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Axial oscillation of a planetoid in Restricted Three Body Problem : The circular Sitnikov problem

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Abstract. Two equally massive primaries are assumed to be moving in circular orbits in Cartesian x-y plane. A planetoid is assumed to be on the z-axis. This is a particular case of the restricted three body problem with mass ratio μ=1/2, known as the circular Sitnikov problem. Motion of the planetoid is calculated using Lindstedt-Poincare' perturbation and Green's function method. It is found that the planetoid oscillates nonlinearly along the z-axis. We present analytic solutions up to 2nd order of approximation and compare the solutions with earlier results of other authors. A solution of the exact problem is also discussed.

Keywords : Restricted three body, Sintikov problem, Celestial mechanics

1. Introduction

In celestial mechanics, one of the most prestigious problems is the problem of three bodies. One special case of the problem of three bodies is the restricted problem of three bodies (RTBP) which can be enunciated as follows : Two bodies, called primaries, revolve round their centre of mass in circular orbits while a third body, generally called a planetoid, of mass infinitesimally small so that it does not have gravitational influence on the motion of the primaries, moves in their field; the restricted problem of three bodies is to determine the motion of the planetoid (Whittaker 1988). Since the conception for the first time in 1772 by Euler, RTBP has been studied extensively within the framework of Newtonian mechanics in connection with the motions of satellites, comets, asteroids and recently, fictitious planets in binary star systems. Recently, RTBP is also being studied in the framework of the general theory of relativity (Maindl et al. 1994). As such, the stage for RTBP encompasses systems in the solar system through galaxies to the cosmos. In a previous paper (Faruque 2002), we have discussed a special problem in RTBP, namely, linear motion of a planetoid. In this special problem, we assume that two equally massive primaries are in circular motion around their centre of mass. Then, the mass ratio (mass of the smaller primary divided by the total mass of the system) is μ=1/2 and the centre of mass lies halfway between the primaries. We fix the

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coordinate system in such a way that the centre of mass is the origin of a Cartesian coordinate system while the primaries lie on the x-axis. The system is rotating with the primaries, i.e., the coordinate system is synodic. The planetoid is along the z-axis with no velocity component along either the x-or the y-axis. With only z-component of velocity, the planetoid remains along this axis. This is because there is no centrifugal or Coriolis force on the planetoid though these forces usually appear in a rotating coordinate system. This can be proved easily. The problem thus reduces to finding the linear and axial motion of the body along the z-axis.

