Bull. Astr. Soc. India (2002) 30, 951–971

On mass and velocity distributions in members of star clusters : An analytical approach

S. Chatterjee,[1*] Brijesh Kumar,[2] and Ram Sagar,[1,2]
[1] *Indian Institute of Astrophysics, Bangalore 560 034, India*
[2] *State Observatory, Manora Peak, Nainital 263 129, India*

Received 17 June 2002; accepted 18 November 2002

Abstract. Dynamical evolution of stellar mass distribution in star clusters is analysed by considering simultaneously the effects of dynamical friction, stochastic heating and the gravitational potential due to mass distribution in the clusters. A simple expression is suggested for the dynamical friction which adequately describes it in both high and low velocity ranges and the effect of mass distribution in the cluster on the dynamics of a test mass can be described by an anharmonic potential. With the help of energy considerations we describe the dispersions in position and in velocity for stars of different masses showing how the slowing down and mass segregation of stars evolve in the cluster. Results are presented with the parameters for the star clusters of our Galaxy.

Keywords : Star clusters, dynamical friction, dynamical evolution

1. Introduction

Properties of star clusters form some of the most fascinating areas of astrophysics, particularly due to their close contact with issues related to star formation, mass function and stellar evolution. Such systems have stars over a wide range of masses (m) which lie in the range $0.1 \leq m/M_{\odot} < 10$. The distribution of these masses in real space as also in velocity space are signatures of various effects that arise in the context of dynamical evolution of the star in the gravitational potential of all the masses in the cluster. In what follows, we give a simple quantitative estimate of the evolution of velocity of any star of mass M, (for the moment only those whose life times are longer than the age of the cluster). The

*e-mail: chat@iiap.ernet.in

method is completely analytical and thus amenable to simple computational schemes. This is made possible firstly because a simple and yet accurate approximation for the Chandrasekhar dynamical friction formula could be found and the same type of formulae also found to fit the gravitational field in the cluster. This choice enables us to integrate the equation of motion analytically and its usefulness with reference to stellar dynamics in a spherically symmetric potential is demonstrated in the subsequent sections with special reference to open clusters and globular clusters. The relaxation process is considered to be due to dynamical friction, the violent relaxation process being completed within few thousands of years. The mass burning upto this time since the birth of the star is indeed negligible and $v(0)$, $M(0)$ are thus the quantities that obtain after the completion of violent relaxation. The gas and dust content in clusters being small, the drag forces due to dust and gas are neglected and only dynamical friction due to stars is considered. As is known, the dynamical friction is caused by the fluctuations in the trajectory of the star in question, due to encounters (not physical collisions) with other stars present in the cluster. The other important parameter that contributes to the dynamics of a test particle is the gravitational field arising out of the mass distribution (assumed Gaussian) in the cluster. We show that for a density profile $\rho(r)$ decaying with r, the gravitational potential $\phi(r)$ too can be adequately described by a function similar to that used to approximate the Chandrasekhar dynamical friction expression. A combination of these two approximations helps to integrate the equation of motion through simple mathematical operations. The integrated equation thus considers the dynamical evolution of a test mass M due to combined effects of dynamical friction and its oscillation in an anharmonic $\phi(r)$ due to mass distribution in the cluster. It is indeed necessary to explain the motivation behind the present analytical work. This is mainly motivated by the recognition that "there is plenty of room for the development of new analytic techniques", "bringing into play more physically motivated reasoning in the choice of fitting functions" (Hut, 1996). The results that follow do give tractable analytical methods to estimate various physical processes in the evolution of a test mass in a cluster and may prove to be useful for the future theoretical understanding of the existing observational data.

The main principles involved in the dynamics of the system are discussed in the next two sections. In section 2, we determine the effect of dynamical friction by considering stellar encounters. Essentially, the method is similar to what is given by Binney and Tremaine (1987) in which a King type velocity distribution has been incorporated. The main scheme for integrating the equation of motion is contained in sections 3 and 4 while the numerical scheme of solution is given in section 5. An interesting feature of our scheme is that it can easily accommodate modification due to mass loss arising out of various mechanisms that appear in the course of stellar evolution and is described in section 6. The effect of the above phenomena in describing mass segregation in the cluster is described in section 7 which is followed by results and discussions in section 8 of the paper.

2. Dynamical friction with King type velocity distribution

The role of gravitational interaction between the masses is considered in terms of stellar encounters. This, as is understood from the celebrated work of Chandrasekhar leads to (1) slowing down of the velocity of the particle in the direction of its initial velocity $v_{\|}(0)$ and (2) enhancements of its velocity $v_{\perp}$, in the component perpendicular to $v_{\|}$. The role of interparticle interaction is incorporated through the encounters between the test mass M, travelling with velocity v_M and the scatterers m, travelling with velocity v_m. As is well known, the inverse square nature of the interaction leads to a change Δv_M, which is given by,

$$\Delta v_M = \frac{2m|V_0|\alpha}{(M+m)}\left[1+\frac{b^2V_0^4}{G^2(M+m)^2}\right]^{-1/2} \tag{1}$$

where $V_0 = v_m - v_M$ and the components of unit vector α in the direction of initial V_0 is cos θ_0 and in direction perpendicular to V_0 is sin θ_0, where cos θ_0 and sin θ_0 are given as $G(m+M)/\sqrt{(G^2(m+M)^2+b^2V_0^4)}$ and $b^2V_0^4/\sqrt{(G^2(m+M)^2+b^2V_0^4)}$ respectively. These yield,

$$|\Delta v_{M\|}|^2 = \frac{4m^2V_0^2}{(m+M)^2}\left[1+\frac{b^2V_0^4}{G^2(m+M)^2}\right]^{-2} \tag{2}$$

$$|\Delta v_{M\perp}|^2 = \frac{4m^2V_0^6b^2}{G^2(m+M)^4}\left[1+\frac{b^2V_0^4}{G^2(m+M)^2}\right]^{-2} \tag{2.1}$$

which gives,

$$|\Delta v_M|^2 = \frac{4m^2V_0^2}{(m+M)^2}\left[1+\frac{b^2V_0^4}{G^2(m+M)^2}\right]^{-1} \tag{2.2}$$

In order to determine the changes that occur in time t, we consider the encounter process to be uncorrelated and on an average $2\pi n_0 V_0 b\,db$ encounters occur per unit time for particles passing within an interval b and $b+db$. The physical situation puts the limits of b as $b_{min} \leq b \leq b_{max}$ and the velocity distribution of m follows a truncated Gaussian,

$$f(v_m) = \begin{cases} \frac{An_0}{(2\pi\sigma_m^2)^{3/2}}\left[e^{-v_m^2/2\sigma_m^2} - e^{-v_e^2/2\sigma_m^2}\right] & \text{for } v \leq v_e \\ 0 & \text{for } v > v_e \end{cases} \tag{3}$$

where v_e is the escape velocity of the scatterers from the cluster and A is a normalisation constant, being given by,

$$1/A = \left[\mathrm{erf}(x_e) - \frac{2x_e}{\sqrt{\pi}}e^{-x_e^2} - \frac{4}{3\sqrt{\pi}}x_e^3 e^{-x_e^2}\right] \tag{3.1}$$

and we have defined $x = v_m/\sigma_m\sqrt{2}$ and $x_e = v_e/\sigma_m\sqrt{2}$.

