PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 59, No. 2
August 2002
pp. 175–179

Spin squeezing and quantum correlations

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Abstract. We discuss the notion of spin squeezing considering two mutually exclusive classes of spin-s states, namely, oriented and non-oriented states. Our analysis shows that the oriented states are not squeezed while non-oriented states exhibit squeezing. We also present a new scheme for construction of spin-s states using $2s$ spinors oriented along different axes. Taking the case of $s=1$, we show that the 'non-oriented' nature and hence squeezing arise from the intrinsic quantum correlations that exist among the spinors in the coupled state.

Keywords. Squeezing; spin; quantum correlation.

PACS Nos 03.65.-W; 03.65.Ta

1. Introduction

It is well-known that the state of a harmonic oscillator is said to be squeezed if the variance Δx^2 or Δp^2 is less than $\frac{1}{2}$ which is the minimum uncertainty limit. Although squeezing is thus unambiguously defined in the case of bosonic systems [1] its definition in the context of spin needs careful consideration. A comparison of the uncertainty relations satisfied by the components of the spin operator $\vec{S}$,

$$\Delta S_x^2 \Delta S_y^2 \geq \frac{\langle S_z \rangle^2}{4}, \quad x,y,z \quad \text{cyclic}, \tag{1}$$

with

$$\Delta x^2 \Delta p^2 \geq \frac{1}{4}, \tag{2}$$

would naturally suggest that a spin state could be regarded as squeezed if ΔS_x^2 or ΔS_y^2 is smaller than $|\langle S_z \rangle|/2$, where the expectation value and the variances are calculated in some arbitrary coordinate system. Indeed this has been used as the squeezing criterion in the literature [2]. Such a definition does not take into consideration the existence of quantum correlations and is coordinate dependent. In an attempt to arrive at a proper criterion for squeezing, Kitagawa and Ueda [2] have considered a model in which a spin-s

state is visualized as being built out of $2s$ elementary spin-$\frac{1}{2}$ states. A coherent spin-s state (CSS) $|\theta,\phi\rangle$ can then be thought of as having no quantum correlations as the constituent $2s$ elementary spins point in the same direction $\hat{n}(\theta,\phi)$, which is the mean spin direction.

2. State classification and squeezing

In order to discuss squeezing, we begin with the squeezing condition itself. Referring to [2,4] we adopt the following definition: A spin-s state is squeezed in the spin component normal to the mean spin direction $\hat{n}$ if

$$\Delta\left(\vec{S}.\hat{n}_{\perp}\right)^2 < \frac{|\langle\vec{S}.\hat{n}\rangle|}{2}, \quad \hat{n} = \frac{\langle\vec{S}\rangle}{\sqrt{\langle\vec{S}\rangle.\langle\vec{S}\rangle}}, \quad \hat{n}.\hat{n}_{\perp} = 0. \tag{3}$$

It is easy to see that the familiar angular momentum states $|sm\rangle_{\hat{n}}$ are not squeezed. But one can however consider superpositions of the states $|sm\rangle_{\hat{k}}$ of the form

$$|\psi\rangle = \sum_m C_m |sm\rangle_{\hat{k}}, \tag{4}$$

and investigate if these exhibit squeezing or not. For this purpose, we classify such states into two mutually exclusive classes, namely, the oriented and non-oriented states, which together exhaust all pure states in the $2s+1$ dimensional spin space of the system.

An oriented spin state by definition is a state $|\psi\rangle$ of the form

$$|\psi\rangle = |sm'\rangle_{\hat{k}'} = \sum_m D^s_{mm'}(\alpha\beta\gamma)|sm\rangle_{\hat{k}}. \tag{5}$$

Here D^s denote the standard rotation matrices and α, β, γ are the Euler angles taking $\hat{i}\hat{j}\hat{k}$ to $\hat{i}'\hat{j}'\hat{k}'$. If we now calculate the variance perpendicular to the mean spin direction, it indeed turns out to be exactly equal to

$$\Delta\left(\vec{S}.\hat{n}_{\perp}\right)^2 = \frac{1}{2}\left(s(s+1) - m'^2\right), \tag{6}$$

which is never less than $\frac{1}{2}|\langle\vec{S}.\hat{n}\rangle|$. Thus no oriented pure state is a squeezed state.

