ holds. Therefore, $n < 0{\cdot}6n_0$ or the Mach number of the flow, $M > 0{\cdot}5$. Typical solar flux tubes ($\rho \sim 3 \times 10^{-7}$ g/cc, $\beta \sim 1$, $p \sim 1{\cdot}4 \times 10^5$ dyn/cm^{-2}) with an underlying ultra subsonic flow, satisfy the condition for class III solution. For stable flux tubes of $B \sim 1500$ G and $v \sim 3$ km/sec, the parameters are of the order, $v_1 \approx 0{\cdot}16$, $\varepsilon_0 \gtrsim 0{\cdot}1$, indicating that the flow is of class III. The restricted types of equilibria, taking into account open field boundary conditions, which may be present in a plasma immediately after disruptive or turbulent process, have been discussed here. The parameter range for the solar flux tubes (in the photosphere) indicate that the radial density need not be peaked but can have a hollow (increasing density) profile which pertains to class III solutions. The validity of this analysis depends upon the thermal conductivity being dominant in the turbulent process.

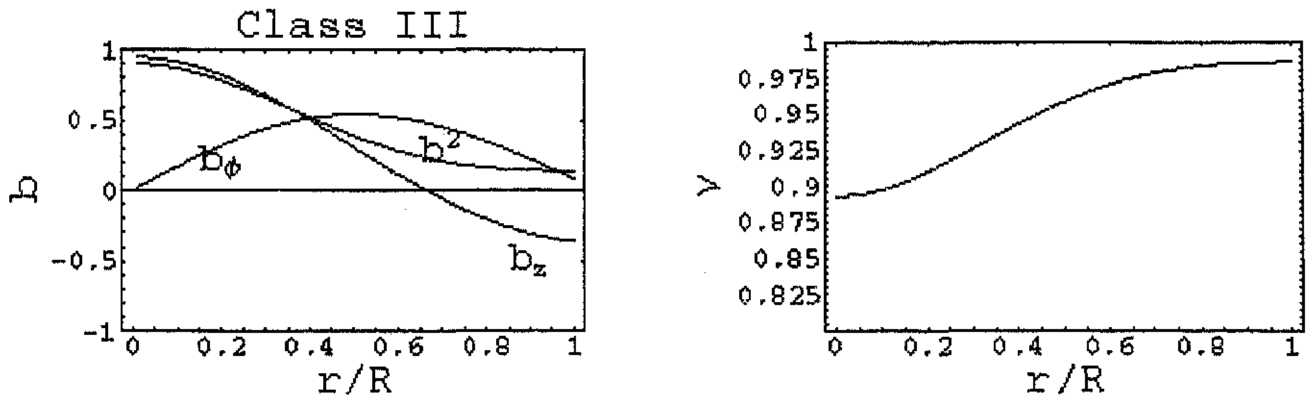

Figure 1. Class III solution with decreasing B and increasing density for the choice of parameters, $v_1 = 0{\cdot}16$, $\mu = 3$ and $\varepsilon_0 = 0{\cdot}1$.

The second variation of E^* indicates that the flows are stable if $\tau > 0$. Further, we plan to include other invariants like angular momentum and explore axisymmetric systems and other geometries. The predictions can be tested by numerical simulations and future observations.

Acknowledgement

We thank Prof. Richard Canfield for useful discussions during the meeting.

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