The velocity distribution of the scatterers as given in equations (3, 3.1) indeed satisfies the consistency condition that the cluster contains only those stars with velocity which cannot escape from the potential well created by the mass distribution in the cluster. The estimate for v_e can be easily made by integrating the Poisson equation and putting $v_e^2/2 = \phi(\infty)$, being thus consistent with the nature of the mass distribution. For an accurate determination of the dynamical friction one must keep in mind the mass spectrum and the mass dependence of the velocity dispersion σ_m. In the scheme presented below, we assume that the mass spectrum can be represented by an average mass m and an average velocity dispersion σ_m.

It then follows from the equations (1), (2.2) and (3) that on account of encounters there appears a diminution of $v_{\|}$ being given by,

$$\frac{dv_M}{dt} = \int_0^\infty \int_{b_{min}}^{b_{max}} \Delta v_m \, 2\pi b db \, V_0 \, f(v_m) \, 4\pi v_m^2 dv_m$$

$$= -\frac{2\pi G^2 m(m+M)}{2\sqrt{2}\sigma_m^3}\left[\ln(\chi_{max}/\chi_{min})\right] A \, n_0(m) \, v_M F_{ch}(x) \tag{4}$$

$$\frac{d|\Delta v_M|^2}{dt} = \int_0^\infty \int_{b_{min}}^{b_{max}} |\Delta v_M|^2 \, 2\pi b db \, V_0 \, f(v_m) \, 4\pi v_m^2 dv_m$$

$$= \frac{4\pi G^2 m^2}{\sqrt{2}\sigma_m}\left[\ln(\chi_{max}/\chi_{min})\right] A \, n_0(m) \, x^2 F_{ch}(x) \tag{4.1}$$

with $\chi \cong 1 + b^2\sigma_m^2/G^2(m+M)^2$ and for all practical purposes we shall approximate $\ln(\chi_{max}/\chi_{min}) \approx 2\ln\Lambda$ with $\Lambda = b_{max}/b_{min}$. The $F_{ch}(x)$ is given as

$$F_{ch}(x) = \begin{cases} (\mathrm{erf}(x) - \frac{2x}{\sqrt{\pi}}e^{-x^2} - \frac{4}{3\sqrt{\pi}}x^3 e^{-x_e^2})/x^3 & \text{for } x \leq x_e \\ (\mathrm{erf}(x) - \frac{2x_e}{\sqrt{\pi}}e^{-x_e^2} - \frac{4}{3\sqrt{\pi}}x_e^3 e^{-x_e^2})/x^3 & \text{for } x > x_e \end{cases} \tag{4.2}$$

It can be seen that the above equations converge to the Chandrasekhar formulae when $x_e \gg 1$. In the low velocity range the dynamical friction $\propto \boldsymbol{v}_M$ and goes as $\boldsymbol{v}_M/|\boldsymbol{v}_M|^3$ for the high velocity range. Further, it can be seen from simple algebra that v_M^2 evolves as,

$$v_M^2(t+\delta t) = v_M^2(t) + 2\boldsymbol{v}_M(t)(d\boldsymbol{v}_M/dt)_t\delta t + (d\langle v_M^2(t)\rangle/dt)_t\delta t \tag{5}$$

The second and third terms in equation (5) are consistent with the cooling term and heating term respectively in Bekenstein and Maoz (1992) (see equations 3.2.5 and 4.16, therein) and are also consistent with equations (161-163) in Nelson and Tremaine (1999). In a future work we plan to re-examine their works in the light of equations (9-11) in the present paper. From equations (4.1) and (4.2) it can be seen that the second term in (5) is always negative while the last term in (5) is positive. The former gives a slowing down due to dynamical friction being $\propto m(m+M)$ while the latter gives stochastic heating and varies as m^2. Both originate in the inverse square interaction and act jointly, taking the system to equilibrium, being thus a manifestation of the fluctuation-dissipation theorem. For accurate evolution of the velocity v_M^2 we have to follow the formalism given in the next section. Full details of these calculations will be puplished shortly. We, however, present a summary in the next section in view of its importance on the question of evolution to equipartition.

2.1 On approach to equipartition

The question concerns, how the kinetic energy $\frac{1}{2}Mv_M^2(t)$ evolves in presence of encounters. It can be shown that on a single encounter the change in the kinetic energy of the mass M is given by (Binney and Tremaine 1987),

$$\Delta(Mv_M^2/2) = (M/2)\left[2\boldsymbol{v}_M.\Delta\boldsymbol{v}_M + |\Delta\boldsymbol{v}_M|^2 + 2\Delta\boldsymbol{v}_M.\boldsymbol{v}_C\right] \tag{6}$$

where $\boldsymbol{v}_M$ is the change of velocity on encounter and $\boldsymbol{v}_c = (m\boldsymbol{v}_m + M\boldsymbol{v}_M)/(m+M)$ is the velocity of the center of mass of the system. Knowing the components $\Delta v_{\|}$ and $\Delta v_{\perp}$ as given in equations (1-3) we find the r.h.s. of equation (6) to be,

$$-(M/2)\left[(v_{M\|} - v_{m\|})(Mv_{M\|} + mv_{m\|})\frac{2G^2(m+M)}{G^2(m+M)^2 + b^2V_0^4} + \right.$$

$$(v_{M\parallel} - v_{m\parallel})^3 (M v_{M\perp} + m v_{m\perp}) \frac{2bG}{G^2(m+M)^2 + b^2 V_0^4} \Bigg] \tag{7}$$

If we have a situation in which particles m randomly encounter the test particle M at values of b which are +ve and −ve equally frequently, we have to use the averages $\langle b \rangle = 0$ and $\langle v_{M\parallel} v_{m\parallel} \rangle = 0$ (e.g. b +ve means that the projectile approaches from above and it's track is bent downwards, while b −ve means that it approaches from below and the track is bent upwards, which essentially makes $\Delta v_{M\perp}$ to be +ve and −ve equally frequently, being thus zero on an average), so that

$$\Delta(M v_M^2/2) = -\left[Mm/(m+M)^2\right] \left[M v_{M\parallel}^2 - m v_{m\parallel}^2\right] \frac{2G^2(m+M)^2}{G^2(m+M)^2 + b^2 V_0^2} \tag{8}$$

It shows that $M v_M^2/2$ increases for $M v_{M\parallel}^2 < m v_{m\parallel}^2$ and decreases for $M v_{M\parallel}^2 > m v_{m\parallel}^2$, so that in the former case the kinetic energy of M increases with scattering while it decreases in the latter case. Since there are $2\pi b\ db\ V_0 n f(v_m)$ collisions per unit time, we find

$$(d/dt)(M v_M^2/2) = -2\pi^2 G^2 m M (M v_{M\parallel}^2 I_1 - m I_2) n \tag{9}$$

where

$$I_1 = \int_0^\infty \frac{1}{V_0^3} \ln \left[\frac{G^2(M+m)^2 + b_{max}^2 V_0^4}{G^2(M+m)^2 + b_{min}^2 V_0^4}\right] f(v_m) 4\pi v_m^2 dv_m \tag{10}$$

$$I_2 = \int_0^\infty \frac{v_{m\parallel}^2}{V_0^3} \ln \left[\frac{G^2(M+m)^2 + b_{max}^2 V_0^4}{G^2(M+m)^2 + b_{min}^2 V_0^4}\right] f(v_m) 4\pi v_m^2 dv_m \tag{11}$$

Dimensionally we note that $I_2/I_1 \simeq \langle v_{m\parallel}^2 \rangle$. Equation thus expresses the evolution of the kinetic energy $M v_M^2/2$ of the test star and shows that its rate of change vanishes when the equipartition i.e. $M v_{M\parallel}^2 = m \langle v_{m\parallel}^2 \rangle$ is reached. Full details of this evolutionary process, particularly, with respect to $\boldsymbol{v}_M$ will be presented in a future publication.