In literature this problem is known as the circular Sitnikov problem or the MacMillan problem which is the special (e=0) case of the Sitnikov (1960) problem. In the Sitnikov problem one assumes that the planetoid moves on the axis perpendicular to the orbital plane of the equal mass primaries, which move on Keplerian ellipses with eccentricity e between 0 and 1, around their centre of mass. The circular Sitnikov problem, known more as the MacMillan problem, was originally discussed by MacMillan (1911) who showed that this problem is integrable using elliptic integrals. Later on this problem is revisited in Belbruno et al. (1994) where periodic orbits of the MacMillan problem are regard as particular orbits of the Circular Spatial RTBP in order to generate families of periodic orbits of the RTBP. From a similar point of view, Olle and Pacha (1999) studied the Sitnikov and MacMillan problem as well as Planar Isosceles RTBP in order to get families of periodic orbits of the more general Spatial Elliptic RTBP. On the other hand, the problem known as the Sitnikov problem was initiated by Sitnikov (1960) who gave the first qualitative results for some oscillatory motions along the z-axis. Subsequently, many authors reanalyzed them. Perdios et al (1988) studied stability and bifurcations of straight line motions of the infinitesimal body. Dvorak (1993) studied, by numerical means, motion of the planetoid limited to a small region around the barycentre of the primaries and found that invariant curves exist for very small oscillations centering the barycentre. Jie Liu & Yi-Sui Sun (1990) replaced the differential equations by mapping and derived the existence of an hyperbolic invariant set. Hagel (1992) and Hagel & Trenkler (1993) carried out analytical approach for bounded small amplitude solutions. Our aim in this paper is similar, but we consider the circular Sitnikov problem and try to shed more light on this problem considering motion of the planetoid limited to amplitudes much less than the distance between the primaries. This condition puts the problem in a fashion that is more suitable for analytical study. Analytic solution to such restricted case is also important because the exact solution derived by MacMillan (1911) using quadrature involving elliptic integral of the third kind and discussed also by Szebehely (1967) cannot be put as $z=z(t)$ in a closed form. This is one of the reasons MacMillan (1911) presented also a series solution for $z(t)$. It is this solution by MacMillan to which we refer the reader for a critical assessment of our study of circular Sitnikov problem presented in this article. The way we present the problem is more like a physical problem than a mathematical one and this is evident in the content of section II. The procedure we follow in this paper is close to that of MacMillan (1911) with the marked difference in the expansion parameter of perturbation method. While MacMillan used a certain parameter constructed out of the amplitude of motion as the expansion parameter we use a parameter that is constructed out of the angular frequency of orbit of the primaries and invariant distance between the primaries. Moreover, we use Lindstedt-Poincare' perturbation theory to arrive at our solution at the 2nd order approximation. The differences in the formulation of the problem resulted in a nominal difference of the solution. This will be indicated in the text in due course. As mentioned above, this work is a continuation of our earlier work (Faruque 2002) where we have presented solution up to 2nd order approximation using Green's function technique. This method furnishes a Fourier series for the position of the planetoid. However, this series suffers limitation which incited us to use Lindstedt-Poincare' perturbation theory. Nevertheless, we present the results of Green's function technique to show the power of this method and its applicability in solving such celestial methanical problems. Moreover, the way we proceed to solve the problem using Green's function has merits over the same method used in other analogous problems by other authors. This shall be indicated in due course. The paper is organized as follows: In section II, we put the equations of motion and analyze them graphically. In section III, we present the solution up to 2nd order of approximation using Lindstedt-Poincare' method and compare the solution with existing ones. Results of Green's function method and compare the solution with existing ones. Results of Green's function method are included in section IV. In section V, we present a brief note on a solution of the exact problem. Finally, section VI contains discussion and conclusion.

2. Equations of motion

As mentioned in the introduction, the x-axis is the line containing the primaries and z-axis is perpendicular to their place of motion. The origin of the coordinate system is located at the centre of mass. The frame rotates with angular velocity

$$\vec{\omega} = \sqrt{\frac{G(m_1 + m_2)}{a^3}}\,\hat{k} = \sqrt{\frac{2GM}{a^3}}\,\hat{k} \tag{1}$$

where $m_1 = m_2 = M$, the mass of primary and 'a' is the distance between the primaries. The planetoid has coordinate

$$\vec{z} = z\hat{k} \tag{2}$$

The equation of motion of the planetoid is

$$\frac{d^2z}{dt^2} = -\frac{\partial V}{\partial z} \tag{3}$$

with

$$V(z) = -\frac{2GM}{\sqrt{\frac{a^2}{4} + z^2}} \tag{4}$$

as the potential. In this special case, there is no Coriolis or centrifugal force on the planetoid. We assume that the planetoid, initially, has no velocity component along either x- or y-axis. The planetoid moves along the z-axis with the Jacobian integral (Szebehely 1967) as its total energy, which is

$$\frac{1}{2}\left(\frac{dz}{dt}\right)^2 - \frac{2GM}{\sqrt{\frac{a^2}{4} + z^2}} = E \tag{5}$$

In Figure 1, we plot the potential. To draw this graph, we have put $G = 1$, $a = 1$, and $M = 1/2$. The potential is symmetric and in the region close to z=0, it resembles the potential of a harmonic oscillator. Motion of the planetoid will be bounded for E in the range $-2<E<0$. Note that $E = -2$ corresponds to $E = -(4GM)/a$. Clearly, the motion is governed by the value of E. As E increases, the planetoid moves periodically in greater portion of the z-axis centering $z = 0$. However, as larger z is covered by the planetoid, motion would appear not to be simple harmonic.

