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A new representation of Güttler's theory of electromagnetic scattering by a composite sphere

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The multipole expansion coefficients for electromagnetic scattering by a stratified core-mantle sphere, suitable for fast and accurate evaluation of the radiation scattering parameters, have been recast.

Die Koeffizienten der Multipolentwicklung für die elektromagnetische Streuung an einer geschichteten Kern-Mantel-Sphäre, die für eine schnelle und genaue Berechnung der Streuparameter der Strahlung geeignet sind, wurden neu berechnet.

1. Introduction

It has been well established fact that the fast and accurate evaluation of the Mie coefficients for single homogeneous spheres can be highly facilitated by using the ratios of Riccati-Bessel functions and the logarithmic derivatives (see, for example, SHAH 1977a, b; WISCOMBE 1980). A similar comprehensive treatment for stratified core-mantle (i.e. composite) spheres has not been given in the literature so far. In what follows we present a new representation of GÜTTLER's (1952) original formulas for the multipole expansion coefficients a_n and b_n which come about as a result of solving the problem of electromagnetic scattering by a composite sphere having arbitrary radii and indices of refraction of the core and the mantle. The main idea is to obtain expressions, analogous to the case of single sphere (SHAH 1977a, b), which can lead to an efficient and accurate method of numerical evaluation of the expansion coefficients on the modern fast computers. In general, the structure of the complex analytical formulas should be such that, in addition to economy of the computer time, the propagation of the errors due to truncation during arithmatic operations can be minimized for arbitrary input parameters.

2. Requisite functions

With reference to the electromagnetic scattering theory for core-mantle spheres developed by GÜTTLER (1952), we explain below the symbols, notation and definitions used in the present work. The scattering particle is shown schematically in Figure 1. The index of refraction of the surrounding medium is assumed to be unity. The scattering process is considered to be coherent. A prime over a function denotes the derivative of the function with respect to the argument concerned.

λ = the wavelength of the incident and the scattered light,

r_1 and r_2 = the radii of the core and the mantle, respectively,

m_1 and m_2 = the indices of refraction of the core and the mantle, respectively.

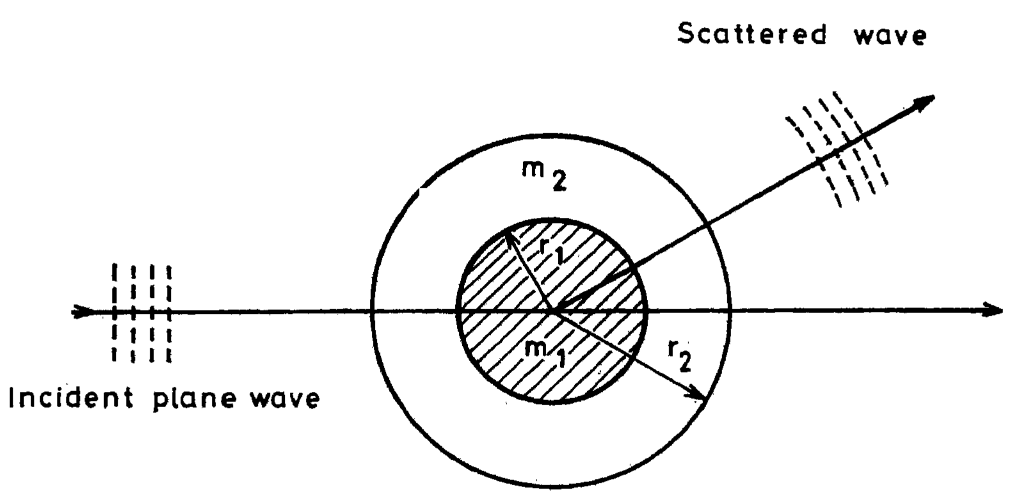

Fig. 1. Stratified sphere with core (radius = r_1 and index of refraction, $m_1 = m_1' - im_1''$) and mantle (r_2, $m_2 = m_2' - im_2''$).

Note that, in general, m_1 and m_2 are complex and wavelength dependent. Thus we can write

$$m_1 = m_1' - \mathrm{i}m_1'' ,$$

and

$$m_2 = m_2' - \mathrm{i}m_2'' .$$

The size-to-wavelength parameters are defined by

$$x_1 = \frac{2\pi r_1}{\lambda}, \tag{1}$$

and

$$x_2 = \frac{2\pi r_2}{\lambda}. \tag{2}$$

The complex arguments (z) for various Riccati-Bessel functions are defined by

$$z_{ij} = m_i x_j, \qquad i = 1, 2, \qquad j = 1, 2 . \tag{3}$$

The first and second kinds of Riccati-Bessel functions, viz. $S_n(z)$ and $C_n(z)$, respectively, with argument z and integer order n are defined as follows:

$$S_n(z) = \left(\frac{\pi z}{2}\right)^{1/2} J_{n+1/2}(z), \tag{4}$$

$$C_n(z) = -\left(\frac{\pi z}{2}\right)^{1/2} N_{n+1/2}(z) . \tag{5}$$

Where $J_{n+1/2}$ and $N_{n+1/2}$ are the spherical Bessel and Neumann functions, respectively, as defined by WATSON (1966) and MCLACHLAN (1961). The usual recurrence relations for $S_n(z)$ and $C_n(z)$ have been adopted. The numerical techniques include the calculation of $S_n(z)$ by downward recursion and $C_n(z)$ by upward recursion as in the case of homogeneous sphere (SHAH 1977a, b; WISCOMBE 1980).