It is to be noted that the Chandrasekhar formula of which equations (4.2) and (13.2) are variants actually considers the density to be uniform. We have applied here the same formalism to an r dependent density profile in view of Kandrup's result (Kandrup 1981) that in an r dependent density profile the Chandrasekhar formula can be adopted by using the local number density $n(\boldsymbol{r})$ for the average n, appearing in the Chandrasekhar formula . Thus on averaging over all possible orbits (that lie at different r' s) one may use the averaged number density, as we have done, to estimate the net effect of dynamical friction. Chandrasekhar's simplification in considering the initial $\boldsymbol{v}_M(0)$ and final

$v_M(\infty)$ to be straight lines have also been investigated and the corrections are found to be marginal (Kandrup 1983). Furthermore, Del Popolo and Gambera (1999) have shown that the numerical value of the dynamical friction can be changed from the Chandrasekhar formula if a power law dependence $n(r) \sim r^{-p}$ ($p > 0$ see equation (12) in this paper) be considered. We have , however, not incorporated this effect since such a power law dependence makes the number density $n(\boldsymbol{r})$ blow up for $r \to 0$. In the computed results, given below, all the parameters have been taken from well known observations.

3. Principles and Approximations

We consider a test mass M, with an initial velocity $v(0)$, undergoing motion in the cluster. This motion takes place in a gravitational field $\phi(\boldsymbol{r},t) = \phi_0(\boldsymbol{r},t) + \delta\phi(\boldsymbol{r},t)$ of which the first part describes slowly varying (in time) average gravitational field due to all stars in the cluster, while $\delta\phi(\boldsymbol{r},t)$ is a rapidly varying time dependent potential that arises due to density fluctuations in the cluster owing to random velocities of stars i.e., $\delta\phi(\boldsymbol{r},t)$ gives rise to dynamical friction. The equation of motion of the star consists of the equations $d\boldsymbol{L}/dt = -\boldsymbol{L}\,[\boldsymbol{F}_{dyn}(M,\boldsymbol{v})/|\boldsymbol{v}|]$ and $dL^2/dt = -2L^2\,[F_{dyn}(M,\boldsymbol{v})/|\boldsymbol{v}|]$ for the angular momenta and $\ddot{\mathrm{r}} = -\partial\phi_0/\partial r + (L^2/mr^3) - \boldsymbol{v}_r F_{dyn}(M,\boldsymbol{v})/|\boldsymbol{v}|$ for the radial motion. In what follows we consider $L = 0$ i.e. a star with no angular momentum, oscillating radially across the cluster. The case for non zero L will be dealt with separately in a future work. For the case at hand, the equation of motion (we consider only the radial velocity) thus reads,

$$\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}\phi_0(\boldsymbol{r},t) - \frac{\boldsymbol{v}}{\tau(M,\boldsymbol{v})} = \boldsymbol{F}_0(r) + \delta\boldsymbol{F}_0(\boldsymbol{r},t) - \boldsymbol{F}_{dyn}(M,\boldsymbol{v}) \tag{12}$$

$$\frac{d\boldsymbol{v}}{dt} \simeq -\boldsymbol{\nabla}\phi_0(\boldsymbol{r}) - \frac{\boldsymbol{v}}{\tau(M,\boldsymbol{v})} \tag{12.1}$$

where the first term in the r.h.s. of equation (12) gives the steady gravitational potential field, while the second term is a time dependent field arising due to changes in the density profile of the cluster in course of its dynamical evolution and the third term is the damping due to dynamical friction. By neglecting the $\delta\boldsymbol{F}_0(\boldsymbol{r},t)$ we obtain equation (12.1). The field $\delta\boldsymbol{F}_0(\boldsymbol{r},t)$ though time dependent is a slowly varying quantity as compared to $\nabla\delta\phi(\boldsymbol{r},t)$, which finally leads to dynamical friction. The net effect of the $\delta\phi(r,t)$ term has been employed here by introducing the last term i.e. dynamical friction in equations (12) and (12.1). We must note, however that the dynamical friction does have a role in the origin of the $\delta F_0(\boldsymbol{r},t)$ term also since the density profile evolves with time due to dynamical friction. Due to the slow variation of $\delta\boldsymbol{F}_0(\boldsymbol{r},t)$, this term is ignored in this zeroth order approximation, to be, however, considered in a more complete theory.

For integrating equation (12.1) two cases have to be considered separately. Equation (12.1) can be solved in its completeness by the method given by Van Kampen (1985). The method essentially involves the identification of a smallness parameter ϵ and expansion of the solution v as a power series in ϵ. We find below that the system has a dimensionless parameter ϵ' = relaxation time/time period of oscillation. When relaxation are slow we have $\epsilon = 1/\epsilon'$ and the star behaves as a damped oscillator. For fast relaxation $\epsilon'' \ll 1$ and one can identify $\epsilon = \epsilon'$ as the small parameter and the system behaves as an overdamped oscillator and hence does not demonstrate any to and fro motion. Considering the mass distribution in the cluster to be $\rho(r) = \rho_0 \exp(-\gamma r^2)$ and a velocity distribution as given in equation (5) we find,

$$-F_r = \partial\phi_0/\partial r = \left(\pi^{3/2} G\rho_0/\gamma^{1/2}\right) yF(y) \tag{13}$$

where $y = r\sqrt{\gamma}$ and

$$\frac{1}{\tau(M,v,t)} = \left[\frac{4\pi G^2 n_0 m(m + M(0) - \Delta M(t))\ln(\Lambda)}{2\sqrt{2}\sigma_m^3}\right] F_{ch}(x) \tag{13.1}$$

where $F_{ch}(x)$ has been defined in (4.4), $\Delta M(t) = |\int_0^t (dM/dt)\, dt|$ being the mass reduction upto the time t after the birth of the star. It is clear that $\Delta M(t)$ depends upon the evolutionary phase of the star which is also determined by its initial mass $M(0)$. In what follows we shall be concerned with the stars in the main sequence for which analytical expression for $\Delta M(0)$ will be used. It is seen that for $x \to 0$ $F_{ch}(x) \sim 4/3\sqrt{\pi}$ while for $x \to \infty$ $F_{ch}(x) \sim 1/x^3$. We find that $F_{ch}(x)$ can be approximated as

$$F(x) = 1/[a + bx^6]^{1/2} \tag{13.2}$$

with $a = 3.25$, $b = 1.28$ for globular cluster case and $a = 2.56$, $b = 4.09$ for open cluster case (see section 4) and , with a reliability of 99% as seen by Kolmogorov-Smirnov (KS) test, a simplification, which we shall use here for a very simple computational scheme. This approximation for $F(x)$ also gives on integrating equation (13), and approximate expression for the gravitational potential inside the cluster, which reads