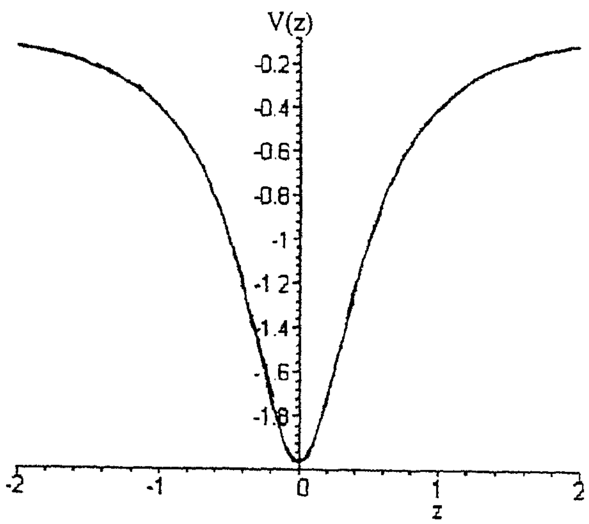

Figure 1. The potential V(z)

Figure 2 is the phase portrait of motion. We have drawn this graph using Eq.(5). As is seen from the graphs, the motion is unbounded for $E \geq 0$. For $E < 0$, there are two turning points ($\dot{z} = 0$). At $E = -2$, the two turning points merge at $z = 0$. The planetoid is forced, then, to remain stationary at the centre of mass.

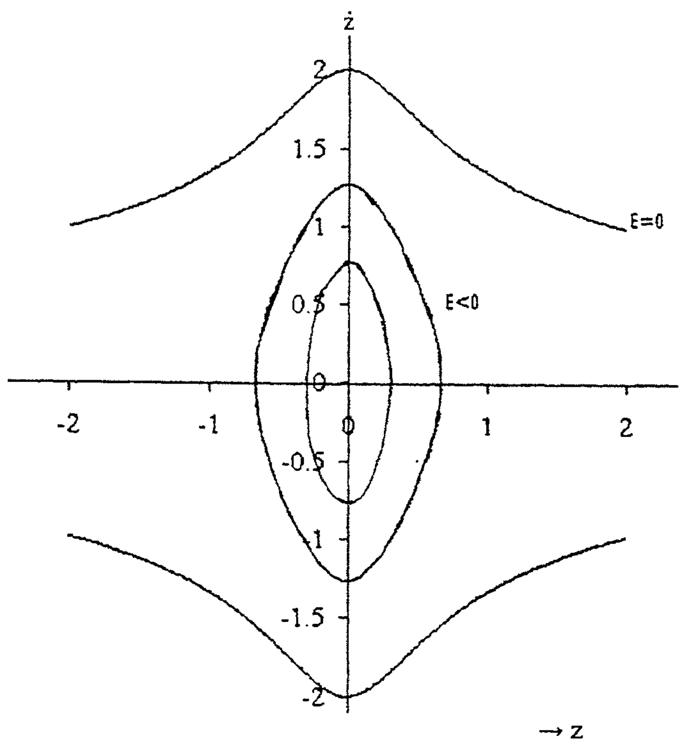

Figure 2. The phase portrait. $\dot{z}$ vs z graph. The plots are drawn using Eq. (5).

For periodic motion with energy $E < 0$, the time period is given by

$$\frac{T}{4} = \int_0^{z_{max}} \frac{dz'}{\left[2\left\{E + \dfrac{2GM}{\sqrt{\dfrac{a^2}{4} + z'^2}}\right\}\right]^{1/2}} \tag{6}$$

where T is the period and z_{max} is the positive root of the radical in the denominator of the integral. The integral can not be evaluated completely in analytic form.

To know the coordinate 'z' of the planetoid as a function of time, we have to solve Eq. (3) analytically. Exact analytic solution of Eq. (3) cannot be found except the solution by quadrature in the form of Eq. (6). We, therefore, look for approximate solutions. In the first approximation, we assume that the energy of the planetoid is such that it is bound to remain near the origin. This happens when E is chosen as close to $-(4GM)/a$.