For the present purpose, we define the following logarithmic derivatives of the Riccati-Bessel functions:

$$A_n(z_{ij}) = \frac{S_n'(z_{ij})}{S_n(z_{ij})}, \qquad i = 1, 2;\ j = 1, 2 , \tag{6}$$

$$B_n(z_{ij}) = \frac{C_n'(z_{ij})}{C_n(z_{ij})}, \qquad i = 1, 2;\ j = 1, 2 , \tag{7}$$

$$A_n(x_i) = \frac{S_n'(x_i)}{S_n(x_i)}, \qquad i = 1, 2 , \tag{8}$$

and

$$B_n(x_i) = \frac{C_n'(x_i)}{C_n(x_i)}, \qquad i = 1, 2 . \tag{9}$$

Note that the arguments z_{12} and x_1 are not needed throughout. Also, for the argument z_{11}, only $A_n(z_{11})$ occurs; other functions, $S_n(z_{11})$, $C_n(z_{11})$ and $C_n'(z_{11})$, are not required. Furthermore, we define the following useful ratios:

$$\alpha_n = \frac{S_n(z_{22})}{S_n(z_{21})}, \tag{10}$$

$$\beta_n = \frac{C_n(z_{22})}{C_n(z_{21})}, \tag{11}$$

$$\gamma_n = \frac{C_n(x_2)}{S_n(x_2)}, \tag{12}$$

and

$$F_n(x_2) = \frac{S_n'(x_2) + \mathrm{i}C_n'(x_2)}{S_n(x_2) + \mathrm{i}C_n(x_2)} . \tag{13}$$

Note that only γ_n is the real quantity among the equations (10) to (13). Using equations (8) and (9), equation (13) can be expressed in the form

$$F_n(x_2) = \frac{A_n(x_2) + \mathrm{i}B_n(x_2)\,\gamma_n}{1 + \mathrm{i}\gamma_n} . \tag{14}$$

It must be appreciated that, unlike TOON and ACKERMAN (1981), our defining quantities in equations (6) to (12) are either single logarithmic derivatives or simple ratios of the special functions. The ensuing advantage is distinctly revealed in the simplicity and elegance of the final formulas.

3. Results and discussion

With the help of the functions defined by equations (6) through (14), we can reduce the Güttler's multipole expansion coefficients a_n and b_n for a composite sphere to simple forms. We shall omit all the algebraic manipulations performed on the original expressions. The final results can be presented in a compact manner as follows:

$$a_n = \frac{\left\{\dfrac{B_n(z_{22}) - m_2A_n(x_2)}{m_1B_n(z_{21}) - m_2A_n(z_{11})}\right\}\beta_n - \left\{\dfrac{A_n(z_{22}) - m_2A_n(x_2)}{m_1A_n(z_{21}) - m_2A_n(z_{11})}\right\}\alpha_n}{\{1 + i\gamma_n\}\left[\left\{\dfrac{B_n(z_{22}) - m_2F_n(x_2)}{m_1B_n(z_{21}) - m_2A_n(z_{11})}\right\}\beta_n - \left\{\dfrac{A_n(z_{22}) - m_2F_n(x_2)}{m_1A_n(z_{21}) - m_2A_n(z_{11})}\right\}\alpha_n\right]} \tag{15}$$

$$b_n = \frac{\left\{\dfrac{m_2B_n(z_{22}) - A_n(x_2)}{m_2B_n(z_{21}) - m_1A_n(z_{11})}\right\}\beta_n - \left\{\dfrac{m_2A_n(z_{22}) - A_n(x_2)}{m_2A_n(z_{21}) - m_1A_n(z_{11})}\right\}\alpha_n}{\{1 + i\gamma_n\}\left[\left\{\dfrac{m_2B_n(z_{22}) - F_n(x_2)}{m_2B_n(z_{21}) - m_1A_n(z_{11})}\right\}\beta_n - \left\{\dfrac{m_2A_n(z_{22}) - F_n(x_2)}{m_2A_n(z_{21}) - m_1A_n(z_{11})}\right\}\alpha_n\right]} \tag{16}$$