$$\phi_0(r) = \phi_0\tilde{\phi}(y) \tag{13.3}$$

where

$$\phi_0 = \pi^{3/2} G\rho_0/2\gamma \tag{13.4}$$

$$\tilde{\phi}(y) = \int_0^y y' F(y') dy' \tag{13.5}$$

$$\tilde{\phi}(y) \simeq y^2 F(y) + (3b/7) y^7 [F(y)]^3 + ... + ... \tag{13.6}$$

We were not able to find a closed form equation for $\tilde{\phi}(y)$ and the expansion given in equation (13.6) is fairly accurate upto $y = 1.33$, which will correspond to $r \sim 40$ pc for a typical globular cluster and $r \sim 13$ pc for an open cluster with halo. We note on numerically integrating $\tilde{\phi}(y)$ that it is monotonically increasing function and $\tilde{\phi}(y) \to 1/2$ for $y \to \infty$. The escape velocity v_e of particles is then given by $\frac{1}{2} v_e^2 = \phi_0 \tilde{\phi}(\infty)$ i.e. $v_e^2 = \phi_0$. In all the discussions that follow we are concerned with particles that are gravitationaly bound to the cluster and hence obey the restriction $v^2 < v_e^2$. The choice of $\rho(r)$ as a Gaussian is indeed arbitrary but it does satisfy that $\rho(r)$ is peaked at $r = 0$ and falls off rapidly beyond a length $r_0 \sim 1/\sqrt{\gamma}$ from the centre, which is an essential feature of the density profile of the cluster (Nilakshi et al. 2002). This results in a gravitational field $F_r \propto -r$ for $r \ll r_0$ and $F_r \propto -r^{-2}$ for $r \gg r_0$. In what follows we ascribe such a behaviour through equations (13) and (13.2) The choices a and b that we present here are the ideal ones for the Gaussian $\rho(r)$. However, for any other type of decay too similar approximations as given in equations (13) and (13.2) are possible in which the choices of a and b would be decided by the exact nature of the decay of $\rho(r)$ e.g. for an exponential density distribution $\rho(r) = \rho(0) e^{-\kappa r}$ we find that the $F_0(r)$ can be approximated to $F_0 \simeq (\kappa r)/[a_1 + b_1 (\kappa r)^6]^{1/2}$ with 90% confidence as shown by the KS test. This observation thus demonstrates the usefulness of our approximation. For the subsequent calculation described here we continue to follow a Gaussian distribution for the density profile.

4. Method of solution

We note that the system introduces certain time scales. For $x \to 0, F(x) \to 1$, and for $x \to \infty, F(x) \sim (2/\sqrt{\pi})(2/3)x^3$. Hence for low velocities we have a relaxational time scale $\tau(M)$, $1/\tau(M, v) = (4\pi G^2 n_0 m(m+M) \ln(\Lambda)/2\sqrt{2}\sigma_m^3)(4/3\sqrt{\pi})$ and an oscillational time period T_0 which is akin to the crossing time such that $4\pi^2/T_0^2 = (4\pi/3) G\rho_0$. The solution of equation (12.1) must consider two distinct cases. (*a*) $T_0 \ll \tau(M)$ and (*b*) $T_0 \gg \tau(M)$.

4.1 Case *(a)* : $T_0 \ll \tau(M)$; $\epsilon' = \tau(M)/T_0$; $\epsilon = 1/\epsilon'$

We examine the problem in two cases. Firstly we consider a typical globular cluster with $n_0 = 25$ pc^{-3}, $m = 1 M_\odot$, $\sigma_m = 6$ km s^{-1}, $r_0 = 1/\sqrt{\gamma} \sim 30$ pc with $v_e = 22$ km s^{-1}. Next we consider the intermediate age open cluster NGC 2099 (M37) for which $n_0 \approx 3$

pc^{-3}, $m = 1M_{\odot}$ and $\sigma_m \approx 1$ km s^{-1}, $r_0 = 1/\sqrt{\gamma} \sim 10$ pc, being the range upto which the halo is present. For this last case we find $v_e = 2.6$ km s^{-1} (Kalirai et al. 2001, Nilakshi and Sagar 2002). For stars with $M(0) \leq 10M_{\odot}$ one finds $\tau_{relax} = 2\tau(M(0), 0) > 9.2 \times 10^7$ years while $T_{osc} \approx 2.75 \times 10^7$ years. Since $\tau_{relax}/T_{osc} \approx 3$, the star does exhibit oscillations, $r(t) = a(t)\cos(2\pi t/T_{osc})$ where $r(t)$ is the instantaneous radial distance of the star from the center of the cluster. Indeed the nature of F_r shows the potential $\phi_0(r)$ to be anharmonic, this anharmonicity can be incorporated in the calculation by making the oscillational time scales $T(n)$'s to be dependent on the amplitudes of the oscillation for the respective cycles n of oscillations of the particle. Indeed the complete dynamics requires the inclusion of higher harmonics of $2\pi/T(n)$ but as far as the energy loss is concerned it is the slowest oscillational mode that outweighs the contribution from other faster modes, due to rapid elimination of fast variables (Van Kampen 1985). It thus suffices to approximate the motion only in terms of the slowest $2\pi/T(n)$ mode as is done below. It is to be noted that with the passage of time the amplitude of oscillation $a(t)$ will decay due to dynamical friction. T_{osc} is also a time dependent quantity arising due to anharmonicity of the potential $\phi_0(r)$. If however, $\tau_{relax}/T_{osc} < 1$ the system becomes overdamped and such oscillations are not to be found as will be seen in case (b).

The energy loss per cycle due to dynamical friction is then found by noting that power loss is the product of the velocity and the resistive force, so that on defining E_n to be the energy of the particle in its nth cycle,

$$E_{n+1} - E_n = \Delta E_n = -\int_0^{T(n)} F_{dyn}(M(t), v)\ v\ dt$$

$$= -2\sigma_m^2 X_o^2 \beta \int_0^{T(n)} \frac{(m + M_{\odot} - \Delta M\ t)\ x_n^2 \cos^2(2\pi t/T(n))}{[1 + x_n^6 \cos^6(2\pi t/T(n))]^{1/2}} dt \tag{14}$$

where $X_0^2 = (a/b)^{1/3}$, $\beta = 4\pi G^2 n_0 m \ln(\Lambda)/2\sqrt{2}\sigma_m^3/\sqrt{a}$, and $x_n^2 = v_n^2/2\sigma_m^2 X_0^2$, v_n being the amplitude of the velocity in the nth cycle and $T(n)$ is the corresponding time period. The above difference equation can be written as

$$z_{n+1} - z_n = -\frac{\beta}{2} \frac{z_n}{[1 + z_n^6]^{1/2}} (m + M_{\odot} - \Delta M(t)) T(n) \frac{F_N(z_n)}{F_D(z_n)} \tag{15}$$

$$F_N(z_n) = 1 - \frac{5}{32} \frac{z_n^6}{[1 + z_n^6]^{1/2}} + \frac{1965}{2097152} \frac{z_n^{12}}{[1 + z_n^6]^{1/2}} \tag{15.1}$$

$$F_D(z_n) = 1 - \frac{1}{10} \left(\frac{T_0 \sigma_m}{R} \right)^2 z_n^2 \tag{15.2}$$

where $z_n^2 = (48/15)^{1/3} x_n^2$, $t = \sum T(n)$, $R = (\gamma/2)^{-1/2}$ = average radius of the cluster, $(2\pi/T_0)^2 = (4\pi/3)G\rho_0$, T_0 being the oscillation time period of the star in the harmonic limit and t is the total time elapsed after the formation of the star. In equation (15) $F_N(z_n)$ denotes diminution of dynamical friction as the star acquires larger velocity as it passes through the center of the cluster, while $F_D(z_n)$ as given in equation (15.2) gives the increase of the period of oscillation on account of anharmonicity. By defining,

$$F[z(t), z(0)] = \sqrt{(1 + z^6(t))} - \sqrt{(1 + z^6(0))} - \sinh^{-1}(1/z^3(t)) + \sinh^{-1}(1/z^3(0)) \quad (16)$$

Equation (15) can be finally integrated to yield (by writing $t = nT_0$),

$$F[z(t), z(0)][F_D(z)/F_N(z)] = -(3/2)\beta \left(m\, t + M(0)\, t - \int_0^t \Delta M(t')dt' \right) \qquad (16.1)$$

where $z(t)$ thus denotes the variation of z when change over is made from discrete n to continous variable t. Using the above equation (16) we plot $v(t)$ with t for globular clusters and open clusters.