In this case, z_{max} is much smaller than a / 2 and Eq. (3) approximates to

$$\frac{d^2z}{dt^2} = + \omega_r^2 z = 0 \tag{7}$$

where

$$\omega_r = \sqrt{8\omega} \tag{8}$$

Equation (7) has the well known simple harmonic solution

$$z(t) = A \exp(i\omega_r t) \tag{9}$$

wher A, the amplitude is determined by the energy E, which, in this case, is

$$\frac{1}{2}\left(\frac{dz}{dt}\right)^2 + \frac{1}{2}\omega_r^2 z^2 - \frac{4GM}{a} = E \tag{10}$$

Hence, once E is fixed, the motion is predicted according to Eq. (9). In the 2nd approximation, E is chosen such that z_{max} is comparable to a / 2 but still a / 2 is larger. In this case, the equation of motion reads

$$\frac{d^2z}{dt^2} + \omega_r^2 z - \omega_d^2 z^3 = 0 \tag{11}$$

with

$$\omega_d = \frac{4\sqrt{3}}{a}\omega \tag{12}$$

Solution of Eq. (11) can only be found through the use of perturbation technique. The use perturbation methods, we take Eq. (7) as the unperturbed system with the solution (9) as the unperturbed motion of the planetoid. Then, the term $(-\omega_d^2 z^3)$ appears as the perturbation. The effect of this nonlinear term is to shift the frequency as a function of the amplitude and to distort the trajectory $z(t)$, introducing harmonics of the shifted linear oscillator frequency. In the next section we proceed to solve Eq. (11) using Lindstedt-Poincare' perturbation theory.

3. Solution using Lindstedt-Poincare' perturbation theory

The equation of motion in the 2nd approximation, Eq. (11), is very similar to the equation of Duffing oscillator (The difference lies in the sign of the nonlinear term). We write Eq. (11) as

$$\ddot{z} + \omega_r^2 z - \varepsilon z^3 = 0 \tag{13}$$

where $\varepsilon = \omega_d^2$ is the strength of the perturbation. Standard method of expansion of $z(t)$ in powers of the strength ε can be followed and we, therefore, assume

$$z(t) = z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + ... = \sum_{i=0}^{\infty} \varepsilon^i z_i(t) \tag{14}$$

Insertion of this expansion into Eq. (13) and equating the different terms containing the same power of ε results in an infinite set of coupled differential equations. The subsequent equations can be solved using Green's function technique. However, the resulting solution contains trigonometric terms with one secular term (i.e., term proportional to time t). Since the solution should be periodic, the secular term cannot be allowed in the solution. Here, the Green's function technique fails to give us a periodic solution of Eq. (13). In the next section, we shall show that a general method of applying Green's function exists in quantum mechanics that leads to a solution free of secular terms. For now, we proceed with the method due to Lindstedt and Poincare'. This method consists in making a change of independent variable at the same time as the power series expansion of the solution (Hand et al. 1998). We take an independent variable '*s*' defined by

$$s \equiv \omega t \tag{15}$$

and

$$\omega \equiv \omega_r + \varepsilon\omega_1 + \varepsilon^2 \omega_2 + ... \tag{16}$$

Once again

$$z(s) = \sum_{i=0}^{\infty} \varepsilon^i z_i(s) \tag{17}$$

This defines the series for the solution we anticipate for. Wtih the freedom we get from the constants ω_1, ω_2 etc. we can remove all secular terms from the equations for the z_i. With $z'' \equiv d^2z/ds^2$ now, the equation of motion (13) becomes

$$\omega^2 z'' + \omega_r^2 z - \varepsilon z^3 = 0 \tag{18}$$

or

$$\begin{aligned}(\omega_r + \varepsilon\omega_1 + \varepsilon^2\omega_2 + ...)^2 \frac{d^2}{ds^2} (z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + ..) \\ + \omega_r^2(z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + ..) - \varepsilon (z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + ..)^3 = 0\end{aligned} \tag{19}$$