This essentially completes the new representation of the multipole expansion coefficients for a core-mantle composite sphere. Equations (15) and (16) can be recast in the following manner for computational convenience: Define

$$U_1 = m_1A_n(z_{21}) - m_2A_n(z_{11}), \tag{17}$$

$$U_2 = m_1B_n(z_{21}) - m_2A_n(z_{11}), \tag{18}$$

$$V_1 = m_2A_n(z_{21}) - m_1A_n(z_{11}), \tag{19}$$

$$V_2 = m_2B_n(z_{21}) - m_1A_n(z_{11}). \tag{20}$$

The coefficients a_n and b_n can then be expressed as follows:

$$a_n = \frac{\beta_nU_1\{B_n(z_{22}) - m_2A_n(x_2)\} - \alpha_nU_2\{A_n(z_{22}) - m_2A_n(x_2)\}}{\{1 + i\gamma_n\}[\beta_nU_1\{B_n(z_{22}) - m_2F_n(x_2)\} - \alpha_nU_2\{A_n(z_{22}) - m_2F_n(x_2)\}]}, \tag{21}$$

$$b_n = \frac{\beta_nV_1\{m_2B_n(z_{22}) - A_n(x_2)\} - \alpha_nV_2\{m_2A_n(z_{22}) - A_n(x_2)\}}{\{1 + i\gamma_n\}[\beta_nV_1\{m_2B_n(z_{22}) - F_n(x_2)\} - \alpha_nV_2\{m_2A_n(z_{22}) - F_n(x_2)\}]}. \tag{22}$$

The comparisons between U_1 and V_1 in equations (17) and (19) as well as between U_2 and V_2 in equations (18) and (20) show that the indices of refractions m_1 and m_2 are interchanged. Also, on comparing the expansion coefficients a_n and b_n either in equations (15) and (16) or in equations (21) and (22), one may notice the positions of m_2 occurring in a complementary manner other things being identical except for the interchange of U and V. These interesting features, although anticipated but not quite obvious in the literature available so far, are akin to those observed in the case of the Mie theory for a homogeneous sphere.

Once the coefficients a_n and b_n have been evaluated from the present representation, one can obtain all the electromagnetic radiation scattering parameters such as extinction, scattering and absorption cross-sections, scattering phase function, radiation pressure, asymmetry factor etc. by adopting the appropriate formulas used in the case of a single homogeneous sphere. A special computer code in FORTRAN language, viz. PROGRAM BELJAY, has been already developed by SHAH (1983) to compute the Riccati-Bessel functions of the first and second kinds with complex argument as well as the related ratios and logarithmic derivatives in the range $n = 1$ to the necessary highest order $n = N_{max}$.

The expansion coefficients which occur in the Mie theory for single spheres can be derived from equations (21) and (22) by letting $m_1 = m_2 = m$. This automatically implies that $r_1 = r_2 = a$, say, and $x_1 = x_2 = x = 2\pi a/\lambda$. Thus the functions corresponding to equations (4) to (14) reduce to the following simple forms:

$$S_n(x_2) = S_n(x),$$

$$C_n(x_2) = C_n(x),$$

$$S_n(z_{22}) = S_n(z_{21}) = S_n(z),$$

$$C_n(z_{22}) = C_n(z_{21}) = C_n(z),$$

$$A_n(x_2) = A_n(x),$$

$$A_n(z_{11}) = A_n(z_{22}) = A_n(z_{21}) = A_n(z),$$

$$B_n(x_2) = B_n(x),$$

and

$$B_n(z_{22}) = B_n(z_{21}) = B_n(z)$$

where $z = mx$, and x are the only two arguments concerned. Furthermore, we have

$$\alpha_n = 1,$$

$$\beta_n = 1,$$

$$\gamma_n = \frac{C_n(x)}{S_n(x)}$$

and

$$F_n(x_2) = F_n(x),$$

Now, it may be noted that U_1 and V_1 both reduce to zero. From the remaining expressions for a_n and b_n, it can be further seen that U_2 and V_2, respectively, cancel out. Thus, substituting the above reduced forms, equation (21) leads to

$$a_n = \frac{A_n(z) - mA_n(x)}{\left\{1 + i\,\frac{C_n(x)}{S_n(x)}\right\}[A_n(z) - mF_n(x)]}. \tag{23}$$

This can be expressed more succinctly as follows:

$$a_n = \frac{A_n(z) - mA_n(x)}{A_n(z)\,G_n(x) - mH_n(x)} \tag{24}$$

where

$$G_n(x) = 1 + i\,\frac{C_n(x)}{S_n(x)} \tag{25}$$

and

$$H_n(x) = A_n(x) + i\,\frac{C_n'(x)}{S_n(x)}. \tag{26}$$

Similarly,

$$b_n = \frac{mA_n(z) - A_n(x)}{mA_n(z)\,G_n(x) - H_n(x)}. \tag{27}$$

Equations (24) and (27) can be identified exactly with the Mie coefficients for a single homogeneous sphere as represented by various authors (see, for examples, MIE 1908; DEBYE 1909, VAN DE HULST 1957, KERKER 1969, DORSCHNER 1970, SHAH 1977a, b, SHIFRIN 1951, 1968).

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