As a first step we solve equation (16) by taking as a zeroth order approximation $F_N = 1 = F_D$. In the next step we solve equation (16) again by taking $F_N(z) = F_N(z0(t))$ and $F_D(z) = F_D(z0(t))$. The resulting $z(t)$ values are next used to calculate $F_N(z)$ and $F_D(z)$, and solve for the new $z(t)$ by using these values of $F_N(z)$ and $F_D(z)$ in equation (16). This iterative scheme of calculation is now being extended to take care of iteration upto any desired level. Also the scheme can take care of the effects of mass loss if the rate of mass loss (dM/dt) can be specified for the star in different stages of its evolution.

4.2 Case *(b)* : $T_0 \gg \tau(M)$; $\epsilon' = T_0/\tau(M)$; $\epsilon = \epsilon'$

This is a case when the star relaxes within a single oscillational time period. This is indeed an overdamped case. Here we take only the second term in equation (12.1) as the zeroth order term while the first term is a perturbation. This is easily integrated in the zeroth order approximation to be,

$$F[z(t), z(0)] = \sqrt{(1 + z^6(t))} - \sqrt{(1 + z^6(0))} + \sinh^{-1}(1/z^3(t)) - \sinh^{-1}(1/z^3(0))$$

$$= -3\beta \left(m\, t + M(0)\, t - \int_0^t \Delta M(t')dt' \right) \tag{17}$$

Since in case (b) we have discarded the role of harmonic potential, it describes the motion for a free particle under damping due to dynamical friction. Comparison of equation (16.1) with equation (17) shows that the factor 3/2 on the r.h.s. of equation (16.1) is replaced by 3 in equation (17). This is because when we deal with a particle in absence of the harmonic potential its variation in velocity is due to damping alone. For a particle in a harmonic (also anharmonic potential) the oscillation makes v to oscillate and the dynamical friction also tunes itself accordingly. In effect the damping is faster in case (b)

The above equation (17) can be easily solved numerically. This is a solution which gives $v(t) \sim v(0)\exp(-t/\tau(m))$ for $v \ll \sigma_m$ and $[v(t)^3 - v(0)^3] = -3\beta t$ for $v \gg \sigma_m$ as is known for motion under dynamical friction, in absence of a potential. Next we define

$$r(t) = \int_0^t v(t')dt' \tag{18}$$

which can be calculated from solution of $v(t)$ as given in equation (17). We now use the numerical values of $r(t)$ for the first term in equation (12.1), being thus treated as a perturbation. We note that the role for the first term in equation (12.1) is to provide an extra acceleration:

$$F_0(t) = (\pi^{3/2} G\rho_0/\gamma^{1/2})\ \ (\gamma^{1/2} r(t))\ \ F(\gamma^{1/2} r(t)) \tag{19}$$

We now add this extra term to equation (12.1) and integrate. This leads to

$$F[z(t), z(0)] = \sqrt{(1+z^6(t))} - \sqrt{(1+z^6(0))} - \sinh^{-1}(1/z^3(t)) + \sinh^{-1}(1/z^3(0))$$

$$= -3\beta \left(m\, t + M(0)\, t - \int_0^t \Delta M(t')dt' \right) - (1/\sqrt{2}\sigma_m X_0) \int_0^t F_0(t)dt \tag{20}$$

where the last term in equations (20) describes the effects of the gravitational field and is calculated from the solution of equations (17) and using the solution to numerically evaluate equations (18, 19) and the second term in equation (20). The r.h.s. of equation (20) being thus known in its entirety, the same numerical method used for solution of equation (17) can be repeated to find the solution of equation (20).

5. Numerical scheme of solution

Time development of the velocity of the star in the potential of all other stars is contained in the dimensionless quantity $z(t)$ which has to be solved from equation (16) when $T_0 \ll \tau(M)$ and from equations (17) and (20) for $T_0 \gg \tau(M)$. It is , however, necessary to note that in both cases the formal schemes are the same. Essentially, we have to solve equations of the type,

$$f(z(t), z(0)) = \chi(t) \tag{21}$$

In case (a) $f[z(t), z(0)] = F[z(t), z(0)][F_D(z)/F_N(z)]$ while in case (b) f[z(t),z(0)] = F[z(t),z(0)] and the $\chi(t)$ are the r.h.s. of equations (16), (17) or (20) as the case may be. The numerical scheme of solution is thus essentially the same and is described below.

Given any initial condition $z(0)$ we tabulate the l.h.s. of equation (21) for different z and $\chi(t)$ is also caculated for different t. It is then easy to find graphically the point $z(t)$ at which $f[z(t), z(0)]$ equals $\chi(t)$ which is shown in figure 1. One must, however, observe that while evaluation of the l.h.s. of equation (21) is a simple numerical work, evaluation of the r.h.s. of equation (21) needs inputs on mass loss from various sources.

6. Calculation of $\Delta M(t)$

We assume t to begin from ths star formation stage and $\Delta M(t)$ must contain the mass loss from all the sources e.g. mass burning, winds, mass ejection etc. A notable feature of the present theory is that the mass appears only on the r.h.s. of the equations (16), (17) and (20), while the dynamical variables like v appear only on the l.h.s. This greatly helps the numerical work and yet the evolutionary history and dynamical history of the particle are both accommodated in one single equation. It is clear that the mass reduction in the main sequence occurs due to mass burning of upto 10% of core hydrogen, converting only a fraction of 0.0007 of the total hydrogen mass into energy. The quantity $\int_0^t \Delta M(t')dt'$ due to mass burning can thus be neglected in comparison to $M(0)t$, within the main sequence life time of the star. In this phase however, some mass is also lost due to stellar winds, the total mass loss due to mass burning and stellar winds is taken from Schaller (1992) for solar composition and the quantity $\int_0^t \Delta M(t')dt'$ can thus be computed. This goes as an input to the r.h.s. of equations (16, 16.1, 17, 20). Beyond the main sequence life time, the mass loss $\Delta M(t)$ becomes extremely complex for analytical calculation (Wilson 2000, Kudritzki and Puls 2000, Iben and Renzini 1983, Iben 1974). However the $\Delta M(t)$ values provided by Schaller (1992) incorporates all these processes for stars in various phases of their evolution and has been incorporated to find $\int_0^t \Delta M(t')dt'$ in our computation and thus to find the dynamical evolution of stars of different masses present in the cluster.