Equating coefficients of same power of ε in Eq. (19), we get the following set of equations :

$$\omega_r^2 \frac{d^2 z_0}{ds^2} + \omega_r^2 z_0 = 0 \tag{20}$$

$$\omega_r^2 \frac{d^2 z_1}{ds^2} + \omega_r^2 z_1 = z_0^3 + 2\omega_r \omega_1 z_0 \tag{21}$$

$$\omega_r^2 \frac{d^2 z_2}{ds^2} + \omega_r^2 z_2 = 3z_0^2 z_1 + 2\omega_r \omega_1 \left(z_1 - \frac{z_0^3}{\omega_r^2} \right) + (2\omega_r \omega_2 - 3\omega_1^2) z_0 \tag{22}$$

Now, we assume for simplicity, the initial conditions

$$z(0) = A, \tag{23}$$
$$\dot{z}(0) = 0$$

Solution of (20) satisfying the above initial conditions is

$$z_0 = A \cos s \tag{24}$$

Putting this into Eq. (21), we get

$$z_1'' + z_1 = \left[\frac{3A^3}{4\omega_r^2} + \frac{2\omega_1 A}{\omega_r} \right] \cos s + \frac{A^3}{4\omega_r^2} \cos 3s \tag{25}$$

It can be shown that the first term on the right hand side of Eq. (25), namely, the cos s term acts as a "resonant" driving term that leads to a secular term in z_1. Therefore, we must set the coefficient of cos s to be zero and thereby eliminating the secular term. Hence, we get

$$\omega_1 = -\frac{3A^2}{8\omega_r} \tag{26}$$

Equation (25) then reduces to

$$z_1'' + z_1 = \frac{A^3}{4\omega_r^2} \cos 3s \tag{27}$$

Solution of Eq. (27) is

$$z_1 = \frac{A^3}{32\omega_r^2} (\cos s - \cos 3s) \tag{28}$$

That this is true can be checked easily. Now $z_1(0) = 0$, and $(z_0 + z_1)$ at t = 0 is 'A', satisfying our initial condition. Insertion of (24), (26) and (28) into (22) leads to

$$z_2'' + z_2 = \left(\frac{2\omega_2}{\omega_r} A - \frac{15}{128} \frac{A^5}{\omega_r^4} \right) \cos s + \frac{3A^5}{16\omega_r^4} \cos 3s - \frac{3A^5}{128\omega_r^4} \cos 5s \tag{29}$$

Again, we set the coefficient of cos s to be zero to eliminate secular term in z_2. This leads to

The result above [Eq.(34)] agrees well with literature (MacMillan 1911). To compare with MacMillan's result [the first unnumbered equation in page 13 of MacMillan's paper], we note that the solution in this paper is first written in terms of a function $\xi = 1 - z / a$, where z in that paper is the same as our z and a is his amplitude which in our work is A. We note also that a in Eq.(34) is equal to 2 in MacMillan's paper. When these conversions are made and the parameter $\mu = a^2 / (1 + a^2)$ in MacMillan's paper is replaced by a^2 ($= A^2$ in our work), since we impose small amplitude condition (see the discussion below Eq. (10) in this paper), we get good agreement. The differences lie in the weight of 3rd coefficient of cos ωt, 2nd coefficient of cos 3 ωt, and in that of cos 5 ωt. Considering the magnitudes of these terms, we note that the difference is insignificant quantitiatively. In the qualitative part, appearance of only cosines of odd multiples of ωt (τ, in MacMillan's paper) and sign of the different trigonometric terms are exactly same in both works. The minor difference is caused by our approximation of the equation of motion which, under the conditions we impose, can be safely done. In conclusion, we observe that for most purposes an accurate enough description of the position of the planetoid can be obtained from 2nd order approximation we have demonstrated in this section. The angular frequency [Eq.(35)] will be compared with that of MacMillan's (1911) in section V.

In the next section, we shall present the solution using Green's function technique. We shall show that the solution comes out as a series like (34).