Any normalized spin-s state $|\psi\rangle$ of the form (4) is, in general, specified by $4s$ real independent parameters. The oriented states described above are specified at the most by the three independent Euler angles α, β and γ. Since $4s > 3$, for $s \geq 1$, there exist states which are not oriented. In other words, there exist states which can not be identified as eigen states of S^2 and S_z with respect to any choice of the axis of quantization. We refer to such states as non-oriented. While an oriented state is characterized by a single direction, viz., the axis of quantization (specified by two real variables θ,ϕ) in the physical space, a non-oriented state could be characterized by more than one direction. In order to see whether squeezing exists for a non-oriented state we now start with an arbitrary state $|\psi\rangle$ and first determine its mean spin direction $\hat{z}_0$. The most general spin-1 state that possesses a non-zero mean spin value $\langle\vec{S}\rangle$, can be written in the form

$$|\psi\rangle = \cos\delta\ |1,1\rangle_{\hat{z}_0} + \sin\delta\ |1,-1\rangle_{\hat{z}_0}, \qquad 0 < \delta < \pi, \tag{7}$$

where $|1,m_0\rangle_{\hat{z}_0}$ are the angular momentum states specified with respect to $\hat{z_0}$. This state is obviously non-oriented for all values of δ other than $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$. For such a state referred to the frame $x_0y_0z_0$, the squeezing conditions for S_{x_0} and S_{y_0} are respectively given by

$$1+\sin 2\delta < |\cos 2\delta| \tag{8}$$

and

$$1-\sin 2\delta < |\cos 2\delta|. \tag{9}$$

These conditions are indeed separately valid for the entire range except for $\delta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, which implies that a non-oriented state $|\psi\rangle$ is indeed a squeezed state.

3. Quantum correlations

Having thus identified the squeezed states in the spin-1 case, it is of interest to analyse in quantitative terms if squeezing in spin systems arises from the existence of quantum correlations. This can be done by employing the model in which a spin-s state is constructed using $2s$ spin-$\frac{1}{2}$ states. Majorana's geometric realization [5] of a spin-s state as a constellation of $2s$ points on a sphere leads to Schwinger's idea [6] of realising $|sm\rangle$ states in the form

$$|sm\rangle = \frac{\left(a_+^\dagger\right)^{s+m}\left(a_-^\dagger\right)^{s-m}}{\left((s+m)!\,(s-m)!\right)^{\frac{1}{2}}}|00\rangle, \tag{10}$$

where $a_+^\dagger, a_-^\dagger$ are the creation operators for the spin 'up' and spin 'down' states, respectively. It must be noted here that spin 'up' and spin 'down' states as well as $|sm\rangle$ states are all referred to the same axis of quantization.

At this point, we would like to generalize this realization by taking the $2s$ 'up' spinors $u(\theta_l,\phi_l),\ l=1,\ldots,2s$, where the kth spinor is specified with respect to an axis of quantization $\hat{Q}_k(\theta_k\phi_k)$ in the physical space. Coupling $2s$ spin-$\frac{1}{2}$ states in this way leads to a spin-s state in the form (4), where the coefficients C_m are given by

$$C_m = N_s d_m, \quad N_s^{-1} = \left(\sum_{m=-s}^{s}|d_m|^2\right)^{1/2} \tag{11}$$

and

$$\begin{aligned} d_m = \sum_{m_1,\ldots,m_{2s-1}} & C(\tfrac{1}{2}\tfrac{1}{2}1;\ m_1m_2\mu_1)C(1\tfrac{1}{2}\tfrac{3}{2};\ \mu_1m_3\mu_2)\cdots C(s-\tfrac{1}{2}\tfrac{1}{2}s;\ \mu_{2s-2}m_{2s}m) \\ & \times D^{\frac{1}{2}}_{m_1\frac{1}{2}}(\phi_1\theta_1 0)\cdots D^{\frac{1}{2}}_{m_{2s}\frac{1}{2}}(\phi_{2s}\theta_{2s}0). \end{aligned} \tag{12}$$

Thus our construction of a spin-s state $|\psi\rangle$ is done using $2s$ spin-$\frac{1}{2}$ states which are specified with respect to $2s$ different directions, $\hat{Q}_1,\hat{Q}_2,\ldots,\hat{Q}_{2s}$ in general. In particular, if $\hat{Q}_1 = \pm\hat{Q}_2 = \cdots = \pm\hat{Q}_{2s}$, then our construction specializes to the realization suggested by Schwinger and employed in refs [2–4]. Indeed, in this particular case, the spin state realized is nothing but an oriented state $|sm\rangle$. The significance of our construction lies in the fact that if $\hat{Q}_l \neq \pm\hat{Q}_m$ for at least two quantization directions, the state realized is a non-oriented state of spin-s.