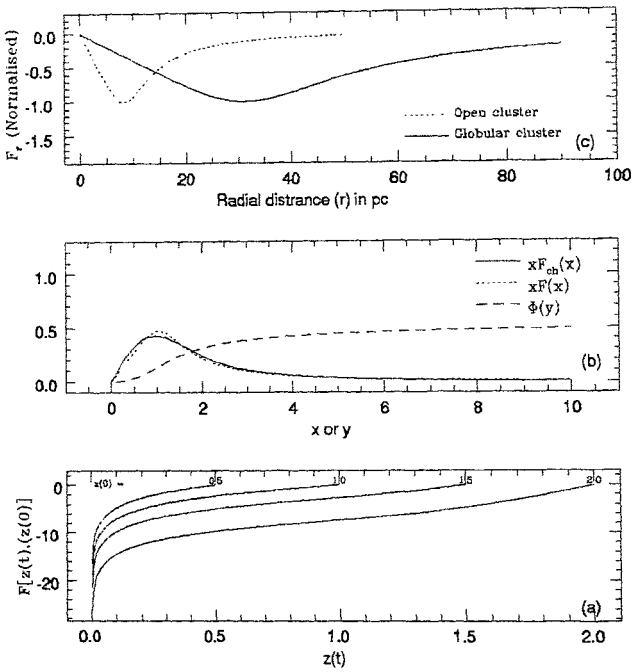

Figure 1. (a) F[z(t),z(0)] for different z(0) values, (b) Approximations for $xF_{ch}(x)$,$xF(x)$, $\Phi(y)$ for the globular cluster case (c) Normalised gravitational field in the open cluster (NGC 2099) and a typical globular cluster.

7. On the question of mass segregation

The role of dynamical friction is to slow down fast moving particles and to finally equilibrate the system so that the velocities of the different masses reach their respective equipartition values. This energy loss by the test mass will result in a gradual shrinking of the amplitude $a(t)$ of its oscillation inside the cluster. This thus results in a segregation of the heavier mass towards the center of the cluster. How the mass segregation evolves with time can be measured by noting the variation of $\langle r(t)\rangle$ with t from the virial theorem. By applying the virial theorem, $2\langle T\rangle = \langle -r\partial V/\partial r\rangle$, we find, (on noting that $2\langle T\rangle = 2.\frac{1}{2}\langle v^2\rangle = 2.\frac{1}{2}\langle \frac{1}{2}v_n^2\rangle$, where averaging $\langle\cdots\rangle$ is done over a cycle),

$$v^2/2 = (\pi^{3/2}G\rho_0/\gamma)\left[\frac{\gamma\langle r^2\rangle}{[a+b\gamma^3\langle r^2\rangle^3]^{1/2}}\right] \tag{22}$$

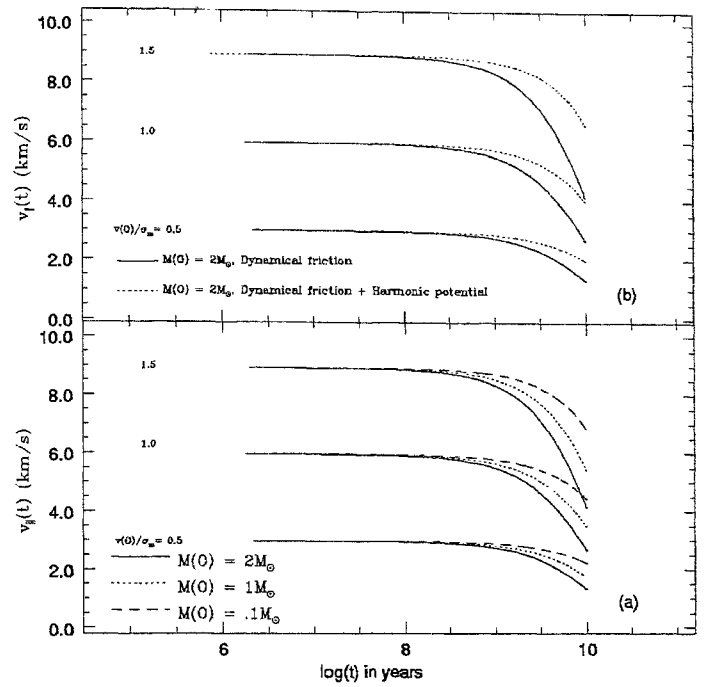

Figure 2. (a) Evolution of $v_\parallel$ for different masses at different initial velocities under dynamical friction alone. (b) Effect of dynamical friction on the evolution of velocities under harmonic potential.

where v^2 (valid for $v^2 < v_e^2$) is calculated from equations (14-16) and $\langle r \rangle \sim \langle r^2 \rangle^{1/2}$. The variation of $\langle r \rangle$ with t for different masses M is a question of importance in studying the mass segregation in the cluster.

8. Results and discussions

Investigations contained in the present paper give a method to study the dynamics of a star in a potential due to a spherically symmetric distribution of masses, - a situation that arises in star clusters - $v_\parallel$ of the star being damped by the dynamical friction with $v_\perp$ being enhanced by the stochastic heating. We have considered the mass distribution in the cluster to be falling off in a Guassian fashion with the distance r from the center of the star cluster. Such a variation is chosen because close to the center of the cluster, the gravitational potential is expected to be harmonic, while the anharmonic part takes over only at longer distances. To calculate the dynamical friction, the Chandrasekhar type formula which uses King type distribution of velocities of stars has been used. However

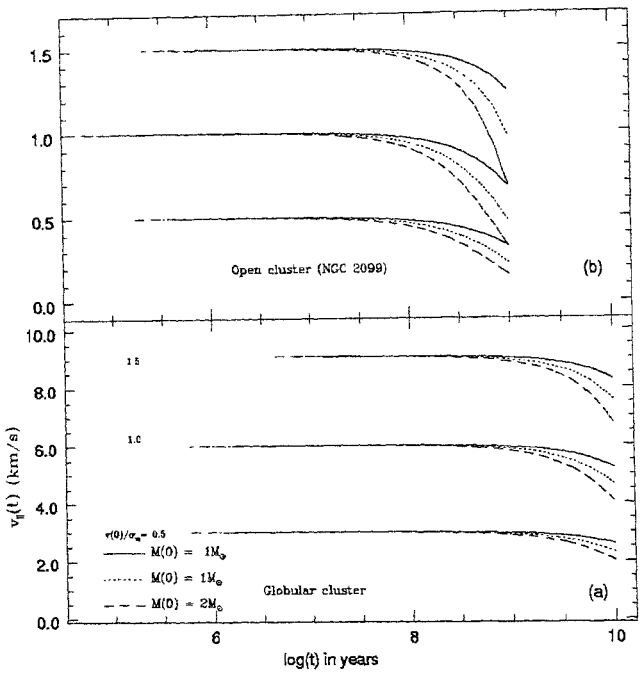

Figure 3. Evolution of velocities for a globular cluster (a), an open cluster (b) The captions of panel (a) applies to (b) also.

this modified Chandrasekhar formula being quite involved, we have approximated it by a simple power law type of expression (13.2), which has an excellent fit with the modified Chandrasekhar formula, as seen from Kolmogorov-Smirnov test. It is indeed this identification of a simple formula which fits the Chandrasekhar formula very satisfactorily - that makes our numerical scheme very simple, fast and easy to check and yet accurate. The computational scheme is further facilitated by the fact that it is adequate to approximate the gravitational field F_r by equation (13) and the gravitational potential $\phi(r)$ by equations (13.3 - 13.6). The results that follow are based on the above inputs and implies that we are considering an epoch after the completion of the violent relaxation phase, at which an initial mass segregation described by $\rho(r)$ has taken place. We have chosen b_{min} = average interparticle distance and $b_{max} = \sqrt{(3/2\gamma)}$.