4. Solution using Green's function

To solve the equation of motion (11) using Green's function method, we write Eq. (11) as follows

$$\ddot{z} + \omega_r^2 z = (\omega_d^2 z^2)\, z = -\rho(z(t)) \tag{36}$$

We assume the solution as

$$z = A \exp(i\omega_r t) + z_s(t). \tag{37}$$

The first term satisfies the homogeneous equation (7). Hence, we have

$$\frac{d^2 z_s}{dt^2} + \omega_r^2 z_s = \omega_d^2 z^3 \equiv -\rho(z(t)) \tag{38}$$

Now, we define the Green's function G through

$$\frac{d^2 G}{dt^2} + \omega_r^2 G = -\delta(t - t') \tag{39}$$

which yields, for z_s, the integral equation

$$z_s(t) = \int G(t,t')\, \rho(t')\, dt'. \tag{40}$$

The formal solution of Eq. (39), the Green's function, can be written as

$$G(t, t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp\{i\omega(t - t')\}}{\omega^2 - \omega_r^2 - i0} \, d\omega \tag{41}$$

Since, the Dirac delta function, $\delta(t - t')$, can be represented by

$$\delta(t - t') = \frac{1}{2\pi} \int \exp\{i\omega(t - t')\} d\omega \tag{42}$$

the function (41) is the required Green's function. This can be verified easily using (41) and (42) in Eq. (39). Now, we can visualize an interaction picture and consider the term $(\omega_d^2 z^2)$ in Eq. (36) as the interaction. Then, the first iterated solution of Eq. (36), using

$$\rho(t) = \int \delta(t - t')\rho(t')dt' \tag{43}$$

is

$$\begin{aligned} z(t) &= A \exp(i\omega_r t) - \omega_d^2 \int G(t, t')(z(t'))^3 \, dt' \\ &+(-\omega_d^2) \int\int G(t, t')(z(t'))^2 \, G(t', t'')(z(t''))^2 \, z(t'')dt'' \, dt' \end{aligned} \tag{44}$$

The interpretation of the terms in Eq. (44) are as follows : There is an interaction which brings the planetoid, whose position is given in the first approximation by a simple harmonic term (Eq. (9)), from $z(t'')$ to $z(t')$ and then to $z(t)$. And at the instant t'' and t', the interactions are as $\omega_d^2 z^2(t'')$, $\omega_d^2 z^2(t')$ and the propagators are $G(t', t'')$, $G(t, t')$. All such single interactions and double interactions and ad infinitum, combined with the initial position (Eq.(9)) gives the position of the planetoid at time t.

Now, the Green's function can be found using the standard method of residue theorem. We find the Green's function as

$$G(t, t') = \frac{i\exp\{i\omega_r(t - t')\}}{2\omega_r} \tag{45}$$

At this stage we need to know the functions $z(t')$ and $z(t'')$ to be put on the right side of (44). To the first approximation, we can use the solution (9) of the homogeneous equation (7), i.e.,

$$z(t') = A \exp(i\omega_r t') \tag{46}$$

etc. Finally, substitution of (45) and (46) in (44) gives, after integration, the analytic solution in the 2nd approximation :

$$z(t) = \left(A + \frac{3A^3}{2a^2} + \frac{9A^5}{8a^4}\right) \exp(i\omega_r t) - \left(\frac{3A^3}{2a^2} + \frac{9A^5}{4a^4}\right) \exp(3i\omega_r t) + \frac{9A^5}{8a^4} \exp(5i\omega_r t) + \ldots \tag{47}$$

The real part of Eq. (47) furnishes the solution that automatically gives $z(0) = A$ and $\dot{z}(0) = 0$. We note that solution (47) has the same appearance as solution (34) found using Lindstedt-Poincare method. The difference lies in the frequency and coefficients of the terms. Otherwise, number of terms and nature of variation of the terms with A and a are exactly the same. Also, we did not impose initial conditions to arrive at (47); the initial conditions are outcome of the solution once we fix the unperturbed solution. However, validity of (47) is limited: It applies to motion for which A/a is small. This is learned from Eq.(35) where it is seen that $\omega \approx \omega_r$ when A/a is small.