Considering in particular the simplest case of $s = 1$, we note that such a construction can be carried out using two spinors specified with respect to $\hat{Q}_1(\theta_1\phi_1)$ and $\hat{Q}_2(\theta_2\phi_2)$ so that the spin-1 state

$$|\psi\rangle = N_1 \sum_{m_1,m} D^{1/2}_{m_1 1/2}(\phi_1\theta_1 0) D^{1/2}_{m_2 \frac{1}{2}}(\phi_2\theta_2 0) C(\tfrac{1}{2}\tfrac{1}{2}1; m_1 m_2 m)|(\tfrac{1}{2}\tfrac{1}{2})1m\rangle \tag{13}$$

in the lab frame $\hat{i}\hat{j}\hat{k}$ is non-oriented if $\hat{Q}_1 \neq \pm\hat{Q}_2$. The mean spin direction $\hat{z}_0$ for such a state happens to be along the bisector of the two directions $\hat{Q}_1$ and $\hat{Q}_2$. The squeezing condition for S_{x_0} now takes the form

$$\cos^2\theta < |\cos\theta| \tag{14}$$

which is satisfied for all θ except when $\theta = 0,\ \pi/2,\ \pi$. The absence of squeezing for $\theta = 0, \pi/2, \pi$ is obvious as the two axes then merge together giving an oriented state. Thus in all other cases the state $|\psi\rangle$ is squeezed in the spin component S_{x_0} and is non-oriented by construction.

We now establish explicitly for $s = 1$, the connection between squeezing and the spin–spin correlations that exist between the component spinors. Any spin-1 state constructed using the two spinors is said to possess spin correlations if the matrix C^{12} defined through its elements

$$C^{12}_{\mu\nu} = \Delta(S_{1\mu}S_{2\nu}) = \langle S_{1\mu}S_{2\nu}\rangle - \langle S_{1\mu}\rangle\langle S_{2\nu}\rangle \tag{15}$$

is non-zero. Here $S_{1\mu}$ and $S_{2\nu}$ are the spin components associated with the two spinors and the angular brackets denote the expectation values with respect to the coupled state. For the state $|\psi\rangle$ in (7), the correlation matrix is diagonal in the frame $x_0y_0z_0$ with the 'diagonal' or the 'eigen' correlation elements given by

$$C^{12}_{x_0x_0} = -\left[\frac{\sin^2\theta}{4(1+\cos^2\theta)}\right] = -C^{12}_{y_0y_0}, \quad C^{12}_{z_0z_0} = \left[\frac{\sin^2\theta}{2(1+\cos^2\theta)}\right]^2. \tag{16}$$

A glance at these expressions shows that when $\theta = 0, \pi/2, \pi$, the values of the correlations are either 0 or $\pm 1/4$. On the other hand for all other values of θ, the eigen correlations satisfy

$$0 < |C^{12}_{ii}| < 1/4, \quad i = x_0, y_0, z_0. \tag{17}$$

In other words, all non-oriented spin-1 states have the eigen correlations restricted to the above range. One can also see that the trace of the correlation matrix is

$$\mathrm{Tr}(C^{12}) = \left[\frac{\sin^2\theta}{2(1+\cos^2\theta)}\right]^2. \tag{18}$$

This being invariant under rotations of the coordinate frames, satisfies the condition

$$0 \le \mathrm{Tr}(C^{12}) \le 1/4. \tag{19}$$

Indeed if a given coupled state has a correlation matrix that satisfies this condition, the state is squeezed. We can find the value of θ through

$$\cos\theta = \pm \left[\frac{1 - 2\sqrt{\mathrm{Tr}(C^{12})}}{1 + 2\sqrt{\mathrm{Tr}(C^{12})}} \right]^{1/2}, \tag{20}$$

which identifies the structure of the state in terms of the two spinors. The four values of θ that satisfy the above equation correspond to the directions $\pm\hat{Q}_1$ and $\pm\hat{Q}_2$. Thus we conclude that the trace condition (19) on the correlation matrix is the necessary and sufficient condition for a spin-1 state to be squeezed.

4. Conclusions

We have classified spin states into two mutually exclusive classes, namely, oriented and non-oriented states, and studied their squeezing properties. It is clear from our analysis that squeezing is exhibited only by non-oriented states. Considering in particular the non-oriented states of a spin-1 system, we have shown that they exhibit squeezing. This has been illustrated in two different ways: first by looking at the non-oriented nature of the spin-1 state itself, and secondly, by introducing a new form of coupling in which two spin-$\frac{1}{2}$ states add up to give the required spin-1 non-oriented state. Our construction gives a quantitative description of the existence of quantum correlations as well as an indication as to how they lead to non-oriented nature and hence to the squeezing behavior.

This intimate relationship between squeezing and 'non-oriented' nature indeed suggests a way to prepare a squeezed state. The non-oriented states are potential candidates for observing squeezing experimentally. A recent study by Ramachandran and Deepak [7] reveals that the collision of a spin-$\frac{1}{2}$ beam with a spin-$\frac{1}{2}$ target, both oriented in different directions, leads to a combined spin state which is non-oriented.

Acknowledgements

The authors (GR and PND) acknowledge with thanks the financial support of CSIR, India.

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