The results of our calculations are found in figures 1-6, which are indeed self explanatory. In our computations we have used the numbers given in section 4 of the paper. The method of solution is illustrated in in figure 1(a) where we show the plot of $F[z(t), z(0)]$ with $z(t)$ for various initial values $z(0)$. As a check we have tried to see the evolution of velocity of a test particle in absence of the gravitational potential which is essentially

the solution of equation (17). We find that in such a case the velocity decay follows $v_{\|}(t) = v_{\|}(0)exp(-t/\tau(m))$ for $v_{\|} \ll \sigma_m$ and $[v_{\|}(t)]^3 = [v_{\|}(0)]^3 - 3\beta t$ for $v_{\|} \gg \sigma_m$ (for low t values) as is expected from the Chandrasekhar dynamical friction. This is illustrated in figure 2(a), where only the effect of dynamical friction is retained. The same scheme of solution when applied to equation (17) yield the result in figure 2(b), where we find that due to oscillations in the potential well, the dynamical friction gets modulated in terms of instantaneous velocity and hence the decay may be slower. The usual mass dependences are also borne out due to the dependence of dynamical friction on the mass M of the projectile. This also has bearing with the fact that lower masses lose small amounts of their mechanical energy, while larger masses lose large amounts of their initial mechanical energy. This has been adequately described by equations (9-11). However, we cannot give at present accurate v dependences of the integrals in (10) and (11) and hence the exact nature of approach towards equipartition cannot be described in the present paper and is reserved for a future work. An approximate scheme is contained in (5), which takes care of dynamical friction as well as stochastic acceleration[1]. It shows the following qualitative behaviour that the decrement in $v^2(t)$ is more pronounced for the higher mass stars. This, indeed results from a competition between the two terms in equation (5) which shows that the net loss of energy due to dynamical friction goes as $m(m+M)$ while the stochastic acceleration - in effect a diffusion in velocity space - goes as m^2 (Chandrasekhar 1943, Spitzer and Schwarzschild 1951, Wielen 1977). For a more exact result in a realistic situation these effects with proper mass spectrum are to be considered (Chatterjee 1991, 1995). This is illustrated in figure 4. These processes will also describe $\delta F_0(r,t)$, which we have ignored in our present approximation. It can indeed be incorporated as a perturbation once we understand the effects of condensation and evaporation from our present work.

The change of $v^2(t)$ for different masses, enables us to make contacts with earlier workers (e.g. Gorti and Bhatt 1996, 1996a, who consider the dynamical effects of protostellar clumps in gas clouds and Aarseth and Heggie 1998). Their numerical works essentially contain the description of the evolution process as seen in equation (5). This evolution of $v^2(t)$ also serves as the means to investigate the mass segregation in clusters. Given the value of v^2, the value of $\langle r \rangle \sim \langle r^2 \rangle^{1/2}$ for a star of given mass M can be found from equation (22), the results being represented in figures 5 and 6. In obtaining these results we have considered only heavier stars i.e. $1 \leq M/M_{\odot} \leq 5$ i.e. $m \ll M$. In obtaining figure 6 we have noted that the heavier stars evolve to their late phase in $\sim 10^8$ years i.e. within a time scale in which very little effect is observed with respect to their dynamical evolution. The subsequent scenarios of their evolution being not known with precise models, we have considered these stars to evolve only dynamically and noted their condensation with time, through the parameter $\langle r(t) \rangle$. In other words figure 6 sug-

[1]Why equation (5) cannot describe approach to equipartition is because the two averages used in calculating $\langle dv_{\|}/dt \rangle$ and $d\langle(\Delta v)^2\rangle/dt$ have been considered to be independent. An averaging subject to the constraint (8) will lead to equipartition and will be discussed in a future paper.

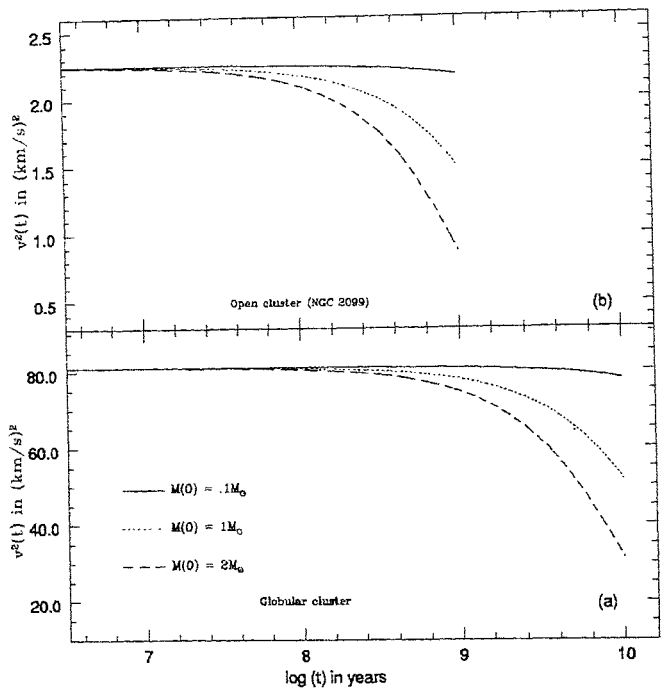

Figure 4. Change in $v^2(t)$ for different masses when $v(0) = 1.5\sigma_m$.

gests the "tentative" locations within which moderately heavy stars or their remnants e.g. neutron stars are to be found as they evolve between 10^5 - 10^{10} years.

We return again to figure 5 and 6, to understand the possible origin of an extended corona in open clusters (Nilakshi et al. 2002, Pandey et al. 2001, Durgapal and Pandey 2001). It is seen that if we have a $1M_\odot$ star with $v(0) = 1.5\sigma_m$, it can occupy upto 11 pc initially (newly born) in an open cluster, while occupy only upto 5 pc after 10^9 years. The corresponding numbers for a typical globular cluster are 21 pc at birth and 20 pc at the end of 10^{10} years. If on the other hand we consider a $0.1M_\odot$ star with $v(0) = 1.5\sigma_m$ at birth, it will continue to live within 11 pc if it be in a typical open cluster and within 21 pc if it be in a typical globular cluster, because of very weak velocity loss due to feebleness of the dynamical friction that it experiences. It is thus possible for such a star to travel far away from the center of the open cluster in view the smallness of the gravitational field that arises in an open cluster. This may thus explain the presence of an extended corona in older open clusters. However, if the corona are integral part of star formation process itself as has been indicated by Nilakshi et al. (2002), then the dynamical evolution will only modify it.

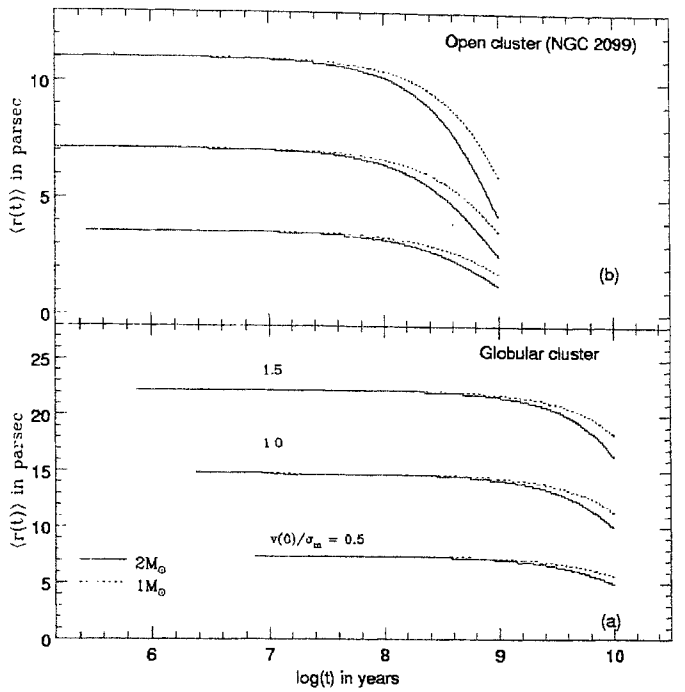

Figure 5. $\langle r(t)\rangle$ versus t for a globular cluster (a) and an open cluster (b). The captions of panel (a) applies to (b) also.