The reason we presented the solution (47) found through the Green's function technique is to show that Green's function method can be applied to this type of problem in a way that does not give rise to secular terms in contrast to the way shown, for example, by Hand et al (1998)

5. Note on solution of the exact problem

In this section, we use $a = 1$, $G = 1$ and $M = 1/2$. The integral of the problem is then

$$\frac{1}{2}\left(\frac{dz}{dt}\right)^2 - \frac{1}{\sqrt{z^2 + \frac{1}{4}}} = -\frac{C}{2} \tag{48}$$

where $C = -2E$, a positive number (see Eq.(5) and Fig.1). The distance between either of the primaries and the planetoid is

$$r = \sqrt{z^2 + \frac{1}{4}} \tag{49}$$

When we put this expression in Eq.(48), we get

$$\dot{r}^2 = \left(\frac{2}{r} - C\right)\left(1 - \frac{2}{4r^2}\right) \tag{50}$$

We now define a function

$$\bar{u} = \frac{1}{r} \tag{51}$$

as the dependent variable. Using this in Eq. (50), we get

$$\left(\frac{d\bar{u}}{dt}\right)^2 = \bar{u}^4\,(2\bar{u} - C)\left(1 - \frac{\bar{u}^2}{4}\right) \tag{52}$$

We now change the scale introducing $u = \frac{1}{2}\bar{u}$ and $c = C/4$. In this notation, we get a quadrature from Eq. (52) :

$$\int_1^u \frac{du}{4u^2\,[(u - c)(1 - u^2)]^{1/2}} = \int_0^t dt \tag{53}$$

where the lower limit is chosen in such a way that at $t = 0$, $u = 1$ or $\bar{u} = 2$ and $r = 1/2$. Consequently, at $t = 0$, $\dot{z} = \sqrt{4 - C}$. The planetoid begins its motion at the origin of the coordinate system. If $C > 4$, motion is not possible at the origin (Compare this with $E < -2$ in Fig. 1). The quadrature of Eq.(53) can be reduced to Legendre form by putting

$$v^2 = \frac{1-u}{1-c}, \; k^2 = \frac{1-c}{2} \tag{54}$$

whence, we get

$$\int_0^v \frac{dv}{(1-2k^2v^2)^2[(1-v^2)(1-k^2v^2)]^{1/2}} = \sqrt{8t} \tag{55}$$

The period is given by

$$\int_0^1 \frac{dv}{(1-2k^2v^2)^2[(1-v^2)(1-k^2v^2)]^{1/2}} = 2^{3/2}\,\frac{T}{4} \tag{56}$$

where T is the period. To evaluate this integral, we put $v = \sin\theta$. Then the integral reduces to

$$2^{3/2}\,\frac{T}{4} = \int_0^{\pi/2} \frac{d\theta}{(1-2k^2\sin^2\theta^2)\,(1-k^2\sin^2\theta)^{1/2}} \tag{57}$$

Employing binomial theorem, we can write this integral as

$$2^{3/2}\,\frac{T}{4} = \int_0^{\pi/2} \left(1 + \frac{9}{2}\,k^2\sin^2\theta + \frac{115}{8}\,k^4\sin^4\theta + \ldots\right) d\theta \tag{58}$$

Using $\int_0^{\pi/2} \sin^p\theta d\theta = \frac{1.3.5...(p-1)}{2.4.6...p}\,\frac{\pi}{2}$, if p is even positive integer, we get

$$T = \frac{\pi}{\sqrt{2}}\left(1 + \frac{9}{4}\,k^2 + \frac{345}{64}\,k^4 + \ldots\ldots\right) \tag{59}$$

The general solution is obtained by inverting the quardature given in Eq. (55) as an elliptic integral of the third kind. However, complete analytic inversion is not possible. Hence, approximation is needed in this solution, too. This is one of the reasons we have presented approximate solutions in the previous sections. Now, the period (59) should be multiplied by $\sqrt{8}$ to arrive at the period [Eq.(8)] in MacMillan's (1911) work. This is due to the definition of the terms we use in calculation. What remains is to show the relation between the period given in (59) and the period corresponding to the frequency in Eq.(35). We note that they are identical up to the 2nd term. This follows since in the period (59), frequency of the frame is unity and distance between the primaries is also unity. When we put the same in Eq.(35), we get

$$T = \frac{\pi}{\sqrt{2}}\left(1 + \frac{9}{2}\,A^2 + \ldots\ldots\ldots\right) \tag{60}$$