The competition of dynamical friction with stochastic heating as given in equation (5) is indeed qualitatively correct when dealing with $M \gg m$ but always leads to a decrease in $v^2(M)$ for all masses. The correct approach should be to obtain $v^2(M)$ from equation (11), which slows down the evolution of $v^2(M)$ as it approaches the equipartition value. This approach as also equation (5) in the present paper do predict the situation that leads to the lighter mass (a) lying very close to the edge of the corona or (b) escaping the gravitational potential of the cluster. Concerning case (a) we note that this has been borne out in several recent works (Durgapal and Pandey 2001, Pandey et al. 2001, Nilakshi et al. 2002) who find a long tail for the corona, owing to the presence of low mass stars . Though the density profile $\rho(r)$ determined by them cannot be fitted to a Gaussian curve and definitive form is not yet established, the long tail in the corona, can still be considered to give rise to gravitational field of the type $xF(x)$ given by equation (13.3), which we have already tested by considering various types of $\rho(r)$'s with the coefficients a and b of the fit being decided by the nature of the $\rho(r)$. The dynamics that can follow from such a fit is in the process of being tested by us and in a preliminary study is seen to be consistent with the models concluded through observations. To estimate evaporation process we ought to solve the Boltzmann equation by using the appropriate scattering

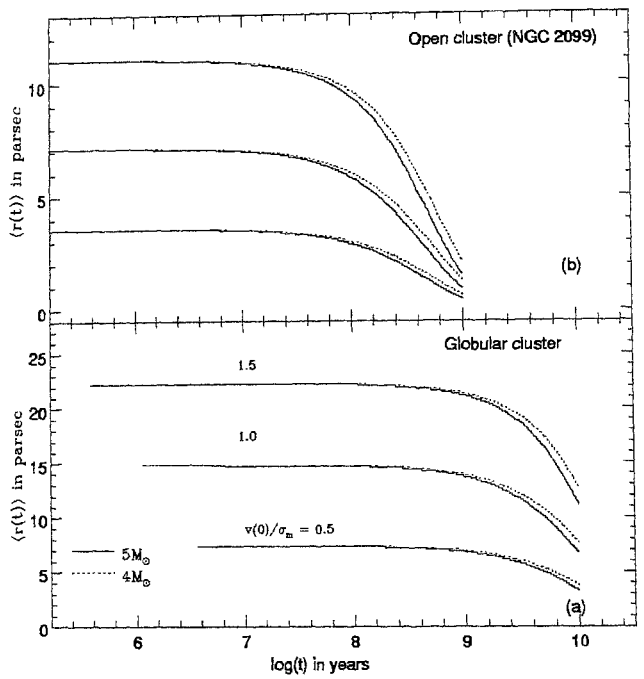

Figure 6. $\langle r(t) \rangle$ versus t for a globular cluster (a) and an open cluster (b). The captions of panel (a) applies to (b) also.

cross section for the $\sigma(v_{\parallel}, v_{\perp})$ an inverse square field (Chandrasekhar 1943, Spitzer and Schwarzschild 1951), but for a particle bound in an anharmonic potential $\phi(r)$ given by equation (13.1). Such a method will at one stroke combine (1) dynamical friction (2) stochastic heating and (3) diffusion in r space in an anharmonic potential $\phi(r)$. The last mentioned has been qualitatively addressed here but a full scale understanding (e.g. of the density profile and role of evaporation in describing the density profile) does deserve such a combined attack. As far as we are aware, this problem has not been tackled till now, to which we will return shortly along with the role of an initial mass function and velocity dispersion on the dynamical evolution of a cluster.

Acknowledgements

The authors would like to thank Professor H.C. Bhatt for useful comments on an earlier version of the manuscript. They thank Dr. A.K. Pandey for his comments on several features of open clusters, as seen by recent observations. One of the authors (SC) would

like to thank Professor D.C.V. Mallik for his encouragement. The referee of the paper is thanked for useful comments.

References

Aarseth S.L. and Heggie D.C., 1998, *Mon. Not. R. Astron. Soc.*, **297**, 794.
Bekenstein J.D. and Maoz E., 1992, *Astrophys. J*, **390**, 79.
Binney J. and Tremaine S.D., 1987, *Galactic Dynamics*, Princeton Univ. Press., p. 416.
Chandrasekhar S., 1943, *Astrophys. J.*, **97**, 263.
Chatterjee S., 1991, *J. Astron. Astrophys.*, **12**, 269.
Chatterjee S., 1995, *Astron. Astrophys.*, **298**, 438.
Durgapal A.K. and Pandey A.K., 2001, *Astron. Astrophys.*, **375**, 840.
Del Popolo A and Gambera M., 1999, *Astron. Astrophys.*, **342**, 34.
Gorti U. and Bhatt H.C., 1996, *Mon. Not. R. Astron. Soc.*, **278**, 611.
Gorti U. and Bhatt H.C., 1996a, *Mon. Not. R. Astron. Soc.*, **283**, 566.
Hut P., 1996, *IAU Symp.*, **174**, 325.
Iben I.Jr., 1974, *Ann. Rev. Astron. Astrophys.*, **12**, 215.
Iben I.Jr., Renzini A., 1983, *Ann. Rev. Astron. Astrophys.*, **21**, 271.
Kalirai J.S. et al., 2001, *Astron. Astrophys. Suppl.*, **199**, 7902.
Kandrup H.E., 1981, *Astrophys. J.*, **244**, 1039.
Kandrup H.E., 1983, *Astrophys. Space Sci.*, **97**, 435.
Kudritzki R.P., Puls J., 2000, *Ann. Rev. Astron. Astrophys.*, **38**, 613.
Nelson R.W. and Tremaine S., 1999, *Mon. Not. R. Astron. Soc.*, **306**, 1.
Nilakshi and Sagar R., 2002, *Astron. Astrophys.*, **381**, 65.
Nilakshi, Sagar R., Pandey A.K., and Mohan V., 2002, *Astron. Astrophys.*, **383**, 153.
Pandey A.K., Nilakshi, Ogura K., Sagar R., and Tarusawa K., 2001, *Astron. Astrophys.*, **374**, 504.
Schaller G., Schaerer D., Meynet G., Maeder A., 1992, *Astron. Astrophys. Suppl.*, **96**, 269.
Spitzer L. Jr and Schwarzschild M., 1951, *Astrophys. J.*, **114**, 385.
Spitzer L. Jr and Schwarzschild M., 1953, *Astrophys. J.*,**118**, 106.
Spitzer L. Jr and Hart M.H., 1971, *Astrophys. J.*, **164**, 399.
Van Kampen N.G., 1985, *Physics Reports*, **124**, 263.
Wielen R., 1977, *Astron. Astrophys.* **60**, 263.
Wilson L.A., 2000, *Ann. Rev. Astron. Astrophys.*, **38**, 573.