Now, A^2 in (60) can be identified as k^2 in (59). This is true since, from Eq.(10), we can write (putting $\dot{z} = 0$, $z = A$, $\omega_r = \sqrt{8}$, $a = 1$, $E = -C / 2$)

$$4 A^2 - 2 = \frac{C}{2}$$

and

$$A^2 = \frac{1 - c}{2} = k^2 \tag{61}$$

Hence, the period of the exact problem and that of the 2nd order approximation are identical upto 2nd order terms. This completes our discussion of the exact solution.

6. Discussion and conclusion

In this article, we have discussed the motion of an infinitesimal body called a planetoid within the framework of the restricted three body problem, known particularly as the circular Sitnikov problem. The geometry of this problem is as follows : Two equally massive primary bodies move in circular orbits around their centre of mass in a Cartesian x-y plane and a third body of infinitesimal mass is on the line perpendicular to the plane of motion of the primaries that passes through the centre of mass. We have analyzed the motion of the planetoid using phase portrait (Fig. 2), which shows that the body would oscillate in simple harmonic fashion if the body is placed near the centroid with only z-component of velocity. The amplitude of this oscillation is determined by the value of the Jacobian integral which, in this case, is simply the total energy of the planetoid. Figure 2 shows also that the planetoid would move periodically for all values of the total energy below zero.

We have presented the position of the planetoid as a function of time up to 2nd order of approximation. In the first approximation, the energy of the planetoid is such that the body remains very close to the centroid and oscillates in simple harmonic motion. The angular frequency of this oscillation is $\sqrt{8}$ times the angular velocity of the primaries. In the 2nd approximation, a nonlinear perturbing force appears in the equation of motion. The equation is solved using the Lindstedt-Poincare' perturbation theory. The coordinate of the planetoid is found to be represented by a Fourier series. The oscillation of the planetoid is thus nonlinear with a frequency that depends on the amplitude of oscillation. This amplitude dependence of frequency is outcome of the method we employed to avoid secular terms in the solution. The solution we found satisfies two specified initial conditions. However, to incorporate the initial conditions we had to put coefficient of one trigonometric term by hand, which otherwise is arbitrary. We solved the equation of motion in the 2nd approximation using also the Green's function method. The solution that follows is free of secular terms whereas other authors (see, for example, Hand et al 1998) found secular terms in solution of similar problems (say Duffing oscillator). The way we employed Green's function method (namely, an interaction picture) thus deserves special attention. Moreover, the solution found through the Green's function method is very similar to the one found through the Lindstedt-Poincare' method. However, the solution found through the latter method is preferable. Solution found by Green's function is applicable when the ratio of amplitude (A) and distance between the primaries (a) is small. In this case, the frequency of oscillation found through the Lindstedt-Poincare theory (Eq.(35)) becomes equal to the frequency found through the Green's function method. We have presented solutions through the two methods to show power of the methods and to get better understanding of the system. Both methods predict nonlinear oscillation of the planetoid along polar axis. The solutions are compared with existing ones and good agreement is found. This indicates that the position of the planetoid can be obtained by 2nd order approximation of the equation of motion when energy of the planetoid is small enough so that the amplitude remains well below the distance between the primaries.

We have also presented a note on a solution, due to MacMillan (1911), of the exact problem for the sake of completeness. The exact problem is solved by quadrature that involves elliptic integral of the third kind. Since, the quadrature can not be fully inverted analytically, approximations are necessary here too. In fact, the period of the exact problem is an indefinite series. However, the period found by the Lindstedt-Poincare' theory in the 2nd approximation is identical to the period of the exact problem up to 2nd order terms. This is shown in the text. In conclusion, we have revisited an old problem and gained helpful insights.

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